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ON THE EXTENSIONS OF $\hat{W}_n$ BY $\hat{\mathscr{G}}^{(\mu)}$ OVER A $Z_{(p)}$-ALGEBRA

By

Yasuhiro NIITSUMA

Abstract. We will give an explicit description of extensions of the group scheme of Witt vectors of length n (resp. the formal group scheme of Witt vectotrs of length n) by the group scheme (resp. the formal group scheme) which gives a deformation of the additive group shceme to the multiplicative group scheme (resp. the additive formal group scheme to the multiplicative formal group scheme) over an algebra for which all prime numbers except a given prime p are invertible.

Introduction

Throughout the paper, p denotes a prime number, $\boldsymbol{Z}_{(p)}$ the localization of $\boldsymbol{Z}$ at the prime ideal (p).

Let W_n (resp. $\hat{W}_n$) denote the group scheme (resp. the formal group scheme) of Witt vectors of length n over $\boldsymbol{Z}$, and W (resp. $\hat{W}$) the group scheme (resp. the formal group scheme) of Witt vectors over $\boldsymbol{Z}$. Let $\boldsymbol{G}_m$ (resp. $\hat{\boldsymbol{G}}_m$) denote the multiplicative group scheme (resp. the multiplicative formal group scheme) over $\boldsymbol{Z}$. Let F be the Frobenius endomorphism of W or of $\hat{W}$ (for the definition see 1.2).

An explicit description of the groups $\mathrm{Ext}^1_A(W_{n,A}, \boldsymbol{G}_{m,A})$ and $\mathrm{Ext}^1_A(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A})$ is given by Sekiguchi-Suwa [6] when A is a $\boldsymbol{Z}_{(p)}$-algebra. More precisely, isomorphisms

$$\mathrm{Ker}[F^n : W(A) \to W(A)] \xrightarrow{\sim} \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}),$$

$$\mathrm{Coker}[F^n : W(A) \to W(A)] \xrightarrow{\sim} H^2_0(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}),$$

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$$\mathrm{Ker}[F^n : \hat{W}(A) \to \hat{W}(A)] \xrightarrow{\sim} \mathrm{Hom}_{A\text{-gr}}(W_{n,A}, \boldsymbol{G}_{m,A}),$$

$$\mathrm{Coker}[F^n : \hat{W}(A) \to \hat{W}(A)] \xrightarrow{\sim} H_0^2(W_{n,A}, \boldsymbol{G}_{m,A})$$

are constructed, using deformations of the Artin-Hasse exponential series. Our aim of this article is to generalize the isomorphisms to those for $\hat{\mathscr{G}}_A^{(\mu)}$ instead of $\hat{\boldsymbol{G}}_{m,A}$. Here $\mu \in A$ and $\mathscr{G}_A^{(\mu)} = \mathrm{Spec}\, A[T, 1/(1+\mu T)]$; this is a group scheme defined by Sekiguchi and Suwa, as a deformation between the additive group scheme $\boldsymbol{G}_a$ and $\boldsymbol{G}_m$, so that $\mathscr{G}^{(0)} = \boldsymbol{G}_a$, $\mathscr{G}^{(1)} \xrightarrow{\sim} \boldsymbol{G}_m$ (for the definition see 3.1). Precisely, our result is as follows.

THEOREM. *Let A be a $\boldsymbol{Z}_{(p)}$-algebra and $\mu \in A$. Then there exist isomorphisms:*

$$\mathrm{Ker}[F^{(\mu)^n} : W^{(\mu)}(A) \to W^{(\mu)}(A)] \xrightarrow{\sim} \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(\mu)}),$$

$$\mathrm{Coker}[F^{(\mu)^n} : W^{(\mu)}(A) \to W^{(\mu)}(A)] \xrightarrow{\sim} H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(\mu)}).$$

Moreover, if μ is nilpotent, then there exist isomorphisms:

$$\mathrm{Ker}[F^{(\mu)^n} : \hat{W}^{(\mu)}(A) \to \hat{W}^{(\mu)}(A)] \xrightarrow{\sim} \mathrm{Hom}_{A\text{-gr}}(W_{n,A}, \mathscr{G}_A^{(\mu)}),$$

$$\mathrm{Coker}[F^{(\mu)^n} : \hat{W}^{(\mu)}(A) \to \hat{W}^{(\mu)}(A)] \xrightarrow{\sim} H_0^2(W_{n,A}, \mathscr{G}_A^{(\mu)}).$$

(See Theorem 3.5 and Remark 3.6. For the definition of $W^{(\mu)}(A)$ and $F^{(\mu)}$, see Section 1.)

Putting $\mu = 1$ in our theorem, we find again the main theorem of [6]. However, we prove the former, starting from the latter. It is crucial to use variants of Witt vectors and to construct deformations of the Artin-Hasse exponential series for an explicit description of the isomorphisms as done in Sekiguchi-Suwa [8].

Now we explain the contents of the article.

In Section 1, paraphrasing the classical theory of Witt vectors we recall the variants of Witt vectors $W^{(M)}(A)$ for a $\boldsymbol{Z}[M]$-algebra A, which is presented in [8]. $W^{(M)}(A)$ is interpreted as the A-valued points of a group scheme $W^{(M)}$ over $\boldsymbol{Z}[M]$. At the end of the section, we recall the exact sequence of groups over $\boldsymbol{Z}[M]$

$$0 \to W^{(M)} \to \prod_{B/A} W_B \to W_A \to 0,$$

where $A = \boldsymbol{Z}[M]$ and $B = A[t]/(t^2 - Mt)$, given in [8].

In Section 2, we recall necessary facts on the Artin-Hasse exponential series and the main result of [6].

In Section 3, we prove the main result, after reviewing the Hochschild cohomology in our case. The theorem can be reduced to the main result of [6] thanks to an exact sequence of formal groups

$$0 \to \hat{\mathscr{G}}_A^{(M)} \to \widehat{\prod_{B/A} \boldsymbol{G}_{m,B}} \to \hat{\boldsymbol{G}}_{m,A} \to 0,$$

where $A = \boldsymbol{Z}[M]$ and $B = A[t]/(t^2 - Mt)$, as done in [8]. Furthermore, in order to give an explicit description we define vaiants of the Artin-Hasse exponential series

$$E_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{T}) \in \boldsymbol{Z}_{(p)}[U_0, U_1, U_2, \dots, M][[T_0, T_1, \dots, T_{n-1}]],$$

modifying the power series

$$E_{p,n}(\boldsymbol{U}; \boldsymbol{T}) \in \boldsymbol{Z}_{(p)}[U_0, U_1, U_2, \dots][[T_0, T_1, \dots, T_{n-1}]]$$

presented in [6]. The definition $E_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{T})$ is parallel to that of $W^{(M)}$ in a sense.

In the section 4, we establish some functorialities, recalling some results of [6].

The last section is devoted to a case over a discrete valuation ring. In general, it is difficult to determine $\mathrm{Ext}_A^1(W_{n,A}, \mathscr{G}_A^{(\mu)})$ if μ is not nilpotent in A. However, $\mathrm{Ext}_A^1(W_{n,A}, \mathscr{G}_A^{(\mu)})$ is isomorphic to the subgroup of $H_{\text{ét}}^1(W_{n,A}, \mathscr{G}_A^{(\mu)})$ formed by the primitive elements when A is a discrete valuation ring. This enables us to give an explicit description of $\mathrm{Ext}_A^1(W_{n,A}, \mathscr{G}_A^{(\mu)})$. Furthermore, we observe a behavior of the canonical map $\mathrm{Ext}_A^1(W_{n,A}, \mathscr{G}_A^{(\mu)}) \to \mathrm{Ext}_A^1(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(\mu)})$ when A is of mixed characteristics 0 and p.

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Notation

Throughout the paper, p denotes a prime integer, $\boldsymbol{Z}_{(p)}$ the localization of $\boldsymbol{Z}$ at the prime ideal (p), and A a $\boldsymbol{Z}_{(p)}$-algebra. All rings are commutative with a unit element 1, unless otherwise stated.

$\boldsymbol{G}_{a,A}$: the additive group scheme over A
$\boldsymbol{G}_{m,A}$: the multiplicative group scheme over A
$W_{n,A}$: the group scheme of Witt vectors of length n over A
W_A: the group scheme of Witt vectors over A
$\hat{\boldsymbol{G}}_{a,A}$: the additive formal group scheme over A
$\hat{\boldsymbol{G}}_{m,A}$: the multiplicative formal group scheme over A
$\hat{W}_{n,A}$: the formal group scheme of Witt vectors of length n over A
$\hat{W}_A$: the formal group scheme of Witt vectors over A

$H_0^2(G,H)$ denotes the Hochschild cohomology group consisting of symmetric 2-cocycles of G with coefficients in H for group schemes or formal group schemes G and H.

For a commutative ring B, $B^{\times}$ denotes the multiplicative group $\boldsymbol{G}_m(B)$.

Contents

1. Recall: Witt Vectors

We start with reviewing necessary facts on Witt vectors. For details, see Demazure-Gabriel [1, Chap. V] or Hazewinkel [3, Chap. III].

1.1. For each $r \geq 0$, we denote by $\Phi_r(\boldsymbol{T}) = \Phi_r(T_0, T_1, \dots, T_r)$ the so-called Witt polynomial

$$\Phi_r(\boldsymbol{T}) = T_0^{p^r} + pT_1^{p^{r-1}} + \cdots + p^r T_r$$

in $\boldsymbol{Z}[\boldsymbol{T}] = \boldsymbol{Z}[T_0, T_1, \dots, T_r]$. We define polynomials

$$S_r(\boldsymbol{X}, \boldsymbol{Y}) = S_r(X_0, \dots, X_r, Y_0, \dots, Y_r),$$

$$P_r(\boldsymbol{X}, \boldsymbol{Y}) = P_r(X_0, \dots, X_r, Y_0, \dots, Y_r)$$

in $\boldsymbol{Z}[\boldsymbol{X}, \boldsymbol{Y}] = \boldsymbol{Z}[X_0, X_1, \dots, X_r, Y_0, Y_1, \dots, Y_r]$ inductively by

$$\Phi_r(S_0(\boldsymbol{X}, \boldsymbol{Y}), S_1(\boldsymbol{X}, \boldsymbol{Y}), \dots, S_r(\boldsymbol{X}, \boldsymbol{Y})) = \Phi_r(\boldsymbol{X}) + \Phi_r(\boldsymbol{Y}),$$

$$\Phi_r(P_0(\boldsymbol{X}, \boldsymbol{Y}), P_1(\boldsymbol{X}, \boldsymbol{Y}), \dots, P_r(\boldsymbol{X}, \boldsymbol{Y})) = \Phi_r(\boldsymbol{X})\Phi_r(\boldsymbol{Y}).$$

The ring structure of the scheme of Witt vectors of length n (resp. of the scheme of Witt vectors)

$$W_{n,\boldsymbol{Z}} = \operatorname{Spec} \boldsymbol{Z}[T_0, T_1, \dots, T_{n-1}] \quad (\text{resp. } W_{\boldsymbol{Z}} = \operatorname{Spec} \boldsymbol{Z}[T_0, T_1, T_2, \dots])$$

is given by the addition

$$T_0 \mapsto S_0(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}),\ T_1 \mapsto S_1(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}),\ T_2 \mapsto S_2(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}), \dots$$

and the multiplication

$$T_0 \mapsto P_0(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}),\ T_1 \mapsto P_1(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}),\ T_2 \mapsto P_2(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}), \dots.$$

We denote by $\hat{W}_{n,\boldsymbol{Z}}$ (resp. $\hat{W}_{\boldsymbol{Z}}$) the formal completion of $W_{n,\boldsymbol{Z}}$ (resp. $W_{\boldsymbol{Z}}$) along the zero section. $\hat{W}_{n,\boldsymbol{Z}}$ (resp. $\hat{W}_{\boldsymbol{Z}}$) is considered as a subfunctor of $W_{n,\boldsymbol{Z}}$ (resp. $W_{\boldsymbol{Z}}$). Indeed, if A is a ring, then

$$\hat{W}_n(A) = \{(a_0, a_1, a_2, \dots) \in W_n(A);\ a_i \text{ is nilpotent for all } i\},$$

$$\hat{W}(A) = \left\{ (a_0, a_1, a_2, \dots) \in W(A);\ \begin{array}{l} a_i \text{ is nilpotent for all } i \text{ and} \\ a_i = 0 \text{ for all but a finite number of } i \end{array} \right\}.$$

1.2. The restriction homomorphism $R : W_{n+1,\boldsymbol{Z}} \to W_{n,\boldsymbol{Z}}$ is defined by the canonical injection

$$\boldsymbol{Z}[T_0, T_1, \dots, T_{n-1}] \hookrightarrow \boldsymbol{Z}[T_0, T_1, \dots, T_n].$$

Note that

$$W_{\boldsymbol{Z}} = \varprojlim_R W_{n,\boldsymbol{Z}}.$$

The Verschiebung homomorphism $V : W_{n,\boldsymbol{Z}} \to W_{n+1,\boldsymbol{Z}}$ (resp. $V : W_{\boldsymbol{Z}} \to W_{\boldsymbol{Z}}$) is defined by

$$T_0 \mapsto 0, T_1 \mapsto T_0, \dots, T_n \mapsto T_{n-1}$$

$$(\text{resp. } T_0 \mapsto 0, T_1 \mapsto T_0, T_2 \mapsto T_1, \dots).$$

Note that V is a homomorphism of group schemes.

Define now polynomials

$$F_r(\boldsymbol{T}) = F_r(T_0, \dots, T_r, T_{r+1}) \in \boldsymbol{Z}[T_0, \dots, T_r, T_{r+1}]$$

inductively by

$$\Phi_r(F_0(\boldsymbol{T}), \dots, F_r(\boldsymbol{T})) = \Phi_{r+1}(T_0, \dots, T_r, T_{r+1})$$

for $r \geq 0$.

We denote by $F : W_{n+1,\boldsymbol{Z}} \to W_{n,\boldsymbol{Z}}$ (resp. $F : W_{\boldsymbol{Z}} \to W_{\boldsymbol{Z}}$) the morphism defined by

$$T_0 \mapsto F_0(\boldsymbol{T}), T_1 \mapsto F_1(\boldsymbol{T}), \dots, T_{n-1} \mapsto F_{n-1}(\boldsymbol{T})$$

$$(\text{resp. } T_0 \mapsto F_0(\boldsymbol{T}), T_1 \mapsto F_1(\boldsymbol{T}), T_2 \mapsto F_2(\boldsymbol{T}), \dots).$$

Then it is verified without difficulty that F is a homomorphism of ring schemes. It is readily seen that $F_r(\boldsymbol{T}) \equiv T_r^p \bmod p$ for $r \geq 0$. Therefore, if A is an $\boldsymbol{F}_p$-algebra, $F : W_{n+1,A} \to W_{n,A}$ (resp. $F : W_A \to W_A$) is nothing but the usual Frobenius endomorphism.

We put

$$S(\boldsymbol{X}, \boldsymbol{Y}) = (S_0(\boldsymbol{X}, \boldsymbol{Y}), S_1(\boldsymbol{X}, \boldsymbol{Y}), S_2(\boldsymbol{X}, \boldsymbol{Y}), \dots),$$

$$P(\boldsymbol{X}, \boldsymbol{Y}) = (P_0(\boldsymbol{X}, \boldsymbol{Y}), P_1(\boldsymbol{X}, \boldsymbol{Y}), P_2(\boldsymbol{X}, \boldsymbol{Y}), \dots),$$

$$F(\boldsymbol{T}) = (F_0(\boldsymbol{T}), F_1(\boldsymbol{T}), F_2(\boldsymbol{T}), \dots).$$

Next we recall the variants of Witt vectors defined in [8. Sect. 1].

1.3. For each $r \geq 0$, we define

$$\Phi_r^{(M)}(\boldsymbol{T}) = \Phi_r^{(M)}(T_0, \dots, T_r) \in \boldsymbol{Z}[M][T_0, \dots, T_r]$$

by

$$\begin{aligned}\Phi_r^{(M)}(\boldsymbol{T}) &= \frac{1}{M}\Phi_r(MT_0, \dots, MT_r) \\ &= M^{p^r-1}T_0^{p^r} + pM^{p^{r-1}-1}T_1^{p^{r-1}} + \cdots + p^{r-1}M^{p-1}T_{r-1}^p + p^r T_r.\end{aligned}$$

Furthermore, we define

$$S_r^{(M)}(\boldsymbol{X}, \boldsymbol{Y}) = S_r^{(M)}(X_0, \dots, X_r, Y_0, \dots, Y_r) \in \boldsymbol{Z}[M][X_0, \dots, X_r, Y_0, \dots, Y_r],$$

$$P_r^{(M)}(\boldsymbol{X}, \boldsymbol{Y}) = P_r^{(M)}(X_0, \dots, X_r, Y_0, \dots, Y_r) \in \boldsymbol{Z}[M][X_0, \dots, X_r, Y_0, \dots, Y_r],$$

$$F_r^{(M)}(\boldsymbol{T}) = F_r^{(M)}(T_0, \dots, T_r, T_{r+1}) \in \boldsymbol{Z}[M][T_0, \dots, T_r, T_{r+1}]$$

by

$$S_r^{(M)}(X_0, \dots, X_r, Y_0, \dots, Y_r) = \frac{1}{M}S_r(MX_0, \dots, MX_r, MY_0, \dots, MY_r),$$

$$P_r^{(M)}(X_0, \dots, X_r, Y_0, \dots, Y_r) = \frac{1}{M}P_r(X_0, \dots, X_r, MY_0, \dots, MY_r),$$

$$F_r^{(M)}(T_0, \dots, T_r, T_{r+1}) = \frac{1}{M}F_r(MT_0, \dots, MT_r, MT_{r+1})$$

respectively.

We put

$$S^{(M)}(\boldsymbol{X},\boldsymbol{Y}) = (S_0^{(M)}(\boldsymbol{X},\boldsymbol{Y}), S_1^{(M)}(\boldsymbol{X},\boldsymbol{Y}), S_2^{(M)}(\boldsymbol{X},\boldsymbol{Y}),\dots),$$
$$P^{(M)}(\boldsymbol{X},\boldsymbol{Y}) = (P_0^{(M)}(\boldsymbol{X},\boldsymbol{Y}), P_1^{(M)}(\boldsymbol{X},\boldsymbol{Y}), P_2^{(M)}(\boldsymbol{X},\boldsymbol{Y}),\dots),$$
$$F^{(M)}(\boldsymbol{T}) = (F_0^{(M)}(\boldsymbol{T}), F_1^{(M)}(\boldsymbol{T}), F_2^{(M)}(\boldsymbol{T}),\dots).$$

1.4. Put $W^{(M)} = \operatorname{Spec} \boldsymbol{Z}[M][T_0, T_1, T_2, \dots]$. Then a morphism

$$\begin{aligned} W^{(M)} \times_{\boldsymbol{Z}[M]} W^{(M)} = \operatorname{Spec} \boldsymbol{Z}[M][&T_0 \otimes 1, T_1 \otimes 1, \\ &T_2 \otimes 1, \dots, 1 \otimes T_0, 1 \otimes T_1, 1 \otimes T_2, \dots] \\ \to W^{(M)} = \operatorname{Spec} \boldsymbol{Z}[M]&[T_0, T_1, T_2, \dots] \end{aligned}$$

defined by

$$T_0 \mapsto S_0^{(M)}(\boldsymbol{T}\otimes 1, 1\otimes \boldsymbol{T}),\ T_1 \mapsto S_1^{(M)}(\boldsymbol{T}\otimes 1, 1\otimes \boldsymbol{T}),$$
$$T_2 \mapsto S_2^{(M)}(\boldsymbol{T}\otimes 1, 1\otimes \boldsymbol{T}),\dots$$

gives an addition on $W^{(M)}$, which induces a structure of a commutative group scheme over $\boldsymbol{Z}[M]$ on $W^{(M)}$ (cf. [8, Sec. 1]).

Furthermore, a morphism

$$\begin{aligned} W_{\boldsymbol{Z}[M]} \times_{\boldsymbol{Z}[M]} W^{(M)} = \operatorname{Spec} \boldsymbol{Z}[M][&T_0 \otimes 1, T_1 \otimes 1, \\ &T_2 \otimes 1, \dots, 1 \otimes T_0, 1 \otimes T_1, 1 \otimes T_2, \dots] \\ \to W^{(M)} = \operatorname{Spec} \boldsymbol{Z}[M]&[T_0, T_1, T_2, \dots] \end{aligned}$$

defined by

$$T_0 \mapsto P_0^{(M)}(\boldsymbol{T}\otimes 1, 1\otimes \boldsymbol{T}),\ T_1 \mapsto P_1^{(M)}(\boldsymbol{T}\otimes 1, 1\otimes \boldsymbol{T}),$$
$$T_2 \mapsto P_2^{(M)}(\boldsymbol{T}\otimes 1, 1\otimes \boldsymbol{T}),\dots$$

gives an action of $W_{\boldsymbol{Z}[M]}$ on $W^{(M)}$, which induces a structure of $W_{\boldsymbol{Z}[M]}$-module on $W^{(M)}$ (cf. [8, Sec. 1]).

Remark 1.5. Let A be a $\boldsymbol{Z}[M]$-algebra. Let $\boldsymbol{a}, \boldsymbol{b} \in W^{(M)}(A)$ and $\boldsymbol{c} \in W(A)$. We will denote sometimes $\boldsymbol{a} + \boldsymbol{b}$, $\boldsymbol{c} \cdot \boldsymbol{a}$ by $\boldsymbol{a} +^{(M)} \boldsymbol{b}$, $\boldsymbol{c} \cdot^{(M)} \boldsymbol{a}$, respectively, to avoid confusion.

1.6. Let A be a $\boldsymbol{Z}[M]$-algebra, and let μ denote the image of M in A. We denote sometimes $W^{(M)} \otimes_{\boldsymbol{Z}[M]} A$ by $W^{(\mu)}$. We define also

$$S_r^{(\mu)}(\boldsymbol{X}, \boldsymbol{Y}) = S_r^{(\mu)}(X_0, \dots, X_r, Y_0, \dots, Y_r) \in A[X_0, \dots, X_r, Y_0, \dots, Y_r],$$

$$P_r^{(\mu)}(\boldsymbol{X}, \boldsymbol{Y}) = P_r^{(\mu)}(X_0, \dots, X_r, Y_0, \dots, Y_r) \in A[X_0, \dots, X_r, Y_0, \dots, Y_r],$$

$$F_r^{(\mu)}(\boldsymbol{T}) = F_r^{(\mu)}(T_0, \dots, T_r, T_{r+1}) \in A[T_0, \dots, T_r, T_{r+1}]$$

by substituting M by μ in $S_r^{(M)}(\boldsymbol{X}, \boldsymbol{Y})$, $P_r^{(M)}(\boldsymbol{X}, \boldsymbol{Y})$, $F_r^{(M)}(\boldsymbol{T})$, respectively.

EXAMPLE 1.6.1. It is clear that

$$S_r^{(1)}(\boldsymbol{X}, \boldsymbol{Y}) = S_r(\boldsymbol{X}, \boldsymbol{Y}), \quad P_r^{(1)}(\boldsymbol{X}, \boldsymbol{Y}) = P_r(\boldsymbol{X}, \boldsymbol{Y}), \quad F_r^{(1)}(\boldsymbol{T}) = F_r(\boldsymbol{T}),$$

and therefore $W_{\boldsymbol{Z}}^{(1)}$ is nothing but the scheme of Witt vectors $W_{\boldsymbol{Z}}$.

EXAMPLE 1.6.2. It follows that

$$S_r^{(0)}(\boldsymbol{X}, \boldsymbol{Y}) = X_r + Y_r, \quad P_r^{(0)}(\boldsymbol{X}, \boldsymbol{Y}) = \Phi_r(\boldsymbol{X}) Y_r, \quad F_r^{(0)}(\boldsymbol{T}) = pT_{r+1}$$

(cf. [8, 1.4]). Hence the group scheme $W_{\boldsymbol{Z}}^{(0)}$ is isomorphic to the direct product $\boldsymbol{G}_{a,\boldsymbol{Z}}^{\boldsymbol{N}}$.

1.7. We define homomorphisms $V : W^{(M)} \to W^{(M)}$ and $F^{(M)} : W^{(M)} \to W^{(M)}$ by

$$T_0 \mapsto 0, T_1 \mapsto T_0, T_2 \mapsto T_1, \dots$$

and

$$T_0 \mapsto F_0^{(M)}(\boldsymbol{T}), T_1 \mapsto F_1^{(M)}(\boldsymbol{T}), T_2 \mapsto F_2^{(M)}(\boldsymbol{T}), \dots$$

respectively.

By abbreviation we denote $F^{(M)}$ by F.

1.8. We define a morphism $\alpha^{(M)} : W^{(M)} \to W_{\boldsymbol{Z}[M]}$ by

$$T_0 \mapsto MT_0, T_1 \mapsto MT_1, T_2 \mapsto MT_2, \dots.$$

Then it is verified without difficulty that $\alpha^{(M)}$ is a group homomorphism.

REMARK 1.9. Let A be a $\boldsymbol{Z}[M]$-algebra, and let $B = A[t]/(t^2 - Mt)$, in which ε denotes the image of t. Then we have $\varepsilon^2 = M\varepsilon$. Defining a ring homomorphism $B \to A$ by $\varepsilon \mapsto 0$, we have also a ring homomorphism $W(B) \to W(A)$ and

$$\operatorname{Ker}[W(B) \to W(A)] = \{(\varepsilon a_0, \varepsilon a_1, \varepsilon a_2, \dots); a_0, a_1, a_2, \dots \in A\}.$$

In [8, Sec. 1], the following theorem is proved: Let $A = \boldsymbol{Z}[M]$, $B = \boldsymbol{Z}[M,t]/(t^2 - Mt)$. Then $W^{(M)}$ is isomorphic to $\operatorname{Ker}\left[\prod_{B/A} W_B \to W_A\right]$, where $\prod_{B/A}$ denotes the Weil restriction functor. More precisely,

(1) $(a_0, a_1, a_2, \dots) \mapsto (\varepsilon a_0, \varepsilon a_1, \varepsilon a_2, \dots)$ gives rise to a $W(A)$-isomorphism

$$W^{(M)}(A) \xrightarrow{\sim} \operatorname{Ker}[W(B) \to W(A)];$$

(2) $F : W(B) \to W(B)$ induces F on $W^{(M)}(A)$;

(3) $V : W(B) \to W(B)$ induces V on $W^{(M)}(A)$.

2. Recall: Hochschild Cohomology

In this section, we recall the main result of Sekiguchi-Suwa [6].

We begin by recalling the necessary facts on the Artin-Hasse exponential series. For details, see [1, Sec. 5] or [6, Sec. 2].

2.1. The Artin-Hasse exponential series $E_p(T) \in \boldsymbol{Z}_{(p)}[[T]]$ is defined by

$$E_p(T) = \exp\left(\sum_{r\geq 0} \frac{T^{p^r}}{p^r}\right).$$

For $\boldsymbol{U} = (U_r)_{r\geq 0}$, we put

$$E_p(\boldsymbol{U}; T) = \prod_{r\geq 0} E_p(U_r T^{p^r}) = \exp\left(\sum_{r\geq 0} \frac{\Phi_r(\boldsymbol{U}) T^{p^r}}{p^r}\right).$$

It is readily seen that

$$E_p(S(\boldsymbol{U}, \boldsymbol{V}); T) = E_p(\boldsymbol{U}; T) E_p(\boldsymbol{V}; T).$$

2.2. For $\boldsymbol{U} = (U_r)_{r\geq 0}$ and $\boldsymbol{T} = (T_r)_{r\geq 0}$, We define a formal power series $E_p(\boldsymbol{U}; \boldsymbol{T}) \in \boldsymbol{Z}_{(p)}[\boldsymbol{U}][[\boldsymbol{T}]]$ by

$$E_p(\boldsymbol{U}; \boldsymbol{T}) = \exp\left(\sum_{r\geq 0} \frac{1}{p^r} \Phi_r(\boldsymbol{U}) \Phi_r(\boldsymbol{T})\right) = \exp\left(\sum_{r\geq 0} \frac{1}{p^r} \Phi_r(P(\boldsymbol{U}, \boldsymbol{T}))\right).$$

It is verified that

$$E_p(S(\boldsymbol{U}, \boldsymbol{V}); \boldsymbol{T}) = E_p(\boldsymbol{U}; \boldsymbol{T}) E_p(\boldsymbol{V}; \boldsymbol{T}).$$

2.3. Let n be a positive integer. We define a polynomial $\Phi_{r,n}(\boldsymbol{T}) = \Phi_{r,n}(T_0, T_1, \dots, T_{n-1})$ in $\boldsymbol{Z}[T_0, T_1, \dots, T_{n-1}]$ by

$$\Phi_{r,n}(\boldsymbol{T}) = \begin{cases} \Phi_r(T_0, T_1, \dots, T_r) & \text{if } r \leq n-1, \\ \Phi_r(T_0, T_1, \dots, T_{n-1}, 0, 0, \dots) & \text{if } r \geq n. \end{cases}$$

In [6, 2.4], a formal power series

$$E_{p,n}(\boldsymbol{U}; \boldsymbol{T}) \in \boldsymbol{Z}_{(p)}[\boldsymbol{U}][[T_0, T_1, \dots, T_{n-1}]]$$

is defined by

$$E_{p,n}(\boldsymbol{U}; \boldsymbol{T}) = E_p(\boldsymbol{U}; T_0, \dots, T_{n-1}, 0, 0, \dots) = \exp\left(\sum_{r \geq 0} \frac{1}{p^r} \Phi_r(\boldsymbol{U}) \Phi_{r,n}(\boldsymbol{T})\right).$$

It is readily seen that

$$E_{p,n}(S(\boldsymbol{U}; \boldsymbol{V}), \boldsymbol{T}) = E_{p,n}(\boldsymbol{U}; \boldsymbol{T}) E_{p,n}(\boldsymbol{V}; \boldsymbol{T}).$$

2.4. Let k, l be integers with $k \geq l \geq 0$. Define a polynomial

$$S_{k,l}(\boldsymbol{X}, \boldsymbol{Y}) = S(X_0, \dots, X_{l-1}, Y_0, \dots, Y_{l-1}) \in \boldsymbol{Z}[X_0, \dots, X_{l-1}, Y_0, \dots, Y_{l-1}]$$

by

$$S_{k,l}(\boldsymbol{X}, \boldsymbol{Y}) = S_k(X_0, \dots, X_{l-1}, 0, \dots, 0, Y_0, \dots, Y_{l-1}, 0, \dots, 0).$$

In [6, 2.7], a formal power series

$$F_{p,n}(\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y}) \in \boldsymbol{Z}_{(p)}[U_0, U_1, U_2, \dots][[X_0, \dots, X_{n-1}, Y_0, \dots, Y_{n-1}]]$$

is defined by

$$\begin{aligned} F_{p,n}(\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y}) &= E_p(\boldsymbol{U}; \tilde{S}_n(\boldsymbol{X}, \boldsymbol{Y})) \\ &= E_p(\boldsymbol{U}; S_{n,n}(\boldsymbol{X}, \boldsymbol{Y}), S_{n+1,n}(\boldsymbol{X}, \boldsymbol{Y}), S_{n+2,n}(\boldsymbol{X}, \boldsymbol{Y}), \dots). \end{aligned}$$

It is readily seen that

(1) $F_{p,n}(\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y}) F_{p,n}(\boldsymbol{U}; S(\boldsymbol{X}, \boldsymbol{Y}), \boldsymbol{Z}) = F_{p,n}(\boldsymbol{U}; \boldsymbol{X}, S(\boldsymbol{Y}, \boldsymbol{Z})) F_{p,n}(\boldsymbol{U}; \boldsymbol{Y}, \boldsymbol{Z})$,

(2) $F_{p,n}(\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y}) = F_{p,n}(\boldsymbol{U}; \boldsymbol{Y}, \boldsymbol{X})$.

Moreover, we have

(3) $F_{p,n}(S(\boldsymbol{U}, \boldsymbol{V}); \boldsymbol{X}, \boldsymbol{Y}) = F_{p,n}(\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y}) F_{p,n}(\boldsymbol{V}; \boldsymbol{X}, \boldsymbol{Y})$.

Now we recall some results of [6]. For generalities of the Hochschild cohomology, see [1, Ch. II.3 and Ch. III.6].

2.5. Let A be a $\boldsymbol{Z}_{(p)}[M]$-algebra. We define a complex

$$0 \to C^1(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \xrightarrow{\partial} C^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \to 0 \to \cdots$$

by

$$C^1(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) = \{F(\boldsymbol{T}) \in A[[T_0, T_1, \dots, T_{n-1}]]; F(\boldsymbol{T}) \equiv 1 \mod \deg 1\},$$

$$C^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A})$$
$$= \{F(\boldsymbol{X}, \boldsymbol{Y}) \in A[[X_0, X_1, \dots, X_{n-1}, Y_0, Y_1, \dots, Y_{n-1}]]; F(\boldsymbol{X}, \boldsymbol{Y}) \equiv 1 \mod \deg 1\}.$$

The boundary map $\partial : C^1(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \to C^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A})$ is given by

$$\partial : F(T_0, \dots, T_{n-1}) \mapsto \frac{F(X_0, \dots, X_{n-1}) F(Y_0, \dots, Y_{n-1})}{F(S_0(\boldsymbol{X}, \boldsymbol{Y}), \dots, S_{n-1}(\boldsymbol{X}, \boldsymbol{Y}))}$$

([6, 2.1]). A formal power series $G(\boldsymbol{X}, \boldsymbol{Y}) = G(X_0, X_1, \dots, X_{n-1}, Y_0, Y_1, \dots, Y_{n-1}) \in C^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A})$ is called a symmetric 2-cocycle if $G(\boldsymbol{X}, \boldsymbol{Y})$ satisfies the following functional equations:

(1) $G(\boldsymbol{X}, \boldsymbol{Y}) G(S(\boldsymbol{X}, \boldsymbol{Y}), \boldsymbol{Z}) = G(\boldsymbol{X}, S(\boldsymbol{Y}, \boldsymbol{Z})) G(\boldsymbol{Y}, \boldsymbol{Z})$,

(2) $G(\boldsymbol{X}, \boldsymbol{Y}) = G(\boldsymbol{Y}, \boldsymbol{X})$.

Let $Z^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A})$ denote the subgroup of $C^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A})$ which consists of the symmetric 2-cocycles. Let $B^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) = \operatorname{Im} \partial$, and define

$$H_0^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) = Z^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) / B^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}).$$

We have two complexes concentrated in degrees 1 and 2,

$$\tilde{C}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) : 0 \to C^1(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \xrightarrow{\partial} Z^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \to 0 \to \cdots,$$

$$\tilde{D}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) : 0 \to W(A) \xrightarrow{F^n} W(A) \to 0 \to \cdots.$$

By [6, 2.8], a morphism of complexes

$$\xi_n : \tilde{D}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \to \tilde{C}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A})$$

is defined by

$$\xi_n^0 : W(A) \to C^1(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}), \quad \xi_n^0(\boldsymbol{a}) = E_{p,n}(\boldsymbol{a}; \boldsymbol{T}),$$

$$\xi_n^1 : W(A) \to Z^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}), \quad \xi_n^1(\boldsymbol{a}) = F_{p,n}(\boldsymbol{a}; \boldsymbol{X}, \boldsymbol{Y}).$$

It is proved by [6, Th. 2.8.1] that this induces isomorphisms,

$$\xi_n^0 : \operatorname{Ker}[F^n : W(A) \to W(A)] \xrightarrow{\sim} \operatorname{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}),$$

$$\xi_n^1 : \operatorname{Coker}[F^n : W(A) \to W(A)] \xrightarrow{\sim} H_0^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}).$$

REMARK 2.6. In [6, 2.1], a complex

$$0 \to C^1(\hat{W}_{n,A}, \hat{G}_{m,A}) \xrightarrow{\partial} C^2(\hat{W}_{n,A}, \hat{G}_{m,A}) \to 0 \to \cdots$$

is defined by

$$C^1(\hat{W}_{n,A}, \hat{G}_{m,A}) = A[[T_0, T_1, \dots, T_{n-1}]]^\times$$

$$C^2(\hat{W}_{n,A}, \hat{G}_{m,A}) = A[[X_0, X_1, \dots, X_{n-1}, Y_0, Y_1, \dots, Y_{n-1}]]^\times.$$

The boundary map $\partial : C^1(\hat{W}_{n,A}, \hat{G}_{m,A}) \to C^2(\hat{W}_{n,A}, \hat{G}_{m,A})$ is given by

$$\partial : F(T_0, \dots, T_{n-1}) \mapsto \frac{F(X_0, \dots, X_{n-1})F(Y_0, \dots, Y_{n-1})}{F(S_0(\boldsymbol{X}, \boldsymbol{Y}), \dots, S_{n-1}(\boldsymbol{X}, \boldsymbol{Y}))}.$$

This definition is different from that of 2.5. But as is pointed out in [8, 3.3.1], the complex defined in 2.5 is quasi-isomorphic to the complex defined in [6, 2.1].

3. Statement and Proof of the Theorem

3.1. Let A be a $\boldsymbol{Z}[M]$-algebra. We define a group scheme $\mathscr{G}_A^{(M)}$ over A by

$$\mathscr{G}_A^{(M)} = \operatorname{Spec} A\left[T, \frac{1}{1+MT}\right]$$

with

(1) the multiplication: $T \mapsto T \otimes 1 + 1 \otimes T + MT \otimes T$;
(2) the unit: $T \mapsto 0$;
(3) the inverse $T \mapsto -\dfrac{T}{1+MT}$.

Moreover, we define an A-homomorphism $\alpha_A^{(M)} : \mathscr{G}_A^{(M)} \to \boldsymbol{G}_{m,A}$ by

$$U \mapsto 1 + MT : A[U, U^{-1}] \to A\left[T, \frac{1}{1+MT}\right].$$

If M is invertible in A, $\alpha_A^{(M)}$ is an A-isomorphism. On the other hand, if $M = 0$ in A, $\mathscr{G}_A^{(M)}$ is nothing but the additive group $\boldsymbol{G}_{a,A}$.

We denote by $\hat{\mathscr{G}}_A^{(M)}$ the formal completion of $\mathscr{G}_A^{(M)}$ along the zero section.

REMARK 3.2. Let A be a $\boldsymbol{Z}[M]$-algebra, and let $B = A[t]/(t^2 - Mt)$, in which ε denotes the image of t. Then we have $\varepsilon^2 = M\varepsilon$. Defining a ring homomorphism $B \to A$ by $\varepsilon \mapsto 0$, we have

$$\operatorname{Ker}[B^\times \to A^\times] = \{1 + \varepsilon a;\ a \in A,\ 1 + Ma \text{ is invertible in } A\}.$$

Hence $\mathscr{G}_A^{(M)}$ is isomorphic to $\mathrm{Ker}\left[\prod_{B/A} \boldsymbol{G}_{m,B} \to \boldsymbol{G}_{m,A}\right]$, where $\prod_{B/A}$ denotes the Weil restriction functor. Furthermore, the inclusion $A \to B$ defines a section of $\prod_{B/A} \boldsymbol{G}_{m,B} \to \boldsymbol{G}_{m,A}$, and therefore, the exact sequence

$$0 \to \mathscr{G}_A^{(M)} \to \prod_{B/A} \boldsymbol{G}_{m,B} \to \boldsymbol{G}_{m,A} \to 0$$

splits.

3.3. Let A be a $\boldsymbol{Z}_{(p)}[M]$-algebra. We shall define a complex

$$0 \to C^1(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) \xrightarrow{\partial} C^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) \to 0 \to \cdots$$

by

$$C^1(\hat{W}_{n,A}, \mathscr{G}_A^{(M)}) = \{F(\boldsymbol{T}) \in A[[T_0, T_1, \dots, T_{n-1}]]; F(\boldsymbol{T}) \equiv 0 \mod \deg 1\},$$

$$C^2(\hat{W}_{n,A}, \mathscr{G}_A^{(M)}) = \{F(\boldsymbol{X}, \boldsymbol{Y}) \in A[[X_0, X_1, \dots, X_{n-1}, Y_0, Y_1, \dots, Y_{n-1}]]; \\ F(\boldsymbol{X}, \boldsymbol{Y}) \equiv 0 \mod \deg 1\}.$$

The boundary map $\partial : C^1(\hat{W}_{n,A}, \mathscr{G}_A^{(M)}) \to C^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)})$ is given by

$$\partial : F(\boldsymbol{T}) \mapsto \frac{F(\boldsymbol{X}) + F(\boldsymbol{Y}) + MF(\boldsymbol{X})F(\boldsymbol{Y}) - F(S(\boldsymbol{X}, \boldsymbol{Y}))}{1 + MF(S(\boldsymbol{X}, \boldsymbol{Y}))}.$$

A formal power series $G(\boldsymbol{X}, \boldsymbol{Y}) = G(X_0, X_1, \dots, X_{n-1}, Y_0, Y_1, \dots, Y_{n-1}) \in C^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)})$ is called a symmetric 2-cocycle if $G(\boldsymbol{X}, \boldsymbol{Y})$ satisfies the following functional equations:

(1) $G(\boldsymbol{X}, \boldsymbol{Y}) + G(S(\boldsymbol{X}, \boldsymbol{Y}), \boldsymbol{Z}) + MG(\boldsymbol{X}, \boldsymbol{Y})G(S(\boldsymbol{X}, \boldsymbol{Y}), \boldsymbol{Z}) = G(\boldsymbol{X}, S(\boldsymbol{Y}, \boldsymbol{Z})) + G(\boldsymbol{Y}, \boldsymbol{Z}) + MG(\boldsymbol{X}, S(\boldsymbol{Y}, \boldsymbol{Z}))G(\boldsymbol{Y}, \boldsymbol{Z})$,

(2) $G(\boldsymbol{X}, \boldsymbol{Y}) = G(\boldsymbol{Y}, \boldsymbol{X})$.

Let $Z^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)})$ denote the subgroup of $C^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)})$ which consists of the symmetric 2-cocycles. Let $B^2(\hat{W}_{n,A}, \mathscr{G}_A^{(M)}) = \mathrm{Im}\, \partial$, and define

$$H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) = Z^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) / B^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}).$$

We have two complexes concentrated in the degree 1 and 2,

$$\tilde{C}^*(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) : 0 \to C^1(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) \xrightarrow{\partial} Z^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) \to 0 \to \cdots,$$

$$\tilde{D}^*(\hat{W}_{n,A}, \mathscr{G}_A^{(M)}) : 0 \to W^{(M)}(A) \xrightarrow{F^n} W^{(M)}(A) \to 0 \to \cdots.$$

3.4. Let A be a $\boldsymbol{Z}[M]$-algebra, and let $B = A[t]/(t^2 - Mt)$, in which ε denotes the image of t. Then we have $\varepsilon^2 = M\varepsilon$. The splitting exact secquence of formal groups

$$0 \to \hat{\mathscr{G}}_A^{(M)} \to \left(\prod_{B/A} \widehat{\boldsymbol{G}_{m,B}}\right) \to \hat{\boldsymbol{G}}_{m,A} \to 0$$

induces a splitting exact sequence of complexes

$$0 \to \tilde{C}^*(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) \to \tilde{C}^*(\hat{W}_{n,B}, \hat{\boldsymbol{G}}_{m,B}) \to \tilde{C}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \to 0.$$

More precisely,

$$C^1(\hat{W}_{n,B}, \hat{\boldsymbol{G}}_{m,B}) \to C^1(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}),$$

$$Z^2(\hat{W}_{n,B}, \hat{\boldsymbol{G}}_{m,B}) \to Z^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A})$$

are induced from the ring homomorphism $B \to A$. Moreover

$$C^1(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) \to C^1(\hat{W}_{n,B}, \hat{\boldsymbol{G}}_{m,B}),$$

$$Z^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) \to Z^2(\hat{W}_{n,B}, \hat{\boldsymbol{G}}_{m,B})$$

are defined by

$$F(\boldsymbol{T}) \mapsto 1 + \varepsilon F(\boldsymbol{T}),$$

$$G(\boldsymbol{X}, \boldsymbol{Y}) \mapsto 1 + \varepsilon G(\boldsymbol{X}, \boldsymbol{Y}),$$

respectively.

On the other hand, we have a commutative diagram with splitting exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W^{(M)}(A) & \longrightarrow & W(B) & \longrightarrow & W(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F^n} & & \downarrow{\scriptstyle F^n} & & \downarrow{\scriptstyle F^n} & & \\
0 & \longrightarrow & W^{(M)}(A) & \longrightarrow & W(B) & \longrightarrow & W(A) & \longrightarrow & 0
\end{array}$$

by Remark 1.9. Obviously the diagram of complexes

$$\begin{array}{ccc}
\tilde{D}^*(\hat{W}_{n,B}, \hat{\boldsymbol{G}}_{m,B}) & \longrightarrow & \tilde{D}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \\
{\scriptstyle \xi_n}\downarrow & & \downarrow{\scriptstyle \xi_n} \\
\tilde{C}^*(\hat{W}_{n,B}, \hat{\boldsymbol{G}}_{m,B}) & \longrightarrow & \tilde{C}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A})
\end{array}$$

is commutative. Hence we obtain a morphism of complexes

$$\xi_n : \tilde{D}^*(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) \to \tilde{C}^*(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}),$$

To sum up, we obtain a commutative diagram of cochain complexes with splitting exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{D}^*(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \longrightarrow & \tilde{D}^*(\hat{W}_{n,B}, \hat{\boldsymbol{G}}_{m,B}) & \longrightarrow & \tilde{D}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \xi_n} & & \downarrow{\scriptstyle \xi_n} & & \downarrow{\scriptstyle \xi_n} & & \\
0 & \longrightarrow & \tilde{C}^*(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \longrightarrow & \tilde{C}^*(\hat{W}_{n,B}, \hat{\boldsymbol{G}}_{m,B}) & \longrightarrow & \tilde{C}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) & \longrightarrow & 0.
\end{array}$$

The most left ξ_n is a quasi-isomorphism since the other two are such, by [6, Th. 2.8.1]. We have thus proved:

THEOREM 3.5. *Let A be a $\boldsymbol{Z}_{(p)}[M]$-algebra. Then there exist isomorphisms*

$$\operatorname{Ker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] \xrightarrow{\sim} \operatorname{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}),$$

$$\operatorname{Coker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] \xrightarrow{\sim} H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}).$$

REMARK 3.6. We can describe explicitly the isomorphisms

$$\xi_n^0 : \operatorname{Ker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] \xrightarrow{\sim} \operatorname{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}),$$

$$\xi_n^1 : \operatorname{Coker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] \xrightarrow{\sim} H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}),$$

induced from

$$\xi_n : \tilde{D}^*(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) \to \tilde{C}^*(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}).$$

Indeed, we define two formal power serieses

$$E_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{T}) \in \boldsymbol{Z}_{(p)}[M, U_0, U_1, U_2, \dots][[T_0, T_1, \dots, T_{n-1}]],$$

$$F_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y}) \in \boldsymbol{Z}_{(p)}[M, U_0, U_1, U_2, \dots][[X_0, \dots, X_{n-1}, Y_0, \dots, Y_{n-1}]]$$

by

$$E_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{T}) = \frac{1}{M}[E_{p,n}(\alpha^{(M)}\boldsymbol{U}, \boldsymbol{T}) - 1],$$

$$F_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y}) = \frac{1}{M}[F_{p,n}(\alpha^{(M)}\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y}) - 1],$$

respectively. Then,

(1) $\boldsymbol{a} \mapsto E_{p,n}^{(M)}(\boldsymbol{a}; \boldsymbol{T})$ gives rise to the isomorphism

$$\xi_n^0 : \mathrm{Ker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] \xrightarrow{\sim} \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)});$$

(2) $\boldsymbol{a} \mapsto F_{p,n}^{(M)}(\boldsymbol{a}; \boldsymbol{X}, \boldsymbol{Y})$ gives rise to the isomorphism

$$\xi_n^1 : \mathrm{Coker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] \xrightarrow{\sim} H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}).$$

This is a consequence of the following Proposition 3.7, 3.8, 3.9 and Corollary 3.11.

Proposition 3.7. *We have*

$$E_{p,n}^{(M)}(\boldsymbol{U} +^{(M)} \boldsymbol{V}; \boldsymbol{T}) = E_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{T}) + E_{p,n}^{(M)}(\boldsymbol{V}; \boldsymbol{T}) + ME_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{T})E_{p,n}^{(M)}(\boldsymbol{V}; \boldsymbol{T}).$$

Proof. It is sufficient to prove that

$$1 + ME_{p,n}^{(M)}(\boldsymbol{U} +^{(M)} \boldsymbol{V}; \boldsymbol{T}) = [1 + ME_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{T})][1 + ME_{p,n}^{(M)}(\boldsymbol{V}; \boldsymbol{T})],$$

that is to say,

$$E_p(\alpha^{(M)}(\boldsymbol{U} +^{(M)} \boldsymbol{V}); \boldsymbol{T}) = E_p(\alpha^{(M)}\boldsymbol{U}; \boldsymbol{T})E_p(\alpha^{(M)}\boldsymbol{V}; \boldsymbol{T}).$$

This is a consequence of the functional equation for $E_{p,n}(\boldsymbol{U}, \boldsymbol{T})$ since $\alpha^{(M)}(\boldsymbol{U} +^{(M)} \boldsymbol{V}) = \alpha^{(M)}(\boldsymbol{U}) + \alpha^{(M)}(\boldsymbol{V})$.

Proposition 3.8. *We have*

$$F_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y}) + F_{p,n}^{(M)}(\boldsymbol{U}; S(\boldsymbol{X}, \boldsymbol{Y}), \boldsymbol{Z}) + MF_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y})F_{p,n}^{(M)}(\boldsymbol{U}; S(\boldsymbol{X}, \boldsymbol{Y}), \boldsymbol{Z})$$
$$= F_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{X}, S(\boldsymbol{Y}, \boldsymbol{Z})) + F_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{Y}, \boldsymbol{Z}) + MF_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{X}, S(\boldsymbol{Y}, \boldsymbol{Z}))F_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{Y}, \boldsymbol{Z})$$

and

$$F_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y}) = F_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{Y}, \boldsymbol{X}).$$

Proof. It is sufficient to prove that

$$[1 + MF_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y})][1 + MF_{p,n}^{(M)}(\boldsymbol{U}; S(\boldsymbol{X}, \boldsymbol{Y}), \boldsymbol{Z})]$$
$$= [1 + MF_{p,n}^{(M)}(\boldsymbol{T}; \boldsymbol{X}, S(\boldsymbol{Y}, \boldsymbol{Z}))][1 + MF_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{Y}, \boldsymbol{Z})]$$

that is to say,

$$F_{p,n}(\alpha^{(M)}\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y})F_{p,n}(\alpha^{(M)}\boldsymbol{U};S(\boldsymbol{X},\boldsymbol{Y}),\boldsymbol{Z})$$
$$= F_{p,n}(\alpha^{(M)}\boldsymbol{U};\boldsymbol{X},S(\boldsymbol{Y},\boldsymbol{Z}))F_{p,n}(\alpha^{(M)}\boldsymbol{U};\boldsymbol{Y},\boldsymbol{Z}).$$

This is a consequence of 2.4 (1). The second assersion follows immediately from 2.4 (2).

PROPOSITION 3.9. *We have*

$$F_{p,n}^{(M)}(S^{(M)}(\boldsymbol{U},\boldsymbol{V});\boldsymbol{X},\boldsymbol{Y})$$
$$= F_{p,n}^{(M)}(\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y}) + F_{p,n}^{(M)}(\boldsymbol{V};\boldsymbol{X},\boldsymbol{Y}) + MF_{p,n}^{(M)}(\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y})F_{p,n}^{(M)}(\boldsymbol{V};\boldsymbol{X},\boldsymbol{Y}).$$

PROOF. It is sufficient to prove that

$$[1 + MF_{p,n}^{(M)}(S^{(M)}(\boldsymbol{U},\boldsymbol{V});\boldsymbol{X},\boldsymbol{Y})] = [1 + MF_{p,n}^{(M)}(\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y})][1 + MF_{p,n}^{(M)}(\boldsymbol{V};\boldsymbol{X},\boldsymbol{Y})],$$

that is to say,

$$F_{p,n}^{(M)}(\alpha^{(M)}S(\boldsymbol{U},\boldsymbol{V});\boldsymbol{X},\boldsymbol{Y}) = F_{p,n}^{(M)}(\alpha^{(M)}\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y})F_{p,n}^{(M)}(\alpha^{(M)}\boldsymbol{V};\boldsymbol{X},\boldsymbol{Y}).$$

This is a consequence of 2.4 (3).

LEMMA 3.10 (cf. [8, Lemma 1.20]). *Let A be a ring, and let $B = A[M,t]/(t^2 - Mt)$, in which ε denotes the image of t. Let $f(T_1,T_2,\dots,T_n) \in A[T_1,T_2,\dots,T_n]$ with $f(0,0,\dots,0) = 0$, and put*

$$f^{(M)}(T_1,T_2,\dots,T_n) = \frac{1}{M}f(MT_1,MT_2,\dots,MT_n).$$

Then $f^{(M)}(T_1,T_2,\dots,T_n) \in A[M][T_1,T_2,\dots,T_n]$ and

$$\varepsilon f^{(M)}(T_1,T_2,\dots,T_n) = f(\varepsilon T_1,\varepsilon T_2,\dots,\varepsilon T_n).$$

COROLLARY 3.11. *Let A be a $\boldsymbol{Z}_{(p)}[M]$-algebra, and let $B = A[t]/(t^2 - Mt)$, in which ε denotes the image of t. Let $\boldsymbol{a} \in W^{(M)}(A)$, and put $\varepsilon\boldsymbol{a} = (\varepsilon a_0,\varepsilon a_1,\varepsilon a_2,\dots)$. Then*:

(1) $1 + \varepsilon E_{p,n}^{(M)}(\boldsymbol{a};\boldsymbol{T}) = E_{p,n}(\varepsilon\boldsymbol{a};\boldsymbol{T})$;
(2) $1 + \varepsilon F_{p,n}^{(M)}(\boldsymbol{a};\boldsymbol{X},\boldsymbol{Y}) = F_{p,n}(\varepsilon\boldsymbol{a};\boldsymbol{X},\boldsymbol{Y})$.

PROOF. We may assume that $A = \boldsymbol{Z}_{(p)}[M][U_0,U_1,U_2,\dots]$ and $\boldsymbol{a} = \boldsymbol{U} = (U_0,U_1,U_2,\dots)$. Put

$$E_{p,n}(\boldsymbol{U};\boldsymbol{T}) = 1 + \sum a_{i_0 i_1\cdots i_{n-1}}(\boldsymbol{U})T_0^{i_0}T_1^{i_1}\cdots T_{n-1}^{i_{n-1}},$$

where $a_{i_0 i_1 \cdots i_{n-1}}(\boldsymbol{U}) \in \boldsymbol{Z}_{(p)}[U_0, U_1, U_2, \dots]$. Put

$$a^{(M)}_{i_0 i_1 \cdots i_{n-1}}(\boldsymbol{U}) = \frac{1}{M} a_{i_0 i_1 \cdots i_{n-1}}(MU_0, MU_1, MU_2, \dots).$$

Then $a^{(M)}_{i_0 i_1 \cdots i_{n-1}}(\boldsymbol{U}) \in \boldsymbol{Z}_{(p)}[M][U_0, U_1, U_2, \dots]$ since $a_{i_0 i_1 \cdots i_{n-1}}(\boldsymbol{U})$ has no constant term. Furthermore

$$E^{(M)}_{p,n}(\boldsymbol{U}; \boldsymbol{T}) = \sum a^{(M)}_{i_0 i_1 \cdots i_{n-1}}(\boldsymbol{U}) T_0^{i_0} T_1^{i_1} \cdots T_{n-1}^{i_{n-1}}.$$

Now by Lemma 3.10 we have

$$\varepsilon a^{(M)}_{i_0 i_1 \cdots i_{n-1}}(\boldsymbol{U}) = a_{i_0 i_1 \cdots i_{n-1}}(\varepsilon U_0, \varepsilon U_1, \varepsilon U_2, \dots).$$

This implies that

$$E_{p,n}(\varepsilon \boldsymbol{a}; \boldsymbol{T}) = 1 + \varepsilon E^{(M)}_{p,n}(\boldsymbol{a}; \boldsymbol{T}).$$

We can prove (2) similarly.

EXAMPLE 3.12.1. $E^{(1)}_{p,n}(\boldsymbol{U}; \boldsymbol{T}) = E_{p,n}(\boldsymbol{U}; \boldsymbol{T}) - 1.$

EXAMPLE 3.12.2. $E^{(0)}_{p,n}(\boldsymbol{U}; \boldsymbol{T}) = \sum_{r \geq 0} U_r \Phi_{r,n}(\boldsymbol{T}).$

Indeed, by the definition we have

$$1 + ME^{(M)}_{p,n}(\boldsymbol{U}; \boldsymbol{T}) = E_{p,n}(\alpha^{(M)} \boldsymbol{U}; \boldsymbol{T})$$

and

$$\log E_{p,n}(\alpha^{(M)} \boldsymbol{U}; \boldsymbol{T}) = \sum_{r \geq 0} \frac{1}{p^r} \Phi_r(\alpha^{(M)} \boldsymbol{U}) \Phi_{r,n}(\boldsymbol{T}).$$

Now note that, for $r \geq 0$,

$$\Phi_r(MU_0, MU_1, \dots, MU_r) \equiv p^r M U_r \mod M^p.$$

Hence we have

$$\log E_{p,n}(\alpha^{(M)} \boldsymbol{U}; \boldsymbol{T}) \equiv \sum_{r \geq 0} MU_r \Phi_{r,n}(\boldsymbol{T}) \mod M^p,$$

and therefore

$$E_{p,n}(\alpha^{(M)} \boldsymbol{U}; \boldsymbol{T}) \equiv 1 + \sum_{r \geq 0} MU_r \Phi_{r,n}(\boldsymbol{T}) \mod M^2.$$

Thus we obtain

$$E_{p,n}^{(M)}(\alpha^{(M)}\boldsymbol{U};\boldsymbol{T}) \equiv \sum_{r\geq 0} U_r \Phi_{r,n}(\boldsymbol{T}) \bmod M.$$

Example 3.12.3. $E_{p,1}^{(0)}(\boldsymbol{U};\boldsymbol{T}) = \sum_{r\geq 0} U_r\Phi_{r,1}(\boldsymbol{T}) = \sum_{r\geq 0} U_r T_0^{p^r}.$

Example 3.13.1. $F_{p,n}^{(1)}(\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y}) = F_{p,n}(\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y}) - 1.$

Example 3.13.2. $F_{p,n}^{(0)}(\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y}) = \sum_{r\geq 0} U_r\Phi_r(\tilde{S}_n(\boldsymbol{X},\boldsymbol{Y})).$

Indeed, by the definition we have

$$1 + MF_{p,n}^{(M)}(\boldsymbol{U},\boldsymbol{X},\boldsymbol{Y}) = F_{p,n}(\alpha^{(M)}\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y})$$

and

$$\log F_{p,n}(\alpha^{(M)}\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y}) = \sum_{r\geq 0} \frac{1}{p^r}\Phi_r(\alpha^{(M)}\boldsymbol{U})\Phi_r(\tilde{S}_n(\boldsymbol{X},\boldsymbol{Y})).$$

Now note that, for $r \geq 0$,

$$\Phi_r(MU_0, MU_1, \dots, MU_r) \equiv p^r MU_r \bmod M^p.$$

Hence we have

$$\log F_{p,n}(\alpha^{(M)}\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y}) \equiv \sum_{r\geq 0} MU_r\Phi_r(\tilde{S}_n(\boldsymbol{X},\boldsymbol{Y})) \bmod M^p,$$

and therefore

$$F_{p,n}(\alpha^{(M)}\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y}) \equiv 1 + \sum_{r\geq 0} MU_r\Phi_r(\tilde{S}_n(\boldsymbol{X},\boldsymbol{Y})) \bmod M^2.$$

Thus we obtain

$$F_{p,n}^{(M)}(\alpha^{(M)}\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y}) \equiv \sum_{r\geq 0} U_r\Phi_r(\tilde{S}_n(\boldsymbol{X},\boldsymbol{Y})) \bmod M.$$

In particular, putting $\boldsymbol{U} = [1] = (1,0,0,\dots)$ and $M = 0$, we obtain

$$F_{p,n}^{(0)}([1];\boldsymbol{X},\boldsymbol{Y}) = S_{n,n}(\boldsymbol{X},\boldsymbol{Y}) = S_n(X_0,\dots,X_{n-1},0,Y_0,\dots,Y_{n-1},0)$$

which is the 2-cocyle of $Z^2(\hat{W}_n, \hat{\boldsymbol{G}}_a)$ defining the extesion $\hat{W}_{n+1}$.

EXAMPLE 3.14. Let A be a $\boldsymbol{Z}_{(p)}[M]$-algebra. The homomorphism of formal groups $\alpha^{(M)} : \hat{\mathscr{G}}_A^{(M)} \to \hat{\boldsymbol{G}}_{m,A}$ induces a morphism of cochain complex

$$\alpha^{(M)} : \tilde{C}^*(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) \to \tilde{C}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}).$$

We can verify the commutativity of the diagrams

$$\begin{array}{ccc} W^{(M)}(A) & \xrightarrow{\alpha^{(M)}} & W(A) \\ {\scriptstyle \xi_n^0}\downarrow & & \downarrow{\scriptstyle \xi_n^0} \\ C^1(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \xrightarrow[\alpha^{(M)}]{} & C^1(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \end{array}$$

and

$$\begin{array}{ccc} W^{(M)}(A) & \xrightarrow{\alpha^{(M)}} & W(A) \\ {\scriptstyle \xi_n^1}\downarrow & & \downarrow{\scriptstyle \xi_n^1} \\ Z^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \xrightarrow[\alpha^{(M)}]{} & Z^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}). \end{array}$$

Moreover, we obtain a commutative diagram of cochain complexes

$$\begin{array}{ccc} \tilde{D}^*(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \longrightarrow & \tilde{D}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \\ {\scriptstyle \xi_n}\downarrow & & \downarrow{\scriptstyle \xi_n} \\ \tilde{C}^*(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \longrightarrow & \tilde{C}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}), \end{array}$$

and therefore commutative diagrams

$$\begin{array}{ccc} \operatorname{Ker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] & \xrightarrow{\alpha^{(M)}} & \operatorname{Ker}[F^n : W(A) \to W(A)] \\ {\scriptstyle \xi_n^0}\downarrow & & \downarrow{\scriptstyle \xi_n^0} \\ \operatorname{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \xrightarrow[\alpha^{(M)}]{} & \operatorname{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \end{array}$$

and

$$\begin{array}{ccc} \operatorname{Coker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] & \xrightarrow{\alpha^{(M)}} & \operatorname{Coker}[F^n : W(A) \to W(A)] \\ {\scriptstyle \xi_n^1}\downarrow & & \downarrow{\scriptstyle \xi_n^1} \\ H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \xrightarrow[\alpha^{(M)}]{} & H_0^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}). \end{array}$$

Assume now the homothety by M is not bijective but injective on A, and put $A_0 = A/(M)$. Then we have a commutative diagram of cochain complexes with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{D}^*(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \longrightarrow & \tilde{D}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) & \longrightarrow & \tilde{D}^*(\hat{W}_{n,A_0}, \hat{\boldsymbol{G}}_{m,A_0}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \xi_n} & & \downarrow{\scriptstyle \xi_n} & & \downarrow{\scriptstyle \xi_n} & & \\
0 & \longrightarrow & \tilde{C}^*(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \longrightarrow & \tilde{C}^*(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) & \longrightarrow & \tilde{C}^*(\hat{W}_{n,A_0}, \hat{\boldsymbol{G}}_{m,A_0}). & &
\end{array}$$

By the snake lemma the exact sequence

$$\begin{aligned}
0 &\to \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) \to \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \to \mathrm{Hom}_{A_0\text{-gr}}(\hat{W}_{n,A_0}, \hat{\boldsymbol{G}}_{m,A_0}) \\
&\xrightarrow{d} H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) \to H_0^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \to H_0^2(\hat{W}_{n,A_0}, \hat{\boldsymbol{G}}_{m,A_0}) \to 0
\end{aligned}$$

arises from the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W^{(M)}(A) & \longrightarrow & W(A) & \longrightarrow & W(A_0) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F^n} & & \downarrow{\scriptstyle F^n} & & \downarrow{\scriptstyle F^n} & & \\
0 & \longrightarrow & W^{(M)}(A) & \longrightarrow & W(A) & \longrightarrow & W(A_0) & \longrightarrow & 0.
\end{array}$$

We conclude the section, by mentioning an analogue of Theorem 3.5 in the case of group schemes.

First we recall two facts stated in [6] and [8].

Remark 3.15 (cf. [6, Th. 2.8.1]). Let A be a $\boldsymbol{Z}_{(p)}$-algebra. Then

(1) $\boldsymbol{a} \mapsto E_{p,n}(\boldsymbol{a}; \boldsymbol{T})$ gives rise to the isomorphism

$$\xi_n^0 : \mathrm{Ker}[F^n : \hat{W}(A) \to \hat{W}(A)] \xrightarrow{\sim} \mathrm{Hom}_{A\text{-gr}}(W_{n,A}, \boldsymbol{G}_{m,A});$$

(2) $\boldsymbol{a} \mapsto F_{p,n}(\boldsymbol{a}; \boldsymbol{X}, \boldsymbol{Y})$ gives rise to the isomorphism

$$\xi_n^1 : \mathrm{Coker}[F^n : \hat{W}(A) \to \hat{W}(A)] \xrightarrow{\sim} H_0^2(W_{n,A}, \boldsymbol{G}_{m,A}).$$

Remark 3.16 (cf. [8]). Let A be a $\boldsymbol{Z}_{(p)}[M]$-algebra, and let $\hat{W}^{(M)}$ denote the functor defined by

$$\hat{W}^{(M)}(A) = \left\{ (a_0, a_1, a_2, \ldots) \in W^{(M)}(A);\ \begin{array}{l} Ma_i \text{ is nilpotent for all } i \text{ and} \\ a_i = 0 \text{ for all but a finite number of } i \end{array} \right\}.$$

Then we have a splitting exact sequence

$$0 \to \hat{W}^{(M)}(A) \to \hat{W}(B) \to \hat{W}(A) \to 0,$$

where $B = A[t]/(t^2 - Mt)$.

Now, we note that if M is nilpotent in A, then we have

$$E_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{T}) \in A[U_0, U_1, U_2, \dots][T_0, T_1, \dots, T_{n-1}],$$

$$F_{p,n}^{(M)}(\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y}) \in A[U_0, U_1, U_2, \dots][X_0, \dots, X_{n-1}, Y_0, \dots, Y_{n-1}].$$

Therefore, combining Remark 3.15 and 3.16 similarly as in the proof of Theorem 3.5, We can prove:

PROPOSITION 3.17. *Let A be a $\boldsymbol{Z}_{(p)}[M]$-algebra. Assume that M is nilpotent in A. Then*

(1) $\boldsymbol{a} \mapsto E_{p,n}^{(M)}(\boldsymbol{a}; \boldsymbol{T})$ *gives rise to an isomorphism*

$$\xi_n^0 : \mathrm{Ker}[F^n : \hat{W}^{(M)}(A) \to \hat{W}^{(M)}(A)] \xrightarrow{\sim} \mathrm{Hom}_{A\text{-gr}}(W_{n,A}, \mathscr{G}_A^{(M)});$$

(2) $\boldsymbol{a} \mapsto F_{p,n}^{(M)}(\boldsymbol{a}; \boldsymbol{X}, \boldsymbol{Y})$ *gives rise to an isomorphism*

$$\xi_n^1 : \mathrm{Coker}[F^n : \hat{W}^{(M)}(A) \to \hat{W}^{(M)}(A)] \xrightarrow{\sim} H_0^2(W_{n,A}, \mathscr{G}_A^{(M)}).$$

4. Functoriality

We establish some functorialities among ξ_n^i $(i = 0, 1, n = 1, 2, \dots)$.

PROPOSITION 4.1. *Let A be a $\boldsymbol{Z}_{(p)}[M]$-algebra. Then:*

(1) *The diagrams*

$$\begin{array}{ccc} \mathrm{Ker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] & \longrightarrow & \mathrm{Ker}[F^{n+1} : W^{(M)}(A) \to W^{(M)}(A)] \\ \downarrow{\scriptstyle \xi_n^0} & & \downarrow{\scriptstyle \xi_{n+1}^0} \\ \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \xrightarrow[R^*]{} & \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n+1,A}, \hat{\mathscr{G}}_A^{(M)}) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Coker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] & \xrightarrow{F} & \mathrm{Coker}[F^{n+1} : W^{(M)}(A) \to W^{(M)}(A)] \\ \downarrow{\scriptstyle \xi_n^1} & & \downarrow{\scriptstyle \xi_{n+1}^1} \\ H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \xrightarrow[R^*]{} & H_0^2(\hat{W}_{n+1,A}, \hat{\mathscr{G}}_A^{(M)}) \end{array}$$

are commutative. Here the first horizontal arrow denotes the canonical injection.

(2) *The diagrams*

$$\begin{array}{ccc}
\mathrm{Ker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] & \xrightarrow{V} & \mathrm{Ker}[F^{n+1} : W^{(M)}(A) \to W^{(M)}(A)] \\
{\scriptstyle \xi_n^0}\downarrow & & \downarrow{\scriptstyle \xi_{n+1}^0} \\
\mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \xrightarrow[F^*]{} & \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n+1,A}, \hat{\mathscr{G}}_A^{(M)})
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Coker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] & \xrightarrow{p} & \mathrm{Coker}[F^{n+1} : W^{(M)}(A) \to W^{(M)}(A)] \\
{\scriptstyle \xi_n^1}\downarrow & & \downarrow{\scriptstyle \xi_{n+1}^1} \\
H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \xrightarrow[F^*]{} & H_0^2(\hat{W}_{n+1,A}, \hat{\mathscr{G}}_A^{(M)})
\end{array}$$

are commutative.

(3) *The diagrams*

$$\begin{array}{ccc}
\mathrm{Ker}[F^{n+1} : W^{(M)}(A) \to W^{(M)}(A)] & \xrightarrow{F} & \mathrm{Ker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] \\
{\scriptstyle \xi_{n+1}^0}\downarrow & & \downarrow{\scriptstyle \xi_n^0} \\
\mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n+1,A}, \hat{\mathscr{G}}_A^{(M)}) & \xrightarrow[V^*]{} & \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)})
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Coker}[F^{n+1} : W^{(M)}(A) \to W^{(M)}(A)] & \longrightarrow & \mathrm{Coker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] \\
{\scriptstyle \xi_{n+1}^1}\downarrow & & \downarrow{\scriptstyle \xi_n^1} \\
H_0^2(\hat{W}_{n+1,A}, \hat{\mathscr{G}}_A^{(M)}) & \xrightarrow[V^*]{} & H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)})
\end{array}$$

are commutative. Here the third horizontal arrow denotes the canonical surjection.

(4) *The diagrams*

$$\begin{array}{ccc}
\mathrm{Ker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] & \xrightarrow{[a]\cdot} & \mathrm{Ker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] \\
{\scriptstyle \xi_n^0}\downarrow & & \downarrow{\scriptstyle \xi_n^0} \\
\mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \longrightarrow & \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)})
\end{array}$$

and

$$\begin{tikzcd}
\mathrm{Coker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] \arrow[r, "F^n[a]\cdot"] \arrow[d, "\xi_n^1"'] & \mathrm{Coker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] \arrow[d, "\xi_n^1"] \\
H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) \arrow[r] & H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)})
\end{tikzcd}$$

are commutative. Here the second and forth horizontal arrows denote the maps induced by endomorphism of $\hat{W}_n$, *defined by*

$$(T_0, T_1, \ldots, T_{n-1}) \mapsto (P_0^{(M)}(\boldsymbol{a}, \boldsymbol{T}), P_1^{(M)}(\boldsymbol{a}, \boldsymbol{T}), \ldots, P_{n-1}^{(M)}(\boldsymbol{a}, \boldsymbol{T})),$$

where $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1}) \in W_n(A)$ *and* $[\boldsymbol{a}] = (a_0, a_1, \ldots, a_{n-1}, 0, 0, \ldots) \in W(A)$.

PROOF. The assertions can be deduced from following proposition as in the proof of the main theorem.

PROPOSITION 4.2. *Let A be a $\boldsymbol{Z}_{(p)}$-algebra. Then:*

(1) *The diagrams*

$$\begin{tikzcd}
\mathrm{Ker}[F^n : W(A) \to W(A)] \arrow[r] \arrow[d, "\xi_n^0"'] & \mathrm{Ker}[F^{n+1} : W(A) \to W(A)] \arrow[d, "\xi_{n+1}^0"] \\
\mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \arrow[r, "R^*"'] & \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n+1,A}, \hat{\boldsymbol{G}}_{m,A})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Coker}[F^n : W(A) \to W(A)] \arrow[r, "F"] \arrow[d, "\xi_n^1"'] & \mathrm{Coker}[F^{n+1} : W(A) \to W(A)] \arrow[d, "\xi_{n+1}^1"] \\
H_0^2(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \arrow[r, "R^*"'] & H_0^2(\hat{W}_{n+1,A}, \hat{\boldsymbol{G}}_{m,A})
\end{tikzcd}$$

are commutative. Here the first horizontal arrow denotes the canonical injection.

(2) *The diagrams*

$$\begin{tikzcd}
\mathrm{Ker}[F^n : W(A) \to W(A)] \arrow[r, "V"] \arrow[d, "\xi_n^0"'] & \mathrm{Ker}[F^{n+1} : W(A) \to W(A)] \arrow[d, "\xi_{n+1}^0"] \\
\mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) \arrow[r, "F^*"'] & \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n+1,A}, \hat{\boldsymbol{G}}_{m,A})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Coker}[F^n : W(A) \to W(A)] \arrow[r, "p"] \arrow[d, "\xi_n^1"'] & \mathrm{Coker}[F^{n+1} : W(A) \to W(A)] \arrow[d, "\xi_{n+1}^1"] \\
H_0^2(\hat{W}_{n,A}, \hat{G}_{m,A}) \arrow[r, "F^*"'] & H_0^2(\hat{W}_{n+1,A}, \hat{G}_{m,A})
\end{tikzcd}$$

are commutative.

(3) *The diagrams*

$$\begin{tikzcd}
\mathrm{Ker}[F^{n+1} : W(A) \to W(A)] \arrow[r, "F"] \arrow[d, "\xi_{n+1}^0"'] & \mathrm{Ker}[F^n : W(A) \to W(A)] \arrow[d, "\xi_n^0"] \\
\mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n+1,A}, \hat{G}_{m,A}) \arrow[r, "V^*"'] & \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{G}_{m,A})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Coker}[F^{n+1} : W(A) \to W(A)] \arrow[r] \arrow[d, "\xi_{n+1}^1"'] & \mathrm{Coker}[F^n : W(A) \to W(A)] \arrow[d, "\xi_n^1"] \\
H_0^2(\hat{W}_{n+1,A}, \hat{G}_{m,A}) \arrow[r, "V^*"'] & H_0^2(\hat{W}_{n,A}, \hat{G}_{m,A})
\end{tikzcd}$$

are commutative. Here the third horizontal arrow denotes the canonical surjection.

(4) *The diagrams*

$$\begin{tikzcd}
\mathrm{Ker}[F^n : W(A) \to W(A)] \arrow[r, "{[\boldsymbol{a}]\cdot}"] \arrow[d, "\xi_n^0"'] & \mathrm{Ker}[F^n : W(A) \to W(A)] \arrow[d, "\xi_n^0"] \\
\mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{G}_{m,A}) \arrow[r] & \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{G}_{m,A})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Coker}[F^n : W(A) \to W(A)] \arrow[r, "{F^n[\boldsymbol{a}]\cdot}"] \arrow[d, "\xi_n^1"'] & \mathrm{Coker}[F^n : W(A) \to W(A)] \arrow[d, "\xi_n^1"] \\
H_0^2(\hat{W}_{n,A}, \hat{G}_{m,A}) \arrow[r] & H_0^2(\hat{W}_{n,A}, \hat{G}_{m,A})
\end{tikzcd}$$

are commutative. Here the second and forth horizontal arrows denote the maps induced by the endomorphism of $\hat{W}_n$, defined by

$$(T_0, T_1, \dots, T_{n-1}) \mapsto (P_0(\boldsymbol{a}, \boldsymbol{T}), P_1(\boldsymbol{a}, \boldsymbol{T}), \dots, P_{n-1}(\boldsymbol{a}, \boldsymbol{T})),$$

where $\boldsymbol{a} = (a_0, a_1, \dots, a_{n-1}) \in W_n(A)$ and $[\boldsymbol{a}] = (a_1, a_1, \dots, a_{n-1}, 0, 0, \dots) \in W(A)$.

(1),(3) and the second diagram of (4) are proved in [6, Lemma 2.9 and Remark 3.7]. Here we verify (2) and the first diagram of (4).

(2): Put $\tilde{F}_n(\boldsymbol{T}) = (F_{n,n}(\boldsymbol{T}), F_{n+1,n}(\boldsymbol{T}), F_{n+2,n}(\boldsymbol{T}), \ldots)$.

The commutativity of the first diagram is a consequence of the the following equality

$$E_{p,n+1}(0, U_0, U_1, \ldots; \boldsymbol{T}) = E_{p,n}(\boldsymbol{U}; F(\boldsymbol{T}))E_p(F^n(\boldsymbol{U}); \tilde{F}_n(\boldsymbol{T})),$$

where $F_{n+i,n}(\boldsymbol{T}) = F_{n+i}(T_0, T_1, \ldots, T_n, 0, 0 \ldots)$. Indeed,

$$\begin{aligned}
&E_{p,n+1}(0, U_0, U_1, \ldots; \boldsymbol{T})E_{p,n}(\boldsymbol{U}; F(\boldsymbol{T}))^{-1} \\
&\quad = \exp\left[\sum_{r\geq 0} \frac{1}{p^r}\Phi_r(\boldsymbol{U})\{\Phi_{r+1,n+1}(\boldsymbol{T}) - \Phi_{r,n}(F(\boldsymbol{T}))\}\right] \\
&\quad = \exp\left[\sum_{r\geq n} \frac{1}{p^r}\Phi_r(\boldsymbol{U})\{p^n F_{n,n}(\boldsymbol{T})^{p^{r-n}} + p^{n+1}F_{n+1,n}(\boldsymbol{T})^{p^{r-n-1}} + \cdots + p^r F_{r,n}(\boldsymbol{T})\}\right] \\
&\quad = \exp\left[\sum_{r\geq 0} \frac{1}{p^r}\Phi_{r+n}(\boldsymbol{U})\{F_{n,n}(\boldsymbol{T})^{p^r} + pF_{n+1,n}(\boldsymbol{T})^{p^{r-1}} + \cdots + p^r F_{r+n,n}(\boldsymbol{T})\}\right] \\
&\quad = E_p(F^n(\boldsymbol{U}); \tilde{F}_n(\boldsymbol{T})).
\end{aligned}$$

The commutativity of the second diagram is a consequence of the following equality

$$\begin{aligned}
&F_{p,n+1}(FV\boldsymbol{U}; \boldsymbol{X}, \boldsymbol{Y})F_{p,n}(\boldsymbol{U}; F(\boldsymbol{X}), F(\boldsymbol{Y}))^{-1} \\
&\quad = E_p(\boldsymbol{U}; \tilde{F}_n(\boldsymbol{X}))E_p(\boldsymbol{U}; \tilde{F}_n(\boldsymbol{Y}))E_p(\boldsymbol{U}; \tilde{F}_n(S(\boldsymbol{X}, \boldsymbol{Y})))^{-1}.
\end{aligned}$$

Indeed, put $F(\boldsymbol{T})_n = (F_0(\boldsymbol{T}), F_1(\boldsymbol{T}), \ldots, F_{n-1}(\boldsymbol{T}), 0, 0, \ldots)$. Then,

$$\begin{aligned}
&p^{n+1}\Phi_r(\tilde{S}_{n+1}(\boldsymbol{X}, \boldsymbol{Y})) - p^n\Phi_r(\tilde{S}_n(F(\boldsymbol{X}), F(\boldsymbol{Y}))) \\
&\quad = \Phi_{r+n+1}(S(\boldsymbol{X}_{n+1}, \boldsymbol{Y}_{n+1})) - \Phi_{r+n+1,n+1}(S(\boldsymbol{X}, \boldsymbol{Y})) \\
&\qquad - \Phi_{r+n}(S(F(\boldsymbol{X})_n, F(\boldsymbol{Y})_n)) + \Phi_{r+n,n}(S(F(\boldsymbol{X}), F(\boldsymbol{Y}))) \\
&\quad = p^n\Phi_r(\tilde{F}_n(\boldsymbol{X})) + p^n\Phi_r(\tilde{F}_n(\boldsymbol{Y})) - p^n\Phi_r(\tilde{F}_n(S(\boldsymbol{X}, \boldsymbol{Y}))).
\end{aligned}$$

Therefore

$$F_{p,n+1}(FV\boldsymbol{U};\boldsymbol{X},\boldsymbol{Y})F_{p,n}(\boldsymbol{U};F(\boldsymbol{X}),F(\boldsymbol{Y}))^{-1}$$

$$= \exp\left[\sum_{r\geq 0}\frac{1}{p^r}\Phi_r(\boldsymbol{U})\{p\Phi_r(\tilde{S}_{n+1}(\boldsymbol{X},\boldsymbol{Y})) - \Phi_r(\tilde{S}_n(F(\boldsymbol{X}),F(\boldsymbol{Y})))\}\right]$$

$$= \exp\left[\sum_{r\geq 0}\frac{1}{p^{r+n}}\Phi_r(\boldsymbol{U})\{p^n\Phi_r(\tilde{F}_n(\boldsymbol{X})) + p^n\Phi_r(\tilde{F}_n(\boldsymbol{Y})) - p^n\Phi_r(\tilde{F}_n(S(\boldsymbol{X},\boldsymbol{Y})))\}\right]$$

$$= E_p(\boldsymbol{U};\tilde{F}_n(\boldsymbol{X}))E_p(\boldsymbol{U};\tilde{F}_n(\boldsymbol{Y}))E_p(\boldsymbol{U};\tilde{F}_n(S(\boldsymbol{X},\boldsymbol{Y})))^{-1}.$$

(4): The commutativity of the first diagram is a consequence of the following equality

$$E_{p,n}(P(\boldsymbol{X}_n,\boldsymbol{U});\boldsymbol{T})E_{p,n}(\boldsymbol{U};P(\boldsymbol{X};\boldsymbol{Y}))^{-1} = E_p(F^n(\boldsymbol{U}),P_{n,n}(\boldsymbol{X},\boldsymbol{Y}),P_{n+1,n}(\boldsymbol{X},\boldsymbol{Y}),\ldots),$$

where

$$P_{n+i,n}(\boldsymbol{X},\boldsymbol{Y}) = P_{n+i}(X_0,X_1,\ldots,X_{n-1},0,0,\ldots,Y_0,Y_1,\ldots,Y_{n-1},0,0,\ldots)$$

and

$$\boldsymbol{X}_n = (X_0,X_1,\ldots,X_{n-1},0,0,\ldots).$$

Indeed,

$$E_{p,n}(P(\boldsymbol{X}_n,\boldsymbol{U});\boldsymbol{T})E_{p,n}(\boldsymbol{U};P(\boldsymbol{X};\boldsymbol{Y}))^{-1}$$

$$= \exp\left[\sum_{r\geq 0}\frac{1}{p^r}\Phi_r(\boldsymbol{U})\{\Phi_{r,n}(\boldsymbol{X})\Phi_{r,n}(\boldsymbol{Y}) - \Phi_{r,n}(\boldsymbol{X},\boldsymbol{T})\}\right]$$

$$= \exp\Bigg[\sum_{r\geq n}\frac{1}{p^r}\Phi_r(U)\{p^nP_{n,n}(\boldsymbol{X},\boldsymbol{Y})^{p^{r-n}} + p^{n+1}P_{n+1,n}(\boldsymbol{X},\boldsymbol{Y})^{p^{r-n-1}}$$
$$+\cdots+p^rP_{r,n}(\boldsymbol{X},\boldsymbol{Y})\}\Bigg]$$

$$= \exp\Bigg[\sum_{r\geq 0}\frac{1}{p^r}\Phi_{n+r}(\boldsymbol{U})\{P_{n,n}(\boldsymbol{X},\boldsymbol{Y})^{p^r} + pP_{n+1,n}(\boldsymbol{X},\boldsymbol{Y})^{p^{r-1}}$$
$$+\cdots+p^rP_{n+r,n}(\boldsymbol{X},\boldsymbol{Y})\}\Bigg]$$

$$= E_p(F^n(\boldsymbol{U});P_{n,n}(\boldsymbol{X},\boldsymbol{Y}),P_{n+1,n}(\boldsymbol{X},\boldsymbol{Y}),\ldots).$$

PROPOSITION 4.3. *The diagram*

$$\begin{array}{ccc} \mathrm{Ker}[F^n : W^{(M)}(A) \to W^{(M)}(A)] & \longrightarrow & \mathrm{Coker}[F^m : W^{(M)}(A) \to W^{(M)}(A)] \\ {\scriptstyle \xi_n^0}\Big\downarrow & & \Big\downarrow{\scriptstyle \xi_m^1} \\ \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(M)}) & \xrightarrow[\partial]{} & H_0^2(\hat{W}_{m,A}, \hat{\mathscr{G}}_A^{(M)}) \end{array}$$

is commutative. Here the first horizontal arrow denotes the map induced by $\boldsymbol{a} \mapsto \boldsymbol{a}$, and ∂ denotes the boundary map defined by the exact sequence of formal group schemes

$$0 \longrightarrow \hat{W}_{n,A} \xrightarrow{V^m} \hat{W}_{n+m,A} \xrightarrow{R^n} \hat{W}_{m,A} \longrightarrow 0.$$

PROOF. The assertion can be deduced from following remark as in the proof of Theorem 3.5.

REMARK 4.4 (cf. [6, Lemma 2.10]). The diagram

$$\begin{array}{ccc} \mathrm{Ker}[F^n : W(A) \to W(A)] & \longrightarrow & \mathrm{Coker}[F^m : W(A) \to W(A)] \\ {\scriptstyle \xi_n^0}\Big\downarrow & & \Big\downarrow{\scriptstyle \xi_m^1} \\ \mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) & \xrightarrow[\partial]{} & H_0^2(\hat{W}_{m,A}, \hat{\boldsymbol{G}}_{m,A}) \end{array}$$

is commutative. Here the first horizontal arrow denotes the map induced by $\boldsymbol{a} \mapsto \boldsymbol{a}$, and ∂ denotes the boundary map defined by the exact sequence of formal group schemes

$$0 \longrightarrow \hat{W}_{n,A} \xrightarrow{V^m} \hat{W}_{n+m,A} \xrightarrow{R^n} \hat{W}_{m,A} \longrightarrow 0.$$

We can obtain the functorialities of the case of group schemes similarly as above.

5. Some Results over a Discrete Valuation Ring

In this section, we treat a case of extensons over a discrete valuation ring as done in Sekiguchi-Suwa [4] and [8].

Throughout the section, A denotes a discrete valuation ring and $\mathfrak{m}$ (resp. K) the maximal ideal (resp. the field of fraction) of A. We denote by π a uniformizing parameter of A and by v the valuation of A normalizing by $v(\pi) = 1$. Furethermore, we fix $\mu \in \mathfrak{m} - \{0\}$ and put $A_0 = A/(\mu)$, $\mathfrak{m}_0 = \mathfrak{m}/(\mu)$.

5.1. Now we assume that G is an affine group scheme over A and F is an fppf-sheaf. Let $\check{\mathscr{H}}^i(F)$ denote the presheaf on $\mathrm{Sch}_{/A}$ defined by $X \mapsto H^i(X, F)$. Then we have an exact sequence

$$0 \to H_0^2(G,F) \to \mathrm{Ext}_A^1(G,F) \to H_0^1(G, \check{\mathscr{H}}^1(F)) \to H_0^3(G,F) \to \mathrm{Ext}_A^3(G,F)$$

(cf. [Ch. III.6, 2.5]).

LEMMA 5.2. *$H_0^i(W_{n,A}, \mathscr{G}_A^{(\mu)}) = 0$ for $i \geq 1$.*

PROOF. Since A is reduced, it is readily seen that $C^i(W_{n,A}, \mathscr{G}_A^{(\mu)}) \simeq (1+\mu A)^\times$ for all $i \geq 1$. And the boundary map is written as follows: $\partial^i(a) = 1$ if i is even, and $\partial^i(a) = a$ if i is odd. It follows immediately that $H_0^i(W_{n,A}, \mathscr{G}_A^{(\mu)}) = 0$ for $i \geq 1$.

COROLLARY 5.3. *$\mathrm{Ext}_A^1(W_{n,A}, \mathscr{G}_A^{(\mu)})$ is isomorphic to the subgroup of $H^1(W_{n,A}, \mathscr{G}_A^{(\mu)})$ formed by the primitive elements.*

PROOF. Recall that $a \in H^1(W_{n,A}, \mathscr{G}_A^{(\mu)})$ is primitive if $\mu^*(a) = \mathrm{pr}_1^*(a) + \mathrm{pr}_2^*(a)$ in $H^1(W_{n,A} \times W_{n,A}, \mathscr{G}_A^{(\mu)})$, where μ is the multiplication and $\mathrm{pr}_i : W_{n,A} \times W_{n,A} \to W_{n,A}$ is the i-th projection.

Applying the exact sequence of 5.1. to $G = W_{n,A}$ and $F = \mathscr{G}_A^{(\mu)}$, we have the exact sequence

$$0 \to H_0^2(W_{n,A}, \mathscr{G}_A^{(\mu)}) \to \mathrm{Ext}_A^1(W_{n,A}, \mathscr{G}_A^{(\mu)})$$
$$\to H_0^1(W_{n,A}, \check{\mathscr{H}}^1(\mathscr{G}_A^{(\mu)})) \to H_0^3(W_{n,A}, \mathscr{G}_A^{(\mu)}).$$

But we have seen that

$$H_0^2(W_{n,A}, \mathscr{G}_A^{(\mu)}) = H_0^3(W_{n,A}, \mathscr{G}_A^{(\mu)}) = 0$$

in 5.2. Hence we obtain an isomorphism

$$\mathrm{Ext}_A^1(W_{n,A}, \mathscr{G}_A^{(\mu)}) \xrightarrow{\sim} H_0^1(W_{n,A}, \check{\mathscr{H}}^1(\mathscr{G}_A^{(\mu)})).$$

By the definition, $H_0^1(W_{n,A}, \check{\mathscr{H}}^1(\mathscr{G}_A^{(\mu)}))$ is nothing but the subset of primitive elements in $H^1(W_{n,A}, \mathscr{G}_A^{(\mu)})$.

LEMMA 5.4. *The group $H^1(W_{n,A}, \mathscr{G}_A^{(\mu)})$ is isomorphic to*

$$\left\{1 + \sum c_{i_0,\dots,i_{n-1}} T_0^{i_0} \cdots T_{n-1}^{i_{n-1}} \in A_0[T_0, \dots, T_{n-1}];\, c_{i_0,\dots,i_{n-1}} \in \mathfrak{m}_0\right\}.$$

PROOF. Since $W_{n,A}$ is flat over A, the sequence

$$0 \longrightarrow \mathscr{G}_A^{(\mu)} \xrightarrow{\alpha^{(\mu)}} \boldsymbol{G}_{m,A} \longrightarrow i_*(\boldsymbol{G}_{m,A_0}) \longrightarrow 0,$$

where $i : \operatorname{Spec} A_0 \to \operatorname{Spec} A$ is the canonical immersion, is exact on the (small) étale site of $W_{n,A}$ (cf. [5]). Thus we obtain an exact sequence

$$\Gamma(W_{n,A}, \boldsymbol{G}_{m,A}) \to \Gamma(W_{n,A_0}, \boldsymbol{G}_{m,A_0}) \to H^1(W_{n,A}, \mathscr{G}_A^{(\mu)}) \to H^1(W_{n,A}, \boldsymbol{G}_{m,A}).$$

Note that we may calculate the cohomology group $H^1(W_{n,A}, \mathscr{G}_A^{(\mu)})$ for the étale topology since $\mathscr{G}_A^{(\mu)}$ is smooth over A (cf. Grothendieck [2]). Since the affine ring of $W_{n,A}$ is a unique factorzation domain,

$$H^1(W_{n,A}, \boldsymbol{G}_{m,A}) = \operatorname{Pic}(W_{n,A}) = 0.$$

Hence the assertion follows from the following calculations:

$$\Gamma(W_{n,A}, \boldsymbol{G}_{m,A}) = A^\times,$$

$$\Gamma(W_{n,A_0}, \boldsymbol{G}_{m,A_0}) = \left\{ a\left(1 + \sum c_{i_0,\dots,i_{n-1}} T_0^{i_0} \cdots T_{n-1}^{i_{n-1}}\right); a \in A_0^\times, c_{i_0,\dots,i_{n-1}} \in \mathfrak{m}_0 \right\},$$

where the canonical map $A^\times \to A_0^\times$ is surjective.

COROLLARY 5.5. *$\operatorname{Ext}_A^1(W_{n,A}, \mathscr{G}_A^{(\mu)})$ is isomorphic to*

$$\left\{ \begin{array}{l} F(\boldsymbol{T}) = 1 + \sum c_{i_0,\dots,i_{n-1}} T_0^{i_0} \cdots T_{n-1}^{i_{n-1}} \\ \qquad \in A_0[T_0, \dots, T_{n-1}] \end{array} ; \begin{array}{l} c_{i_0,\dots,i_{n-1}} \in \mathfrak{m}_0, \\ F(\boldsymbol{X})F(\boldsymbol{Y}) = F(S(\boldsymbol{X}, \boldsymbol{Y})) \end{array} \right\}.$$

Next we give an explicit description of the extensions of $W_{n,A}$ by $\mathscr{G}_A^{(\mu)}$, corresponding to a primitive element.

5.6. Let $F(T_0, \dots, T_{n-1})$ be a polynomial in $A[T_0, \dots, T_{n-1}]$, satisfying the functional equation

1) $F(0, 0, \dots, 0) \equiv 1 \mod \mu$;
2) $F(X_0, \dots, X_{n-1})F(Y_0, \dots, Y_{n-1}) \equiv F(S_0(\boldsymbol{X}, \boldsymbol{Y}), \dots, S_{n-1}(\boldsymbol{X}, \boldsymbol{Y})) \mod \mu$.

Put $\boldsymbol{T} = (T_0, T_1, \dots, T_{n-1})$ and we define a smooth affine commutative group scheme $\mathscr{E}_n^{(\mu;F)}$ over A as follow:

$$\mathscr{E}_n^{(\mu;F)} = \operatorname{Spec} A\left[T_0, T_1, \dots, T_{n-1}, T_n, \frac{1}{\mu T_n + F(T_0, \dots, T_{n-1})}\right]$$

1) law of multiplication

$$T_i \mapsto S_i(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}) \quad (0 \leq i \leq n-1),$$

$$T_n \mapsto \mu T_n \otimes T_n + T_n \otimes F(\boldsymbol{T}) + F(\boldsymbol{T}) \otimes T_n$$
$$+ \frac{1}{\mu}[F(\boldsymbol{T}) \otimes F(\boldsymbol{T}) - F(S(\boldsymbol{T} \otimes 1, 1 \otimes \boldsymbol{T}))];$$

2) unit

$$T_i \mapsto 0 \ (0 \leq i \leq n-1), \quad T_n \mapsto \frac{1}{\mu}[1 - F(0, \dots, 0)];$$

3) inverse

$$T_i \mapsto I_i(\boldsymbol{T}) \quad (0 \leq i \leq n-1),$$

$$T_n \mapsto \frac{1}{\mu}\left[\frac{1}{\mu T_n + F(T_0, \dots, T_{n-1})} - F(I_0(\boldsymbol{T}), I_1(\boldsymbol{T}), \dots, I_{n-1}(\boldsymbol{T}))\right],$$

where $I_0(\boldsymbol{T}), I_1(\boldsymbol{T}), \dots, I_{n-1}(\boldsymbol{T})$ are polynomials defining the inverse on W_n. It is well known that if $p > 2$, $(I_0(\boldsymbol{T}), I_1(\boldsymbol{T}), \dots, I_{n-1}(\boldsymbol{T})) = (-T_0, -T_1, \dots, -T_{n-1})$.

Moreover, we define a homomorphism of group schemes

$$\mathscr{G}_A^{(\mu)} = \operatorname{Spec} A\left[T, \frac{1}{1+\mu T}\right] \to \mathscr{E}_n^{(\mu;F)}$$
$$= \operatorname{Spec} A\left[T_0, \dots, T_{n-1}, T_n, \frac{1}{F(T_0, \dots, T_{n-1}) + \mu T_n}\right]$$

by

$$(T_0, \dots, T_{n-1}, T_n) \mapsto \left(0, \dots, 0, T + \frac{1}{\mu}[1 - F(0, \dots, 0)]\right)$$

and a homomorphism

$$\mathscr{E}_n^{(\mu;F)} = \operatorname{Spec} A\left[T_0, \dots, T_{n-1}, T_n, \frac{1}{F(T_0, \dots, T_{n-1}) + \mu T_n}\right] \to W_{n,A}$$
$$= \operatorname{Spec} A[T_0, \dots, T_{n-1}]$$

by

$$(T_0, \dots, T_{n-1}) \mapsto (T_0, \dots, T_{n-1}).$$

Then the sequence of group schemes

$$0 \to \mathscr{G}_A^{(\mu)} \to \mathscr{E}_n^{(\mu;F)} \to W_{n,A} \to 0$$

is exact, and its class correspondents to $[F(T_0,\dots,T_{n-1}) \bmod \mu] \in H^1(W_{n,A}, \mathscr{G}_A^{(\mu)})$.

5.7. From 5.5 and 5.6, $F \mapsto [\mathscr{E}_n^{(\mu;F)}]$ defines an isomorphism

$$\partial : \mathrm{Hom}_{A_0\text{-gr}}(W_{n,A_0}, \boldsymbol{G}_{m,A_0}) \xrightarrow{\sim} \mathrm{Ext}_A^1(W_{n,A}, \mathscr{G}_A^{(\mu)}).$$

Now note that $F(T_0,\dots,T_{n-1})$ is invertible in $A[[T_0,\dots,T_{n-1}]]$. Then

$$(T_0,\dots,T_{n-1},T_n) \mapsto \left(T_0,\dots,T_{n-1},\frac{T_0}{F(T_0,\dots,T_{n-1})}\right)$$

defines an isomorphism of formal groups

$$\hat{\mathscr{E}}_n^{(\mu;F)} = \mathrm{Spf}\, A[[T_0,\dots,T_{n-1},T_n]] \xrightarrow{\sim} \hat{\mathscr{E}} = \mathrm{Spf}\, A[[T_0,\dots,T_{n-1},T_n]],$$

where $\hat{\mathscr{E}}$ is the extension of $\hat{W}_{n,A}$ by $\hat{\mathscr{G}}_A^{(\mu)}$ defined by the 2-cocycle

$$(\partial F)(\boldsymbol{X},\boldsymbol{Y}) = \frac{1}{\mu}\left[\frac{F(\boldsymbol{X})F(\boldsymbol{Y})}{F(S(\boldsymbol{X},\boldsymbol{Y}))} - 1\right] \in Z_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(\mu)}).$$

Furthermore, defining a homomorphism

$$\mathrm{Ext}_A^1(W_{n,A}, \mathscr{G}_A^{(\mu)}) \to H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(\mu)})$$

by

$$[\mathscr{E}_n^{(\mu;F)}] \mapsto (\partial F)(\boldsymbol{X},\boldsymbol{Y}),$$

we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_{A_0\text{-gr}}(W_{n,A_0}, \boldsymbol{G}_{m,A_0}) & \xrightarrow{\partial} & \mathrm{Ext}_A^1(W_{n,A}, \mathscr{G}_A^{(\mu)}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_{A\text{-gr}}(\hat{W}_{n,A}, \hat{\boldsymbol{G}}_{m,A}) & \longrightarrow & \mathrm{Hom}_{A_0\text{-gr}}(\hat{W}_{n,A_0}, \hat{\boldsymbol{G}}_{m,A_0}) & \xrightarrow{d} & H_0^2(\hat{W}_{n,A}, \hat{\mathscr{G}}_A^{(\mu)}), & &
\end{array}$$

where d is the homomorphism in Example 3.14.

Assume now that A is of mixed characteristics 0 and p. Let $\boldsymbol{a} \in \mathrm{Ker}[F^n : W(K) \to W(K)]$. Then a_r $(r \geq n)$ is determined inductively by

$$\Phi_r(\boldsymbol{a}) = \Phi_{r-n}(F^n(\boldsymbol{a})) = 0.$$

EXAMPLE 5.7.1. If $v(\mu) > v(p)/(p-1)+1$, then the canonical map

$$\mathscr{E} \mapsto \hat{\mathscr{E}} : \mathrm{Ext}^1_A(W_{n,A}, \mathscr{G}^{(\mu)}_A) \to \mathrm{Ext}^1_A(\hat{W}_{n,A}, \hat{\mathscr{G}}^{(\mu)}_A)$$

is not injective.

Indeed, take $(a_0, a_1, \ldots, a_{n-1})$ such that

(1) $v(a_i) > v(p)/(p-1)$ for any i;

(2) $v(a_i) < v(\mu)$ for some i.

Define $\boldsymbol{a} \in W(K)$ inductively by $\Phi_r(\boldsymbol{a}) = 0$ for $r \geq n$. Then $\boldsymbol{a}$ is an element of $\mathrm{Ker}[F^n : W(A) \to W(A)]$ and $\lim_{i\to\infty} v(a_i) = \infty$. Therefore $\boldsymbol{a}$ is an element of $\mathrm{Ker}[F^n : \hat{W}(A_0) \to \hat{W}(A_0)]$. On the other hand, $\boldsymbol{a} \not\equiv (0,0,\ldots) \bmod \mu$ since $v(a_i) < v(\mu)$ for some i.

These imply that

(1) the class $[\mathscr{E}_n^{(\mu;F)}]$ is not trivial in $\mathrm{Ext}^1_A(W_{n,A}, \mathscr{G}^{(\mu)}_A)$,

(2) the image of $[\hat{\mathscr{E}}_n^{(\mu;F)}]$ is trivial in $\mathrm{Ext}^1_A(\hat{W}_{n,A}, \hat{\mathscr{G}}^{(\mu)}_A)$.

Here $F(T) = E_{n,p}(\boldsymbol{a}; \boldsymbol{T}) \bmod \mu \in A_0[T_0, \ldots, T_{n-1}]$.

EXAMPLE 5.7.2. If $p \nmid v(p)$ and $v(\mu) \leq (2p-1)v(p)/(p^3-p^2)$, then the reduction map

$$\mathrm{Hom}_{A-\mathrm{gr}}(\hat{W}_{2,A}, \hat{\boldsymbol{G}}_{m,A}) \to \mathrm{Hom}_{A_0-\mathrm{gr}}(\hat{W}_{2,A_0}, \hat{\boldsymbol{G}}_{m,A_0})$$

is zero, and therefore, the canonical map

$$\mathscr{E} \mapsto \hat{\mathscr{E}} : \mathrm{Ext}^1_A(W_{2,A}, \mathscr{G}^{(\mu)}_A) \to \mathrm{Ext}^1_A(\hat{W}_{2,A}, \hat{\mathscr{G}}^{(\mu)}_A)$$

is injective.

Indeed, take $a_0, a_1 \in A$ and define $\boldsymbol{a} = (a_0, a_1, a_2, \ldots) \in \mathrm{Ker}[F^2 : W(K) \to W(K)]$ inductively by $\Phi(\boldsymbol{a}) = 0$. Especially $a_2 = a_0^{p^2}/p^2 + a_1^p/p$. Then $v(a_0^{p^2}/p^2) \neq v(a_1^p/p)$ since $p \nmid v(p)$, which implies that $v(a_2) = \min\{v(a_0^{p^2}/p^2), v(a_1^p/p)\}$. Furthermore it is verified that, if $v(a_0) \geq (2p-1)v(p)/(p^3-p^2)$ and $v(a_1) \geq v(p)/(p-1)$, then $v(a_i) \geq v(p)/(p-1)$ for any i, which implies that $\boldsymbol{a} \equiv (0,0,0,\ldots) \bmod \mu$. On the other hand, it is verified that, if $v(a_0) < (2p-1)v(p)/(p^3-p^2)$ or $v(a_1) < v(p)/(p-1)$, then $\lim_{r\to\infty} v(a_r) = -\infty$, which implies that $\boldsymbol{a} \notin \mathrm{Ker}[F^2 : W(A) \to W(A)]$. Hence the result.

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Department of Mathematics
Chuo University
1-13-27 Kasuga, Bunkyo-ku
Tokyo 112-8551, JAPAN
E-mail address: niitsuma@gug.math.chuo-u.ac.jp