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# TOTAL CURVATURE OF NONCOMPACT PIECEWISE RIEMANNIAN 2-POLYHEDRA

By

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**Abstract.** In this paper, we treat *piecewise Riemannian* 2*-polyhedra* which are combinatorial 2-polyhedra such that each 2-simplex is isometric to a triangle bounded by three smooth curves on some Riemannian 2-manifold. We will introduce the *total curvature* $C(X)$ of a piecewise Riemannian 2-polyhedron $X$ not only in the compact case but also in the noncompact case, and obtain some generalizations of the Gauss-Bonnet theorem and the Cohn-Vossen theorem.

Furthermore, we will show that the difference between $C(X)$ and some value concerning to the topology of $X$ coincides with some expanding growth rate of $X$.

## §1. Introduction

It is well-known as the Gauss-Bonnet theorem that the total curvature of a compact Riemannian 2-manifold $M$ without boundary is equal to $2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of $M$, and also known as the Cohn-Vossen theorem that the total curvature of noncompact $M$ is not greater than $2\pi\chi(M)$. These theorems are very famous and elegant, and it is important to generalize them to wider classes of objects. There are many approaches to do it. For example, Banchoff's result [1] is one of excellent generalizations, whose object is a piecewise linear finite polyhedron of an arbitrary dimension, and Ballmann-Buyalo's result [2] is on a cocompact piecewise Riemannian 2-polyhedron. But these results essentially treat the case of compact objects and the total curvature

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is determined only by the Euler characteristic. We would like to investigate the noncompact case, which could probably lead to more attractive results.

The purpose of our study in this paper is to investigate some properties of *noncompact piecewise Riemannian 2-polyhedra* which are combinatorial infinite 2-polyhedra such that each 2-simplex is isometric to the face of a triangle consisting of three smooth curves on some Riemannian 2-manifold. First, we will introduce the *total curvature* $C(X)$ of a compact piecewise Riemannian 2-polyhedron $X$ and prove the following generalization of the Gauss-Bonnet theorem.

THEOREM 3.2. *Let $X$ be a compact piecewise Riemannian 2-polyhedron and $\mathscr{B}X$ be the closure of the set of all free faces of $X$. Then*

$$C(X) + \sum_{p \in \mathscr{B}X} k(p) + \int_{\mathscr{B}X} \kappa d_X = 2\pi\chi(X),$$

*where $\chi(X)$ is the Euler characteristic of $X$, $k(p)$ the singular curvature at $p$ and $\kappa$ the geodesic curvature.*

Note that, for a Riemannian manifold $M$ with boundary, the boundary $\partial M$ coincides with the closure of the set of all free faces of $M$ for any triangulation of $M$. Therefore, for a piecewise Riemannian 2-manifold $X$, we shall consider $\mathscr{B}X$ as the boundary of $X$. And we also note that, for a point $p$ on the boundary of a Riemannian manifold, $k(p)$ means the exterior angle at $p$.

We would like to notice that in the case of $\mathscr{B}X = \emptyset$, the above theorem coincides with the Gauss-Bonnet Formula 2.3 (the case $\Gamma = \{id\}$) in [2]. However, to investigate the noncompact case, it is important how to consider and treat "boundaries" of finite 2-polyhedra. Therefore we will introduce precise definitions of the boundary $\mathscr{B}X$ and the total curvature $C(X)$, and prove Theorem 3.2.

Next, let $X$ be a finitely connected complete piecewise Riemannian 2-polyhedron without free faces. Since $X$ is finitely connected, the topological ideal boundary $X_\infty$ of $X$ is defined naturally. For such an noncompact $X$, we will define the *total curvature* $C(X)$ and *w-total curvature* $\tilde{C}(X)$. If $X$ admits total curvature $C(X)$, then $X$ also admits w-total curvature $\tilde{C}(X)$, and then $C(X) = \tilde{C}(X)$ provided $\mathscr{B}X = \emptyset$. For Riemannian 2-manifolds, to admit total curvature is equivalent to admit w-total curvature. But for 2-polyhedra, there is an essential difference, and we will show it in Section 4. Then we will prove the following theorem of the Cohn-Vossen type.

THEOREM 4.5. *If $X$ admits total curvature $C(X)$, then*

$$C(X) \leqq 2\pi\chi(X) + \pi\chi(X_\infty).$$

We will also illustrate that the above theorem dose not hold under w-total curvature.

Furthermore, concerning the above generalized Cohn-Vossen theorem, we will investigate the significance of the difference between the total curvature and the upper estimate:

$$2\pi\chi(X) + \pi\chi(X_\infty) - C(X).$$

Let $X$ be a finitely connected, noncompact complete piecewise Riemannian 2-polyhedron admitting total curvature $C(X)$. Then there is a compact domain $K$ of $X$ with a piecewise smooth boundary such that $X\backslash K$ is homeomorphic to $X_\infty \times \mathbf{R}$. We will divide $X\backslash K$ into some suitable simplices $\{e_\lambda\}$ $(\lambda \in \Lambda)$. For a precise definition, see Section 5. For each surface component $e_\lambda$ of $X\backslash K$, let $d_\lambda$ be the interior metric on the closure of $e_\lambda$ induced from the piecewise Riemannian metric $d$ on $X$, and let $c_t := \bigcup_\lambda \{x \in e_\lambda \mid d_\lambda(x, K) = t\}$ and $K_t := \bigcup_\lambda \{x \in e_\lambda \mid d_\lambda(x, K) \leqq t\} \cup K$. We denote by $L(t)$ the length of $c_t$ and by $A(t)$ the area of $K_t$. Then we have

Theorem 5.3.

$$\lim_{t \to \infty} \frac{L(t)}{t} = \lim_{t \to \infty} \frac{2A(t)}{t^2} = 2\pi\chi(X) + \pi\chi(X_\infty) - C(X)$$

This is a generalization of Fiala's result in [5] (cf. Hartman [6] and Shiohama [11]), and we would like to suggest to refer to [12] concerning to an isoperimetric problem for infinitely connected Riemannian manifolds.

We would like to prove the above theorem under simpler situations. However, for example, it is not true for $c_t := \{x \in X\backslash K \mid d(x, K) = t\}$, or for $c_t$ being a distance sphere from an arbitrary fixed point on $X$. We will also illustrate the counter example in this case.

## § 2. Preliminaries

We begin with reviewing relevant basic terminologies. For a metric space $(X, d)$ and an interval $I \subset \mathbf{R}$, a curve $\alpha : I \to X$ is called a *geodesic* if it is locally distance minimizing. In what follows we assume that $\alpha$ is parameterized by arc length. If it is globally distance minimizing, then we call $\alpha$ a *minimizing geodesic*. In particular, a minimizing geodesic defined on $[0, \infty)$ is called a *ray*. We sometimes identify geodesics with their images.

Now we introduce the definition of a piecewise Riemannian 2-polyhedron. Let $X$ be a 2-dimensional locally finite simplicial complex. In this paper, the word "simplex" means an open simplex. In what follows, we also denote the point-set of union of all the simplices of $X$, the *polyhedron* of $X$, by the same symbol $X$. We introduce a natural metric $d$ on a 2-dimensional polyhedron, simply 2-polyhedron, $X$ as follows.

First for each 2-simplex $\Delta$, we take a metric $d_\Delta$ on it such that $(\Delta, d_\Delta)$ is isometric to some triangle bounded by a piecewise smooth simple closed curve on a Riemannian 2-manifold whose break points are corresponding to vertices of $\Delta$. Here we agree that the induced metric on a 1-simplex adjacent to some 2-simplices is independent of the choice of adjacent 2-simplices. Namely, if a 1-simplex $c$ is adjacent to 2-simplices $\Delta_1, \dots, \Delta_n$, then the induced metric $d_i$ from $\Delta_i$ on $c$ coincides with each other. For each 1-simplex which is not a proper face of any 2-simplex, we may choose any metric. For any pair of points $x, y \in X$, $\gamma : [a, b] \to X$ is called a piecewise smooth curve from $x$ to $y$ if $\gamma(a) = x$, $\gamma(b) = y$ and there is a sequence $a = t_0 < t_1 < \cdots < t_k = b$ such that $\gamma|_{[t_{i-1}, t_i]}$ is contained in a closure of some 2-simplex for each $i$ and is a smooth curve with respect to the Riemannian metric on the simplex. The length of $\gamma$ is denoted by $l(\gamma) := \sum_{i=1}^{k} l(\gamma|_{[\tau_{i-1}, \tau_i]})$, where $l(\gamma|_{[t_{i-1}, t_i]})$ is the length with respect to the differentiable structure on the simplex. Now we define the metric $d$ by

$$d(x, y) := \inf\{l(\gamma) \mid \gamma \text{ is a piecewise smooth curve from } x \text{ to } y\}.$$

DEFINITION 2.1. We call such a space $(X, d)$ a *piecewise Riemannian* 2-*polyhedron* and $d$ a *piecewise Riemannian metric*. If $d$ is a complete metric, then $(X, d)$ is called a *complete* piecewise Riemannian 2-polyhedron.

An $i$-simplex $\Delta$ of a polyhedron $X$ is called a free face if there is only one $(i+1)$-simplex of $X$ which contains $\Delta$ as a face. For a piecewise Riemannian 2-polyhedron $X$, the closure of the point-set of union of free faces is denoted by $\mathscr{B}X$. In our case, free faces are either 1-dimensional or 0-dimensional. The complement of it, $X \backslash \mathscr{B}X$, is denoted by $\mathscr{I}X$. It is clear that the definitions are independent of the choice of divisions of $X$.

A piecewise Riemannian 2-polyhedron $X$ is said to be *piecewise linear* if each 2-simplex is isometric to a geodesic triangle on the Euclidean plane $\mathbf{R}^2$.

For a point $p$ on a piecewise Riemannian 2-polyhedron $X$, we denote by $\mathscr{R}_p$ the set of all minimizing geodesics emanating from $p$. For $\alpha, \beta \in \mathscr{R}_p$ we define the *angle* at $p$ as follows: For an arbitrarily constant $k$, we denote by $M(k)$ the 2-dimensional space form of constant sectional curvature $k$. For a geodesic triangle $\Delta(\alpha(s)p\beta(t))$, let $\tilde{\Delta}(\alpha(s)p\beta(t))$ be a geodesic triangle sketched in $M(k)$ whose corresponding edges have same length as $\Delta(\alpha(s)p\beta(t))$, and let $\tilde{\angle}_k(\alpha(s)p\beta(t))$ be the angle at $p$ of $\tilde{\Delta}(\alpha(s)p\beta(t))$. Then it is clear that the limit

$$\angle_p(\alpha,\beta) := \lim_{s,t\to 0} \tilde{\angle}_k(\alpha(s)p\beta(t))$$

exists, which is independent of the choice of $k$. We call it the *angle* at $p$ subtended by $\alpha$ and $\beta$. It is easily seen that the angle $\angle_p$ is a pseudo-metric on $\mathscr{R}_p$ and induces an equivalence relation $\sim$ defined as follows: $\alpha \sim \beta$ if and only if $\angle_p(\alpha,\beta) = 0$. The completion of the metric space $(\mathscr{R}_p/_\sim, \angle_p)$ is denoted by $(\Sigma_p, \angle_p)$ and is called the *space of directions* at $p$. For a subset $Y$ of $X$, let

$$\mathscr{R}_p^Y := \{\alpha \in \mathscr{R}_p \mid \alpha([0,\varepsilon]) \subset Y \text{ for some } \varepsilon > 0\}.$$

The *space of directions with respect to* $Y$, denoted by $\Sigma_p^Y$, is the completion of the metric space $\mathscr{R}_p^Y/_\sim$.

For a point $p$ on a piecewise Riemannian 2-polyhedron $X$, the *regular curvature* $K(p)$ is defined as follows: $K(p)$ is the Gaussian curvature if $p$ is on some open 2-simplex of $X$ or $K(p) = 0$ otherwise.

For $p \in X$, we will define another curvature. Fix a subdivision of $X$ in which $p$ is a vertex. Then let

$$k(p) = \pi(2 - \chi(LK(p))) - L(\Sigma_p),$$

where $\chi(LK(p))$ is the Euler characteristic of the point-set of the linked complex $LK(p)$ of $p$, that is $\chi(LK(p)) := a_p - b_p$, where $a_p$ is the number of 1-simplices adjacent to $p$ and $b_p$ the number of 2-simplices adjacent to $p$, and $L$ is the 1-dimensional Hausdorff measure on $\Sigma_p$. By definition, $LK(p)$ is the sum of simplices $\sigma$ on $X$ such that the cone with vertex $p$ and base $\sigma$ is also a simplex of $X$. $k(p)$ is called the *singular curvature* at $p$ in this paper. It is clear that, if $p$ is not a vertex of $X$, then $k(p) = 0$.

## § 3. Compact Case

Let $X$ be a compact piecewise Riemannian 2-polyhedron, namely a polyhedron of a finite complex with a piecewise Riemannian metric. In this section, we will define the total curvature of $X$, which is a generalization of total curvature of Riemannian manifolds and prove a generalized Gauss-Bonnet theorem.

Let $C(\Delta)$ be the total curvature of a Riemannian 2-manifold $\Delta$, and put

$$e_{reg}(X) := \sum_{\Delta:\text{2-simplex}} C(\Delta),$$

namely the integral of the regular curvature $K$ on $X$, which is called the *regular total curvature*.

Next we will define a *singular total curvature*. For a 2-simplex $\Delta$, there is an isometric triangle $\tilde{\Delta}$ bounded by three smooth curves in a Riemannian 2-manifold $M(\Delta)$. Let $c$ be a 1-dimensional face of $\Delta$ and $\tilde{c}$ the corresponding smooth curve on the boundary $\partial\tilde{\Delta}$. For such a pair $(c, \Delta)$, $\int_c \kappa d_\Delta$ is defined by the integral of the geodesic curvature $\kappa$ on $\tilde{c}$, namely $\int_{\tilde{c}} \kappa d_{M(\Delta)}$. Then we define $e_{sing}(X)$ by

$$e_{sing}(X) := \sum_{p \in \mathscr{I}X} k(p) + \sum_{(c,\Delta)} \int_c \kappa d_\Delta$$

and call it the *singular total curvature* of $X$, where the summation of the second term is taken over all pairs $(c, \Delta)$ of an open 1-simplex $c \subset \mathscr{I}X$ and a 2-simplex $\Delta$ adjacent to $c$.

Now we define the total curvature as follows.

Definition 3.1. The total curvature $C(X)$ is defined by

$$C(X) := e_{reg}(X) + e_{sing}(X).$$

Then we have the following generalized Gauss-Bonnet theorem.

Theorem 3.2. *Let $X$ be a compact piecewise Riemannian 2-polyhedron. Then we have*

$$C(X) + \sum_{p \in \mathscr{B}X} k(p) + \int_{\mathscr{B}X} \kappa d_X = 2\pi\chi(X),$$

*where $\chi(X)$ is the Euler characteristic of $X$.*

Remark 3.3. Since a 1-simplex on $\mathscr{B}X$ is a proper face of the unique 2-simplex, the geodesic curvature at a point of $\mathscr{B}X$ is also uniquely determined. Hence the last term is expressed as above. If $X$ is a Riemannian 2-manifold with boundary $\partial X$, then $\mathscr{B}X$ coincides with $\partial X$ and $k(p)$ is the exterior angle at $p \in \mathscr{B}X$. Therefore it is a generalization of the Gauss-Bonnet theorem on Riemannian 2-manifolds.

Remark 3.4. In the definition of singular total curvature in this paper, points on the closure of the free faces are treated separately from the other points. However Banchoff [1] and Ballmann-Buyalo [2] did not divide them in their definitions of total curvature for compact piecewise linear or Riemannian polyhedra. If we follow their fashion, we should define the total curvature $\tilde{C}(X)$ of a compact piecewise Riemannian 2-polyhedron $X$ by

$$\tilde{C}(X) = e_{reg}(X) + \tilde{e}_{sing}(X),$$

where $\tilde{e}_{sing}(X) := \sum_{p \in X} k(p) + \sum_{(c,\Delta)} \int_c \kappa d_\Delta$ and $c$ is taken over all 1-simplex of $X$. Then we have $\tilde{C}(X) = 2\pi\chi(X)$, cf. Theorem 4 in [1] and Gauss-Bonnet Formula 2.3 in [2].

PROOF OF THEOREM 3.2. Let $\mathscr{F} = \{\Delta_k\}$ be the open 2-simplices of $X$, $\mathscr{S} = \{c_j\}$ the open 1-simplices, and $\mathscr{V}$ the vertices. For a vertex $x \in \mathscr{V}$, $a_x$ denotes the number of 1-simplices adjacent to $x$ and $b_x$ the number of such 2-simplices. Then, using the Gauss-Bonnet theorem for any 2-simplex $\Delta \in \mathscr{F}$, we have

$$\begin{aligned} C(X) &= e_{reg}X + e_{sing}X \\ &= 2(\#\mathscr{F})\pi - \sum_{x \in \mathscr{V}, \Delta \in \mathscr{F}(x \in \mathscr{B}\Delta)} (\pi - L(\Sigma_x^\Delta)) - \sum_{\Delta \in \mathscr{F}} \int_{\mathscr{B}\Delta} \kappa d_\Delta \\ &\quad + \sum_{x \in \mathscr{V} \cap \mathscr{I}X} (2\pi - a_x\pi + b_x\pi - L(\Sigma_x)) + \sum_{(c,\Delta) \in \mathscr{S} \times \mathscr{F}, c \subset \mathscr{I}X} \int_c \kappa d_\Delta, \end{aligned}$$

where $\#\mathscr{F}$ is the cardinal number of $\mathscr{F}$ and $\mathscr{B}\Delta$ is the point-set of union of proper faces of $\Delta$. Note that $\sum_{x \in \mathscr{V} \cap \mathscr{I}X} a_x\pi = \sum_{x \in \mathscr{V}} a_x\pi - \sum_{x \in \mathscr{V} \cap \mathscr{B}X} a_x\pi = 2\pi\#\mathscr{S} - \sum_{x \in \mathscr{V} \cap \mathscr{B}X} a_x\pi$ and $b_x\pi - L(\Sigma_x) = \sum_{\Delta \in \mathscr{F}(x \in \mathscr{B}\Delta)} (\pi - L(\Sigma_x^\Delta))$. Hence

$$\begin{aligned} C(X) &= 2(\#\mathscr{F})\pi - \sum_{x \in \mathscr{V} \cap \mathscr{B}X, \Delta \in \mathscr{F}(x \in \mathscr{B}\Delta)} (\pi - L(\Sigma_x^\Delta)) - \sum_{(c,\Delta) \in \mathscr{S} \times \mathscr{F}, c \subset \mathscr{B}X} \int_c \kappa d_\Delta \\ &\quad + 2\pi\#(\mathscr{V} \cap \mathscr{I}X) - 2\pi\#\mathscr{S} + \sum_{x \in \mathscr{V} \cap \mathscr{B}X} a_x\pi \\ &= 2\pi\chi(X) - 2\pi\#(\mathscr{V} \cap \mathscr{B}X) + \sum_{x \in \mathscr{V} \cap \mathscr{B}X} a_x\pi \\ &\quad - \sum_{x \in \mathscr{V} \cap \mathscr{B}X} (b_x\pi - L(\Sigma_x)) - \int_{\mathscr{B}X} \kappa d_X \\ &= 2\pi\chi(X) - \sum_{x \in \mathscr{V} \cap \mathscr{B}X} \{\pi(2 - a_x + b_x) - L(\Sigma_x)\} - \int_{\mathscr{B}X} \kappa d_X \\ &= 2\pi\chi(X) - \sum_{x \in \mathscr{B}X} k(x) - \int_{\mathscr{B}X} \kappa d_X. \end{aligned}$$

This completes the proof. $\square$

## § 4. Noncompact Case

To begin with, we will introduce two kinds of definitions of total curvature of a noncompact complete piecewise Riemannian 2-polyhedron $X$, which are both natural.

DEFINITION 4.1. Let $\{D_i\}$ be an increasing sequence of compact piecewise Riemannian 2-polyhedra such that $\bigcup D_i = X$. If the limit $\lim_{i\to\infty} C(D_i)$ exists on $[-\infty, \infty]$ and is independent of the choice of $\{D_i\}$, then it is called the *total curvature* of $X$ and is denoted by $C(X)$. If $C(X)$ is defined, then $X$ is said to admit total curvature.

DEFINITION 4.2. Let $e_{reg}(X)$ be the improper integral of $K$ on $X$, and $\tilde{e}_{sing}(X) := \sum_{p\in X} k(p) + \sum_{(c,\Delta)} \int_c \kappa d_\Delta$ provided the sum is absolutely convergent. Note that $\{p \in X \mid k(p) \neq 0\}$ is contained in the set of the vertices of $X$, which is a countable set. If the sum $e_{reg}(X) + \tilde{e}_{sing}(X)$ makes sense, then it is called the *w-total curvature* of $X$ and is denoted by $\tilde{C}(X)$. If $\tilde{C}(X)$ is defined, then $X$ is said to admits w-total curvature.

Definition 4.2 is restated as follows: Let $\{D_i\}$ be an increasing sequence of a compact piecewise Riemannian 2-polyhedron $X$ such that $\bigcup D_i = X$. We denote by $\partial D_i$ the topological boundary of $D_i$ as a subset of $X$. If the limit $\lim_{i\to\infty}\{\tilde{C}(D_i) - \sum_{p\in\partial D_i} k(p) - \sum_{(c,\Delta), c\subset\partial D_i} \int_c \kappa d_\Delta\}$ exists on $[-\infty, \infty]$ and is independent of the choice of $\{D_i\}$, then it is called the *w-total curvature* of $X$ and is denoted by $\tilde{C}(X)$. If $\tilde{C}(X)$ is defined, then $X$ is said to admit w-total curvature.

It is easily seen that if a piecewise Riemannian 2-polyhedron $X$ without free faces admits total curvature $C(X)$, then $X$ admits w-total curvature $\tilde{C}(X)$ and $\tilde{C}(X) = C(X)$.

Note that, for a Riemannian 2-manifold without boundary, above two definitions are equivalent, but admitting total curvature is strictly stronger than admitting w-total curvature for a piecewise Riemannian 2-polyhedron without free faces, which is shown in the following example.

EXAMPLE 4.3. We will illustrate an example which admits w-total curvature and dose not admit total curvature.

Let $X$ be a piecewise linear 2-polyhedron consisting of a flat cylinder attaching a broken flat strip defined as follows:

$$X := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid (x_1, x_2) \in A\}$$
$$\cup \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid (x_1, x_3) \in B, 0 \leqq x_2 \leqq 3\},$$

where $A := \{(x_1, x_2) \in \mathbf{R}^2 \mid 0 \leqq x_1 \leqq 3, x_2 = 0 \text{ or } 3\} \cup \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 = 0 \text{ or } 3, 0 \leqq x_2 \leqq 3\}$ and $B := \{(t+1, t+2n) \in \mathbf{R}^2 \mid 0 \leqq t \leqq 1, n \in \mathbf{Z}\} \cap \{(2-t, t+2n+1) \in \mathbf{R}^2 \mid 0 \leqq t \leqq 1, n \in \mathbf{Z}\}$. See Figure 1. Then $X$ does not admit total curvature, but w-total curvature.

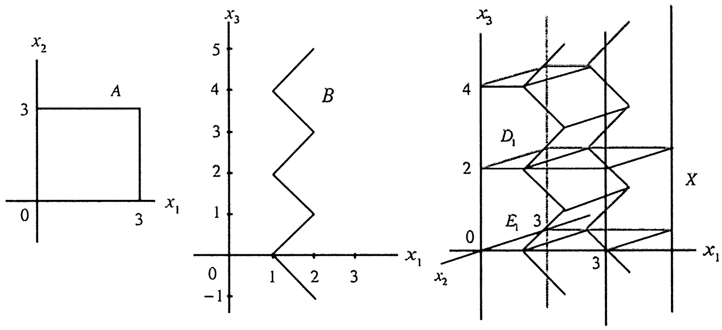


Figure 1. $A$, $B$ and $X$

In fact, since $k(x) = 0$ for any point $x \in X$ and $\int_c \kappa d_\Delta = 0$ for any pair $(c, \Delta)$, $X$ admits w-total curvature

$$\tilde{C}(X) = e_{reg}(X) + \tilde{e}_{sing}(X) = 0.$$

On the other hand, we will take two increasing sequences $\{E_i\}$ and $\{D_i\}$ defined by

$$E_i := \{(x_1, x_2, x_3) \in X \mid -2i \leqq x_3 \leqq 2i\} \quad \text{and}$$
$$D_i := E_i \cup \{(x_1, x_2, x_3) \in X \mid x_1 \leqq x_3 - 2i + 1, 2i \leqq x_3 \leqq 2i + 1\}$$
$$\cup \{(x_1, x_2, x_3) \in X \mid x_1 \leqq 2i + 3 - x_3, 2i + 1 \leqq x_3 \leqq 2i + 2\}.$$

Then we have $C(E_i) = 0$ because $e_{reg}(E_i) = e_{sing}(E_i) = 0$, and $C(D_i) = \pi$ because $k(p) = \pi/2$ for $p = (2, 0, 2i+1)$ or $(2, 3, 2i+1)$. Note that two points $(1, 0, 2i), (1, 3, 2i) \in D_i$ are on $\mathscr{B}D_i$. Therefore we have

$$\lim_{i \to \infty} C(E_i) \neq \lim_{i \to \infty} C(D_i),$$

which implies that $X$ does not admit total curvature.

As to define the Euler characteristic of noncompact piecewise Riemannian 2-polyhedron $X$, there may be several manners. In this paper, we will investigate the following simplest case.

DEFINITION 4.4. A noncompact piecewise Riemannian 2-polyhedron $X$ is said to be *finitely connected*, if it is homeomorphic to a compact 2-polyhedron $\tilde{X}$ with finitely many points $\{p_1, \dots, p_n\}$ removed.

For such a 2-polyhedron $X$, let $L_i$ be the point-set of the linked complex $LK(p_i)$ of $p_i$ on $\tilde{X}$, and $X_\infty$ the disjoint union of $\{L_i\}$. By definition, $LK(p_i)$ is the sum of simplices $\sigma$ on $\tilde{X}$ such that the cone with vertex $p_i$ and base $\sigma$ is also a simplex of $\tilde{X}$. We may assume that $L_i \cap L_j = \varnothing$ for $i \neq j$ by taking a subdivision if necessary. Then there is a large compact set $D$ on $X$ such that $X \backslash D$ is homeomorphic to $X_\infty \times \mathbf{R}$. Since $X$ is homotopic to $D$, the Euler characteristic $\chi(X)$ of $X$ is, by definition, equal to $\chi(\tilde{X}) - n + \chi(X_\infty)$. Note that $\tilde{X}$ is a finite polyhedron but $X$ is not so, that is, the structure of $X$ as a polyhedron is quite different from that of $\tilde{X}$. Now, we have the following theorem of a Cohn-Vossen type.

THEOREM 4.5. *Let $X$ be a finitely connected noncompact complete piecewise Riemannian 2-polyhedron without free faces admitting total curvature. Then we have*

$$C(X) \leqq 2\pi\chi(X) - \pi\chi(X_\infty).$$

REMARK 4.6. If $X$ is a Riemannian 2-manifold without boundary, then $\chi(X_\infty) = 0$. Hence the above theorem coincides with Theorem 6 in [4]. For an odd-dimensional piecewise linear polyhedron $X$ without free faces, on which a total curvature $C(X)$ can be also defined similarly, it holds that

$$C(X) = 0 = 2\pi\chi(X) - \pi\chi(X_\infty).$$

(For the definition and the proof, see §6. Appendix below.)

REMARK 4.7. In Theorem 4.5, it is an essential assumption that $X$ admits total curvature. We will illustrate an example (Example 4.8) of a finitely connected noncompact piecewise Riemannian 2-polyhedron $X$ without free faces admitting w-total curvature such that $\tilde{C}(X) > 2\pi\chi(X) - \pi\chi(X_\infty)$.

PROOF OF THEOREM 4.5. Since $X$ is finitely connected, $\chi(X)$ and $\chi(X_\infty)$ are finite. Therefore if $C(X) = -\infty$, then the statement is clear. So we assume that $C(X) \neq -\infty$. Hence $e^-_{reg}(X) := \int_X K^- d_X < \infty$ and $e^-_{sing}(X) := \sum k^- < \infty$, where $K^- := max\{-K, 0\}$ and $k^- := max\{-k, 0\}$.

Now we will take an increasing sequence $\{D_i\}$ of compact piecewise Riemannian 2-polyhedra such that $X = \bigcup D_i$ and $X \backslash D_i$ is homeomorphic to $\partial D_i \times \mathbf{R}$, where $\partial D_i$ is the topological boundary of $D_i$ as a subset of $X$. This is possible also by the finite connectivity of $X$. Note that $\chi(X) = \chi(D_i)$ and $\chi(X_\infty) = \chi(\partial D_i)$. We may assume that $\partial D_i = \mathscr{B} D_i$ for such a domain $D_i$, since $X$ has no free faces. Then, since $X$ admits total curvature, we have

$$\begin{aligned} C(X) &= \lim_{i \to \infty} C(D_i) \\ &= \lim_{i \to \infty} \left\{ 2\pi\chi(D_i) - \sum_{x \in \mathscr{B} D_i} \{\pi(2 - \chi(LK(x))^{D_i}) - L(\Sigma_x^{D_i})\} - \int_{\mathscr{B} D_i} \kappa d_{D_i} \right\}, \end{aligned}$$

where $\chi(LK(x))^{D_i}$ is the Euler characteristic of the linked complex of $x$ in $D_i$. To conclude the proof, since it holds that $2\pi\chi(D_i) = 2\pi\chi(X)$, it is sufficient to show that

$$\lim_{i \to \infty} \left\{ \pi\chi(\mathscr{B} D_i) - \sum_{x \in \mathscr{B} D_i} \{\pi(2 - \chi(LK(x))^{D_i}) - L(\Sigma_x^{D_i})\} - \int_{\mathscr{B} D_i} \kappa d_{D_i} \right\} \leqq 0.$$

Speaking more precisely, we will show the above inequality restricted to each end of $X$. Fix a number $i_0$ and let $U_1, \ldots, U_m$ be the connected components of $X \backslash D_{i_0}$ and $c_j^i := \mathscr{B} D_i \cap U_j$. We will show that for any $j = 1, \ldots, m$

$$\lim_{i \to \infty} \left\{ \pi\chi(c_j^i) - \sum_{x \in c_j^i} \{\pi(2 - \chi(LK(x))^{D_i}) - L(\Sigma_x^{D_i})\} - \int_{c_j^i} \kappa d_{D_i} \right\} \leqq 0.$$

If $U_j$ is 1-dimensional, then $c_j^i$ consists of a single point $x$. Hence $\chi(c_j^i) = \chi(x) = 1$ and $\chi(LK(x))^{D_i} = 1$, and we agree that $L(\Sigma_x^{D_i})$, $\int_{c_j} \kappa d_X$ are equal to 0. Hence $\lim_{i \to \infty} \{\pi\chi(c_j^i) - \sum_{x \in c_j^i} \{\pi(2 - \chi(LK(x))^{D_i}) - L(\Sigma_x^{D_i})\} - \int_{c_j^i} \kappa d_{D_i}\} = 0$.

If $U_j$ is homeomorphic to a cylinder, then $c_j^i$ is homeomorphic to a circle. Then $U_j$ attached the cone over $c_j^{i_0}$, which is homeomorphic to $\mathbf{R}^2$, admits total curvature. That is, it is a good surface in the sense of [10]. Hence from Theorem 4.1 in [10], it is shown that

$$\begin{aligned} &\lim_{i \to \infty} \left\{ \pi\chi(c_j^i) - \sum_{x \in c_j^i} \{\pi(2 - \chi(LK(x))^{D_i}) - L(\Sigma_x^{D_i})\} - \int_{c_j^i} \kappa d_{D_i} \right\} \\ &= \lim_{i \to \infty} - \left\{ \sum_{x \in c_j^i} (\pi - L(\Sigma_x^{D_i})) + \int_{c_j^i} \kappa d_{D_i} \right\} \leqq 0. \end{aligned}$$

Finally we will deal with the other $U_j$. Let $\{e_\lambda \mid \lambda \in \Lambda\}$ be a cellular decomposition of $U_j$ such that every 1-cell is adjacent to at least three 2-cells. We denote by $a$ and $b$ the number of 1-cells and that of 2-cells of $\{e_\lambda\}$ respectively. Note that there are no vertices in $\{e_\lambda\}$. It is clear that $\chi(c_j^i) = a - b < 0$ and $\chi(LK(x))^{D_i} = 1$ for any $x \in c_j^i$.

For every 2-cell $e \in \{e_\lambda\}$, let $U_e$ be the double of the closure $\bar{e}$ of $e$ identified their boundaries $\partial e$ to each other, which is homeomorphic to $\mathbf{R}^2$. Since $X$ admits total curvature, we can easily seen that $U_e$ also admits total curvature and is a good surface in the sense of [10]. (In Example 4.2, construct $U_e$ similarly. Then there exists $U_e$ which does not admit total curvature.) Therefore similarly as above, we have that

$$\lim_{i\to\infty} \sum_{x \in \widetilde{c_e^i}} (\pi - L(\Sigma_x^{\widetilde{D_i}})) + \int_{\widetilde{c_e^i}} \kappa d_{\widetilde{D_i}} \geqq 0,$$

where $\widetilde{D_i}$ is the closure of a corresponding double of $D_i \cap \bar{e}$ in $U_e$ and $\widetilde{c_e^i}$ is the boundary of $\widetilde{D_i}$ in $U_e$. To sum up, we have

$$\begin{aligned}
&\lim_{i\to\infty} \sum_{e:\text{2-cell}} \left\{ \sum_{x \in \widetilde{c_e^i}} (\pi - L(\Sigma_x^{\widetilde{D_i}})) + \int_{\widetilde{c_e^i}} \kappa d_{\widetilde{D_i}} \right\} \\
&\quad = \lim_{i\to\infty} \left\{ \sum_{e:\text{2-cell}} \sum_{x \in c_e^i} 2(\pi - L(\Sigma_x^{D_i})) + \sum_{x \in c_j^i \setminus \bigcup c_e^i} \{a_x \pi - 2L(\Sigma_x^{D_i})\} + 2\int_{c_j^i} \kappa d_{D_i} \right\} \\
&\quad = 2 \lim_{i\to\infty} \left\{ \sum_{x \in c_j^i} (\pi - L(\Sigma_x^{D_i})) + \int_{c_j^i} \kappa d_{D_i} \right\} + 2(b-a)\pi,
\end{aligned}$$

where $c_e^i := c_j^i \cap e$ and $a_x$ is the number of 1-cells on $c_j^i$ adjacent to $x$. The last equality comes from $\sum_{x \in c_j^i \setminus \bigcup c_e^i} a_x = 2b$. Therefore we have

$$\begin{aligned}
&\lim_{i\to\infty} \left\{ \pi\chi(c_j^i) - \sum_{x \in c_j^i} \{\pi(2 - \chi(LK(x))^{D_i}) - L(\Sigma_x^{D_i})\} - \int_{c_j^i} \kappa d_{D_i} \right\} \\
&\qquad \leqq (a-b)\pi + (b-a)\pi = 0,
\end{aligned}$$

which completes the proof. □

As mentioned in a previous remark, we will give a counter example.

EXAMPLE 4.8. Let $\square_a$ $(a > 0)$ be a trapezoid with bottom angles $\frac{\pi}{2} - a$, and hence the other two angles are equal to $\frac{\pi}{2} + a$. The length of the bottom, the diagonal line and the side line of $\square_a$ are denoted by $\alpha_0$, $\alpha_1$ and $\alpha_2$, respectively. To assemble four $\square_a$, construct a truncated pyramid and make, alternately upside down, a pile $M_1$ of these truncated pyramids attached the square base $\square(p_1, p_2, p_4, p_3)$ as in Figure 2. Naturally, $M_1$ is homeomorphic to $\mathbf{R}^2$. Let $\square_{2a}(q_1, p_1, p_2, r_1)$ be a trapezoid with $\angle p_1 = \angle p_2 = \frac{\pi}{2} + 2a$, $\angle q_1 = \angle r_1 = \frac{\pi}{2} - 2a$, $|p_1 p_2| = \alpha_0$, $|p_1 q_1| = \alpha_1$ and $M_2$ be a pile of trapezoids $\square_{2a}(q_1, p_1, p_2, r_1)$ and $\square_{2a}(q_i, q_{i+1}, r_{i+1}, r_i)$, alternately upside down, the length of whose side line alternates $\alpha_1$ and $\alpha_2$. Then $M_2$ is homeomorphic to a half strip, whose boundary is the broken geodesic joining the points $\{\dots, q_3, q_2, q_1, p_1, p_2, r_1, r_2, \dots\}$. Then $X$ is a piecewise linear polyhedron constructed from $M_1$ and $M_2$ identified the boundary of $M_2$ to the corresponding broken geodesic joining the points $\{\dots, q_3, q_2, q_1, p_1, p_2, r_1, r_2, \dots\}$ on $M_1$ like as in Figure 2.

Then, since $\chi(X) = 1$ and $\chi(X_\infty) = -1$, we have $2\pi\chi(X) - \pi\chi(X_\infty) = 3\pi$. On the other hand, we have $k(p_1) = k(p_2) = \pi$, $k(p_3) = k(p_4) = \frac{\pi}{2} + 2a$ and $k(q_i) = k(r_i) = 0$, where $p_1, \dots, p_4$ are vertices of the bottom of $M_1$ and $q_i$, $r_i$ are the other vertices. Hence $\tilde{C}(X) = 3\pi + 4a > 2\pi\chi(X) - \pi\chi(X_\infty)$.

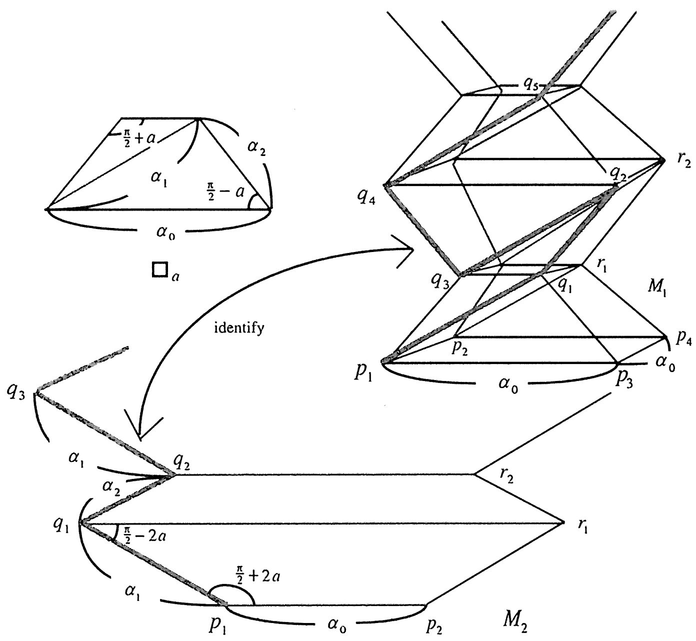


Figure 2. $\square_a$, $M_1$ and $M_2$

REMARK 4.9. Although we have dealt with a noncompact complete piecewise Riemannian 2-polyhedron $X$ without free faces in this section, we have the following result in the case of $\mathscr{B}X \neq \emptyset$ by applying Theorem 4.5: If $X$ is a finitely connected noncompact complete piecewise Riemannian 2-polyhedron admitting total curvature and $\sum_{p \in \mathscr{B}X} k(p) + \int_{\mathscr{B}X} \kappa d_X$ is finite, then

$$C(X) + \sum_{p \in \mathscr{B}X} k(p) + \int_{\mathscr{B}X} \kappa d_X \leqq 2\pi\chi(X) - \pi\chi(X_\infty).$$

In fact, let $\hat{X}$ be a double of $X$ obtained by identifying $\mathscr{B}X$. Then applying Theorem 4.5, we have

$$C(\hat{X}) \leqq 2\pi\chi(\hat{X}) - \pi\chi(\hat{X}_\infty).$$

Here note that

$$\begin{aligned} C(\hat{X}) &= 2C(X) + 2\sum_{p \in \mathscr{B}X} k(p) + 2\int_{\mathscr{B}X} \kappa d_X - \sum_{p \in \mathscr{B}X} \pi(2 - \chi(L(\mathscr{B}X)(p))) \\ &= 2C(X) + 2\sum_{p \in \mathscr{B}X} k(p) + 2\int_{\mathscr{B}X} \kappa d_X - 2\pi\{\chi(\mathscr{B}X) + \chi(\mathscr{B}X_\infty)\}, \end{aligned}$$

where $L(\mathscr{B}X)(p)$ is the linked complex of $\mathscr{B}X$ at $p$, and hence $\chi(L(\mathscr{B}X)(p))$ is the number of edges adjacent to $p$ of $\mathscr{B}X$. On the other hand we have

$$2\pi\chi(\hat{X}) - \pi\chi(\hat{X}_\infty) = 2\pi(2\chi(X) - \chi(\mathscr{B}X)) - \pi(2\chi(X_\infty) - \chi(\mathscr{B}X_\infty)),$$

which leads us to the above inequality. (Compare [14] for the Riemannian case with boundary.)

## §5. Relation Between Total Curvature and Expanding Growth

In this section, we will investigate about the difference of the both sides of the inequality of Theorem 4.5, $\{2\pi\chi(X) - \pi\chi(X_\infty)\} - C(X)$, for a finitely connected noncompact complete piecewise Riemannian 2-polyhedron without free faces admitting total curvature. In the Riemannian case, the difference means the expanding growth rate of a manifold. There are many results around a relation between total curvature and expanding growth. For example, Theorem D in [11] states that the normalized length of geodesic sphere tends to the difference. We will generalize this theorem later.

First, similarly to Fiala [5] and Hartman [6], we will prepare the following

Proposition 5.1. *Let $X$ be a piecewise Riemannian 2-polyhedron homeomorphic to $\mathbf{R}^2$ and $x_0$ a point on $X$. Let $c_t := \{x \in X \mid d(x, x_0) = t\}$ and $L(t)$ be the length of $c_t$. Then $L(t)$ is continuous and differentiable at almost all $t$ and*

$$\frac{dL}{dt}(t) = \int_{c_t} \kappa(s)\, ds - \sum 2 \tan \frac{\theta_i}{2},$$

*where $\kappa$ is a geodesic curvature on $c_t$ and $-\theta_i$ is an exterior angle at a broken point of $c_t$.*

Proof. The proof is essentially the same to [5] and [6]. We will explain in detail only about complicated phenomena caused by dealing with piecewise Riemannian object, but in brief about similar arguments.

For an arbitrarily given $r > 0$, let $K_r = \{x \in X \mid d(x, x_0) \leqq r\}$, the bounded component of $X$ bounded by $c_r = \{x \in X \mid d(x, x_0) = r\}$, and $\mathscr{R}_r$ the set of all maximal minimizing normal geodesic segments emanating from $x_0$ on $K_r$. Note that if two geodesics in $\mathscr{R}_r$ coincide beyond some point, then we consider one of them ends at the point. Geodesics may branch off on a vertex with nonpositive singular curvature or on a point of 1-simplex with negative geodesic curvature.

It seems to be helpful to describe how a minimal geodesic $\gamma$ emanating from $x_0$ will branch off on an 1-simplex $c$ more precisely. We assume that a geodesic $\gamma$ on 2-simplex $e_1$ is contacting to 1-simplex $c \subset \mathscr{B}e_1$ at $p$. Because $c$ has a negative geodesic curvature at $p$ with respect to $e_1$, $\gamma$ does not necessarily branch off at $p$. It depends on the geodesic curvature of $c$ with respect to another adjacent 2-simplex $e_2$. Of course, $\gamma$ branches off if another geodesic curvature of $c$ at $p$ is also negative. When $\gamma$ does not branch off at $p$, minimal geodesics emanating from $x_0$ sufficiently close to $\gamma$ intersect transversely at the intersection with $c$ even if $\gamma$ dose not intersect transversely at $p$. Hence branching along $c$ occurs on at most countable intervals of $c$.

Therefore it is clear that $\mathscr{R}_r$ can be parametrized by a space of directions $\Sigma_{x_0}$ at $x_0$, a suitable subset $\theta_v \subset \Sigma_v$ for such a vertex $v$ as above and such 1-simplices as above. Precisely, $\theta_v$ is defined as follows (see the middle case of Figure 3): Let $A_v \subset \Sigma_v$ be the set of all initial directions of minimizing geodesic segment from $v$ to $x_0$. Then

$$\theta_v = \{x \in \Sigma_v \mid \angle_v(x, y) \geqq \pi \text{ for any } y \in A_v\}.$$

Fix a geodesic segment $\gamma_0 \in \mathscr{R}_r$. Then $\mathscr{R}_r$ is parameterized anti-clockwise and consistently from $\gamma_0$ by $f : [0, L_r] \to \mathscr{R}_r$. Naturally $f(0) = f(L_r) = \gamma_0$. The term "consistently" means as follows: The middle case of Figure 3 implies that there

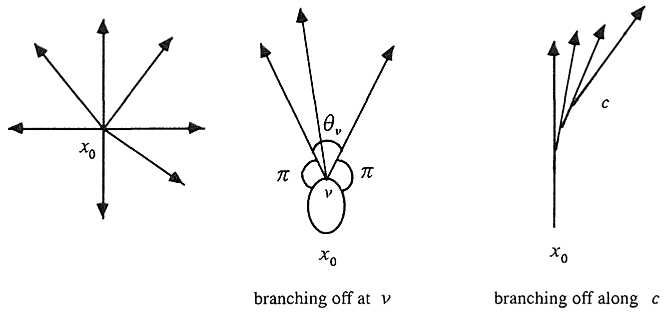


Figure 3. Minimizing geodesics emanating from $x_0$

are many geodesics which coincide to each other beyond $v$. In such a case, we consider that the geodesic parametrized by "the smallest number" is extended beyond $v$ and other geodesics end at $v$.

We will explain the meaning of "the smallest number" and the parametrization of $R_r$ in the following simple example. Let $M$ be a piecewise Riemannian 2-polyhedron homeomorphic to a plane $\mathbf{R}^2$ such that no geodesics emanating from $x_0 \in M$ branch off except at a point $v \in M$ and there are just two minimizing geodesic segments $\gamma_1$, $\gamma_2$ from $x_0$ to $v$. Assume that $\Sigma_{x_0}$ is parametrized anti-clockwise by $\alpha : [0, l_0] \to \Sigma_{x_0}$ with $\gamma_{\alpha(0)} = \gamma_{\alpha(l_0)} = \gamma_1$ and $\gamma_{\alpha(s_0)} = \gamma_2$ $(0 < s_0 < l_0)$, where $\gamma_{\alpha(s)}$ is the geodesic containing in $\alpha(s)$. Then we consider that $\gamma_{\alpha(0)} = \gamma_1$ is extended beyond $v$ and $\alpha(s_0) = \gamma_2$ ends at $v$. In this case, $R_r$ is parametrized as follows: Let $\theta_v \subset \Sigma_v$ be parametrized anti-clockwise by $\beta : [0, l_2] \to \theta_v$ and $\gamma_{\beta(s)}$ is the geodesic containing in $\beta(s)$. Then we can take a parametrization $f : [0, l_1 + l_2] \to R_r$ defined by

$$\begin{cases} f(s) \text{ is the minimizing geodesic segment consisting of } \gamma_1 \text{ and } \gamma_{\beta(s)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 < s < l_2, \\ f(s) \text{ is the minimizing geodesic segment } \gamma_{\alpha(s)} \quad \text{for } l_2 < s < l_2 + l_1. \end{cases}$$

Note that $f(l_2 + s_0) = \gamma_2$ ends at $v$ and the length of $f(s)$ is continuous except 0 and $l_2 + s_0$.

We denote by $\rho(s)$ the length of $f(s)$. Then $f(\rho(s))$ is a cut point of $x_0$ along $f(s)$ except finite points if $\rho(s) < r$, and $\rho$ is continuous except only finite points. Note that $\rho$ is continuous in Riemannian case, but in our case there are some points where $\rho$ is not continuous. See the middle case of Figure 3.

Now let $K_r$ be parametrized by $F(s, t) = (f(s))(t)$, where the domain of $F$ is $D := \{(s, t) \mid s \in [0, L_r], t \in [0, \rho(s)]\}$. Note that there is a division of the domain $D$ into at most countable domains $\{D_i\}$ such that $F$ is a usual geodesic variation on each $D_i$. Similarly to Hartman [6], for almost all $t$ with $0 < t < r$, it holds that $\rho(s) = t$ has a finite number of solutions and $c_t = \{F(s,t) \mid (s,t) \in D\}$ is a set of simple closed curves with finite broken points. If we regard $\partial F/\partial s(s,t) = 0$ for a point $(s,t)$ where $\partial F/\partial s$ is not defined, then $L(t)$ is expressed as $\int_0^{L_r} \|\partial F/\partial s(s,t)\|\, ds$. Computing $dL/dt(t)$ for a suitable reparametrization for such almost all $t$ similarly in [5], we have the conclusion. □

Under the assumption that $X$ admits total curvature, we can obtain more precise observation on a distance sphere $c_t$. Namely, similarly to Shiohama [11], we have

PROPOSITION 5.2. *Let $X$ be a noncompact piecewise Riemannian 2-polyhedron homeomorphic to $\mathbf{R}^2$ admitting total curvature $C(X)$, and $x_0$ be a point on $X$. Let $c_t := \{x \in X \mid d(x, x_0) = t\}$ and $L(t)$ be the length of $c_t$. We denote by $K_t$ the bounded component bounded by $c_t$ and by $A(t)$ the area of $K_t$. Then*

$$\lim_{t\to\infty} \frac{L(t)}{t} = \lim_{t\to\infty} \frac{2A(t)}{t^2} = 2\pi - C(X).$$

PROOF. Since the proof is also essentially the same to [11], we will only explain its outline.

Let $\mathscr{R}$ be the set of rays emanating from $x_0$. Then $X \backslash \mathscr{R}$ is expressed as at most countable disjoint union $\bigcup_{\lambda \in \Lambda} D_\lambda$. Note that $D_\lambda$ is not necessarily to be unbounded. For an unbounded component $D_\lambda$, we have by Lemma 3.2 in [7] that

$$(*) \qquad C(\bar{D}_\lambda) = L(\Sigma_{x_0}^{\bar{D}_\lambda}) - \sum_{x \in \mathscr{B}D_\lambda \backslash \{x_0\}} (\pi - L(\Sigma_x^{\bar{D}_\lambda})),$$

where $\bar{D}_\lambda$ is the closure of $D_\lambda$. This equality $(*)$ is corresponding to Theorem A in [11].

This equality $(*)$ implies that there exists a large number $r$ such that $c_t$ is homeomorphic to a circle for any $t > r$ (confer Theorem B in [11]). In fact, if we assume that $c_t$ is not connected, then we can take an unbounded component $D$ of $X \backslash \mathscr{R}$ such that $C(\bar{D}) = L(\Sigma_{x_0}^{\bar{D}}) - \sum_{x \in \mathscr{B}D \backslash \{x_0\}} (\pi - L(\Sigma_x^{\bar{D}})) + \pi$, which is a contradiction. Furthermore under the assumption that $c_t$ is not a circle, we can take such a component implying a contradiction. Hence $L(t)$ is continuous on $t > r$, and $L(t)$ is differentiable at almost all points. Especially $L(t)$ is absolutely continuous (cf. [13]). Then we have $L(t) = \int_r^t L'(t)\, dt + L(r)$ and $A(t) = \int_r^t L(t)\, dt + A(r)$ for $t > r$.

For any $x \in X$, let $\theta(x) := L(\Sigma_x^{E(x)})$, where $E(x)$ is the maximal bounded component bounded by two minimizing geodesic segments from $x_0$ to $x$. Then from $(*)$, it is seen that for any $\varepsilon > 0$ there is a large number $t(\varepsilon)$ such that $\sum_{x \in c_t} \theta(x) < \varepsilon$ for any $t > t(\varepsilon)$ (confer Theorem C in [11]).

Here we recall the statement of Proposition 5.1: $dL/dt(t) = \int_{c_t} \kappa(s)\, ds - \sum 2 \tan \theta_i/2$. Then, applying Theorem 3.2 and noting that the singular curvature $k(x_i)$ at a broken point $x_i$ on $c_t = \mathscr{B}K_t$ is equal to $-\theta_i$, we have

$$\frac{dL}{dt}(t) = 2\pi\chi(K_t) - C(K_t) - \sum_{p \in \mathscr{B}K_t} k(p) - \sum 2 \tan \frac{\theta_i}{2}$$

$$= 2\pi - C(K_t) - \sum \left\{ 2 \tan \frac{\theta_i}{2} - \theta_i \right\}.$$

Since $\theta_i/2 < \tan \theta_i/2 < \theta_i$ for a small $\theta_i$ (for example for $0 < \theta_i \leqq \pi/3$), we have that $0 \leqq \sum\{2 \tan \theta_i/2 - \theta_i\} < \varepsilon$ for any $t > t(\varepsilon)$ provided $\varepsilon \leqq \pi/3$. Hence for $t > t(\varepsilon)$

$$(**) \qquad 2\pi - C(K_t) - \varepsilon < \frac{dL}{dt}(t) \leqq 2\pi - C(K_t).$$

Now in the case that $C(X) = -\infty$, we have that $L(t), A(t) \to \infty$ and

$$\lim_{t\to\infty} \frac{L(t)}{t} = \lim_{t\to\infty} \frac{dL}{dt}(t) = \infty$$

$$\lim_{t\to\infty} \frac{2A(t)}{t^2} = \lim_{t\to\infty} \frac{d^2A}{dt^2}(t) = \lim_{t\to\infty} \frac{dL}{dt}(t) = \infty,$$

which is the conclusion.

If $C(X) > -\infty$, then there is a large $t'(\varepsilon)$ such that $|C(K_t) - C(X)| < \varepsilon$ for any $t > t'(\varepsilon)$. Put $T(\varepsilon) := \max(r, t(\varepsilon), t'(\varepsilon))$. Then from $(**)$,

$$2\pi - C(X) - 2\varepsilon < \frac{dL}{dt}(t) \leqq 2\pi - C(X) + \varepsilon$$

for any $t > T(\varepsilon)$, and hence

$$\lim_{t\to\infty} \frac{L(t)}{t} = 2\pi - C(X).$$

Furthermore from $A(t) = \int_{T(\varepsilon)}^{t} L(t)\, dt + A(T(\varepsilon))$ and the above estimate of $dL/dt$,

$$\lim_{t\to\infty} \frac{2A(t)}{t^2} = 2\pi - C(X),$$

which is the conclusion. $\square$

Now we will treat a piecewise Riemannian 2-polyhedron. Let $X$ be a finitely connected, noncompact piecewise Riemannian 2-polyhedron without boundary admitting total curvature. Then there is a compact piecewise Riemannian 2-subpolyhedron $K$ of $X$ such that $X \backslash K$ is homeomorphic to $X_\infty \times \mathbf{R}$.

Let $U_1, \dots, U_m$ be the connected components of $X \backslash K$.

For any $i = 1, \dots, m$ and any $t > 0$, we define the sets $c_t^i$ and $K_t^i$ as follows: If $U_i$ is 1-dimensional or is homeomorphic to a cylinder, then $c_t^i := \{x \in U_i \mid d_{U_i}(x, \partial U_i) = t\}$ and $K_t^i := \{x \in U_i \mid d_{U_i}(x, \partial U_i) \leqq t\}$, where $d_{U_i}$ is the interior distance on the closure of $U_i$. For another $U_i$, we will divide it into surface components. By definition, we call a connected component of the set of all points having a neighborhood homeomorphic to a two-dimensional open disk by a *surface component*. Let $\{e_\lambda \mid \lambda \in \Lambda\}$ be the set of all surface components of $U_i$, and for each 2-cell $e_\lambda$ put $c_t^\lambda := \{x \in e_\lambda \mid d_{e_\lambda}(x, K \cap \partial e_\lambda) = t\}$ and $K_t^\lambda := \{x \in e_\lambda \mid d_{e_\lambda}(x, K \cap \partial e_\lambda) \leqq t\}$, and then $c_t^i$ and $K_t^i$ are, by definition, the closure of $\bigcup_\lambda c_t^\lambda$ and $\bigcup_\lambda K_t^\lambda$ on $U_i$, respectively. Note that $c_t^i$ is not necessarily connected.

Theorem 5.3. *Let $L_i(t)$ be the length of $c_t^i$, $A_i(t)$ the area of $K_t^i$, $A(K)$ the area of $K$, and $L(t) := \sum_i L_i(t)$, $A(t) := \sum_i A_i(t) + A(K)$. Then we have*

$$\lim_{t \to \infty} \frac{L(t)}{t} = \lim_{t \to \infty} \frac{2A(t)}{t^2} = 2\pi\chi(X) - \pi\chi(X_\infty) - C(X)$$

Proof. For the connected components $U_1, \dots, U_m$ of $X \backslash K$, we first prove that for some extension $\tilde{U}_i$ of $U_i$ attaching a suitable domain $D_i$,

$$(*) \qquad \lim_{t \to \infty} \frac{L_i(t)}{L} = \lim_{t \to \infty} \frac{2A_i(t)}{t^2} = 2\pi\chi(D_i) - \pi\chi(\partial U_i) - C(\tilde{U}_i).$$

If $U_i$ is 1-dimensional, then $L_i(t) = A_i(t) = 0$, and $\lim_{t \to \infty} L_i(t)/t = \lim_{t \to \infty} 2A_i(t)/t^2 = 0$. Now let $\tilde{U}_i$ be $U_i$ attaching a 2-sphere $D_i$. Note that $2\pi\chi(D_i) - \pi\chi(\partial U_i) - C(\tilde{U}_i) = 4\pi - \pi - 3\pi = 0$, and $(*)$ is satisfied.

In the case that $U_i$ is homeomorphic to a cylinder, let $\tilde{U}_i$ be a piecewise Riemannian 2-polyhedron homeomorphic to $\mathbf{R}^2$ obtained as $U_i$ attaching a suitable closed disk $D_i$ of center $p$ with radius $l$. Then by Proposition 5.2, $\lim_{t \to \infty} L_i(t - l)/t = \lim_{t \to \infty} 2A_i(t - l)/t^2 = 2\pi - C(\tilde{U}_i)$. Note that $\lim_{t \to \infty} L_i(t - l)/t = \lim_{t \to \infty} L_i(t)/t$, $\lim_{t \to \infty} 2A_i(t - l)/t^2 = \lim_{t \to \infty} 2A_i(t)/t^2$ and $2\pi - C(\tilde{U}_i) = 2\pi\chi(D_i) - \pi\chi(\partial U_i) - C(\tilde{U}_i)$.

For the other $U_i$, we took a cellular decomposition $\{e_\lambda \mid \lambda \in \Lambda\}$ of $U_i$, in which the set of 2-cells are the surface components of $U_i$. For each 2-cell $e_\lambda$, construct $U_\lambda$ to be a double of $e_\lambda$. Here we do not identify the points corresponding a point on $\partial U_i$. Hence $U_\lambda$ is homeomorphic to a cylinder. We can attach a suitable closed disk $D_\lambda$ as above case and construct $\tilde{U}_\lambda$. Then by

Proposition 5.2, $\lim_{t\to\infty} 2L_\lambda(t)/t = \lim_{t\to\infty} 4A_\lambda(t)/t^2 = 2\pi - C(\tilde{U}_\lambda)$, where $L_\lambda(t)$ is the length of $c_t^\lambda$ and $A_\lambda(t)$ the area of $K_t^\lambda$.

Now let $D_i$ be $\bigcup D_\lambda$ identified the points corresponding the same point on $\partial U_i$. Then we have $\chi(D_i) = \chi(\partial U_i) + b$, where $b$ is the number of 2-cells of $\{e_\lambda\}$. Furthermore let $\tilde{U}_i$ be $U_i$ attaching $D_i$. It is clear that $2e_{reg}(\tilde{U}_i) - \sum e_{reg}(\tilde{U}_\lambda) = e_{reg}(D_i)$ and $2\sum_{(c,\Delta),\Delta\subset\tilde{U}_i}\int_c \kappa d_\Delta - \sum_\lambda\sum_{(c,\Delta),\Delta\subset\tilde{U}_\lambda}\int_c \kappa d_\Delta = \sum_{(c,\Delta),\Delta\subset D_i}\int_c \kappa d_\Delta$, and from some easily computation we also have $2e_{sing}(\tilde{U}_i) - \sum e_{sing}(\tilde{U}_\lambda) = e_{sing}(D_i)$. Therefore $2C(\tilde{U}_i) - \sum C(\tilde{U}_\lambda) = 2\pi\chi(D_i)$.

Hence $\lim_{t\to\infty} L_i(t)/t = \lim_{t\to\infty} 2A_i(t)/t^2 = \sum_\lambda\{\pi - C(\tilde{U}_\lambda)/2\} = b\pi - C(\tilde{U}_i) + \pi\chi(D_i) = 2\pi\chi(D_i) - \pi\chi(\partial U_i) - C(\tilde{U}_i)$.

Summing up the equality $(*)$ and noting that $\sum\{C(\tilde{U}_i) - 2\pi\chi(D_i)\} = C(X) - 2\pi\chi(K)$, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{L(t)}{t} &= \lim_{t\to\infty}\frac{2A(t)}{t^2}\\ &= \sum_{i=1}^{m}\{2\pi\chi(D_i) - \pi\chi(\partial U_i) - C(\tilde{U}_i)\}\\ &= 2\pi\chi(K) - \pi\chi(\partial K) - C(X)\\ &= 2\pi\chi(X) - \pi\chi(X_\infty) - C(X),\end{aligned}$$

which complete the proof. $\square$

Now, we will illustrate the example mentioned in the introduction.

EXAMPLE. Let $M_1$ be a flat cylinder attaching a closed disk $K$ and $M_2$ a flat truncated sector with vertical angle $\pi/2$, and let $p_1$ be a point on $\partial K$ and $p_2$ the antipodal point of $p_1$ on $\partial K$. On $M_1$, let $l_i$ $(i = 1, 2)$ be a spiral whose angle with $\partial K$ at the starting point $p_i$ is $\pi/4$, and $l_3$ the straight segment from $p_1$ to $p_2$ on $K$. Then we will construct the piecewise Riemannian 2-polyhedron $X$ from $M_1$ and $M_2$ identifying $l_1 \cup l_2 \cup l_3$ with $\partial M_2$ like as Figure 4.

Let $\tilde{c}_t := \{x \in X \mid d(x, K) = t\}$. Then $L(\tilde{c}_t) = l_3\pi + \{2t + l_3\}$ for any $t > 0$. Hence we have that

$$\lim_{t\to\infty}\frac{L(\tilde{c}_t)}{t} = 2.$$

On the other hand, it is clear that $\chi(X) = 1$, $\chi(X_\infty) = -1$ and $C(X) = 5\pi/2$. Therefore it holds that

$$2\pi\chi(X) - \pi\chi(X_\infty) - C(X) = \frac{\pi}{2} < \lim_{t\to\infty}\frac{L(\tilde{c}_t)}{t}.$$

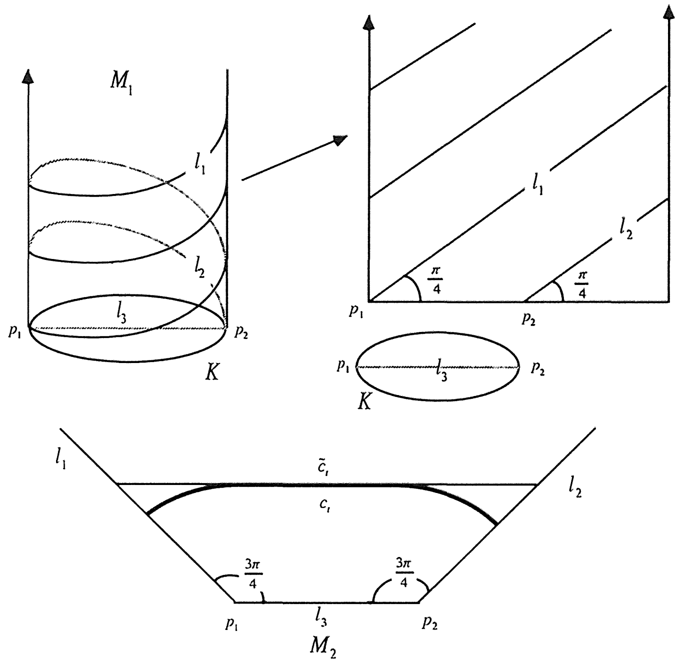


Figure 4. How to construct a counter example

We should note that, for the above $X$ and $K \subset X$, it is easy to see that Theorem 5.3 is true naturally. In fact, if we define $c_t$ as in Theorem 5.3, then we have

$$2\pi\chi(X) - \pi\chi(X_\infty) - C(X) = \lim_{t\to\infty} \frac{L(c_t)}{t} = \frac{\pi}{2}.$$

## 6. Appendix

As we mentioned in Remark 4.6, we will deal with total curvature of finitely connected odd-dimensional piecewise linear manifolds.

First we will introduce the definition of total curvature of compact piecewise linear polyhedron $X$ after Banchoff [1].

DEFINITION 6.1. Let $X$ be a compact piecewise linear polyhedron and $V$ the vertices of $X$. Fix a vertex $v$ and open $i$-simplex $\sigma$ which is adjacent to $v$. Assume that $\sigma$ is embedded in $\mathbf{R}^i$. Let $\mathbf{S}^{i-1}$ be the unit tangent sphere at $v$ and $I := \{x \in \mathbf{S}^{i-1} \mid \gamma_x \cap \sigma \neq \emptyset\}$, where $\gamma_x$ is a geodesic with initial vector $x$. Then the *normalized exterior angle* of $\sigma$ at $v$ is defined by

$$a(v, \sigma) := Vol(A)/Vol(\mathbf{S}^{i-1}),$$

where $A := \{x \in \mathbf{S}^{i-1} \mid \angle(x, y) \leqq \pi/2 \text{ for all } y \in I\}$. Particularly let $a(v, \sigma) = 1$ in the case that $i = 0$, and $a(v, \sigma) = 1/2$ when $i = 1$. Then we define the *curvature* $k(v)$ at $v$ and the *total curvature* $\hat{C}(X)$ of $X$ by

$$k(v) = \sum_{\sigma;\,\text{adjacent to } v} (-1)^{\dim \sigma} a(v, \sigma) \quad \text{and} \quad \hat{C}(X) = \sum_{v \in V} k(v).$$

REMARK 6.2. Since $Vol(\mathbf{S}^n)$ depends on $n$, Banchoff has used the normalized value. For 2-dimensional case, singular curvature defined in Section 2 is the product of $k$ multiplied by $2\pi$.

Banchoff has not distinguished boundary points from interior points in his definition. However as to deal with noncompact polyhedra in the similar way to the Riemannian case, we should redefine total curvature as follows.

DEFINITION 6.3. Let $X$ be a compact piecewise linear $i$-polyhedron and $V$ the vertices of $X$. The closure of the point-set of union of $(i-1)$-simplices which is a proper face of only one $i$-simplex is denoted by $\mathscr{B}X$. The complement of it, $X \backslash \mathscr{B}X$, is denoted by $\mathscr{I}X$. (It is clear that the definitions are independent of the choice of divisions of $X$.) Then the total curvature $C(X)$ of $X$ is defined by

$$C(X) = \sum_{v \in V \cap \mathscr{I}X} k(v).$$

Since it is known as Theorem 4 in [1] that $\hat{C}(X) = \chi(X)$, we have a Gauss-Bonnet type equality, namely $C(X) = \chi(X) - \sum_{v \in V \cap \mathscr{B}X} k(v)$.

Now let $X$ be a noncompact piecewise linear manifold without boundary. Then the total curvature $C(X)$ is defined as $\sum_{v \in V} k(v)$ provided the sum makes sense. In the case of $\dim X = 2$, this definition corresponds to $w$-total curvature.

Furthermore we assume that $X$ is odd-dimensional. It is also well-known by Corollary 2 in [1] that $k(v) = 0$ for any vertex $v \in X$. Hence we have $C(X) = 0$. Turning our attention to Euler characteristic, we have that

$$\chi(X) - \frac{1}{2}\chi(X_\infty) = 0$$

provided $X$ is finitely connected. In fact, by finitely connectedness of $X$, there is a large compact piecewise linear submanifold $K \subset X$ such that $X$ and $X_\infty$ are homeomorphic to $\mathscr{I}K$ and $\mathscr{B}K$ respectively. Let $\tilde{K}$ be a double of $K$ identified on $\mathscr{B}K$. Then we have that $2\chi(K) - \chi(\mathscr{B}K) = \chi(\tilde{K}) = C(\tilde{K}) = 0$, since $\tilde{K}$ is a compact odd-dimensional piecewise linear manifold. Therefore it hold that $C(X) = \chi(X) - \chi(X_\infty)/2$.

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