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SCATTERING FOR NONLINEAR SYMMETRIC HYPERBOLIC SYSTEMS

By

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0 Introduction

In this paper we shall investigate the Cauchy problem and scattering for the following nonlinear symmetric hyperbolic system of first order

$$E(u)\frac{\partial u}{\partial t} = \sum_{j=1}^{n} A_j(u)\frac{\partial u}{\partial x_j} + F(t,x,u), \tag{0.1}$$

where $x \in \boldsymbol{R}^n$, $t \in \boldsymbol{R}^1$, $u = u(t,x)$ is a real $m \times 1$ matrix. $E(u)$ is an $m \times m$ matrix which is real, symmetric and positive definite, $A_j(u)$ $(j = 1, \dots, n)$ are $m \times m$ matrices which are real and symmetric. Moreover we assume that $E(u), A_j(u)$, $F(u) \in C^\infty(\boldsymbol{R}^m)$.

First, in order to obtain the existence of the time global solution of the Cauchy problem for the equation (0.1), we consider the following Cauchy problem for a linear symmetric hyperbolic system of first order with constant coefficients;

$$\begin{cases} E^0\dfrac{\partial u^0}{\partial t} = \displaystyle\sum_{j=1}^{n} A_j^0\frac{\partial u^0}{\partial x_j}, \\ u^0(0,x) = \varphi_0(x), \end{cases} \tag{0.2}$$

where $x \in \boldsymbol{R}^n$, $t \in \boldsymbol{R}^1$, $u^0 = u^0(t,x)$ and $\varphi_0(x)$ are real $m \times 1$ matrices and $\varphi_0(x) \in C_0^\infty(\boldsymbol{R}^n)$. E^0 is a $m \times m$ matrix which is real, symmetric and positive definite. A_j^0 $(j = 1, \dots, n)$ are $m \times m$ matrices which are real, symmetric and constant. We assume that the eigenvalues $\lambda_j(\xi)$ of $\sum_{j=1}^n A_j^0\xi_j$ are non zero, real, distinct and their slowness surfaces are strictly convex. S. Lucente and G. Ziliotti [1] obtain the decay estimate of the solutions of the Cauchy problem (0.2) as $t \to \pm\infty$. By using their estimate and the existence of the local solution (cf: [3]),

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we can prove the existence of the time global solution to the following Cauchy problem

$$\begin{cases} E(u)\dfrac{\partial u}{\partial t} = \displaystyle\sum_{j=1}^{n} A_j(u)\dfrac{\partial u}{\partial x_j} + F(t,x,u), \\ u(0,x) = \varphi(x), \end{cases} \tag{0.3}$$

where $\varphi(x)$ is a real $m \times 1$ matrix, $\varphi(x) \in W^{l,1} \cap H^l$, $l \geq l_0 + 2$ and $l_0 = \left[\frac{n}{2}\right] + 1$, and we get the decay estimate of the solution.

Throughout this paper, we assume the following assumption.

ASSUMPTION 0.1

- $E(u) : m \times m$ matrix which is real, symmetric and positive definite.
- $A_j(u)$ $(j = 1, \dots, n) : m \times m$ matrices which are real and symmetric.
- $E(u), A_j(u) \in C^\infty(\boldsymbol{R}^m)$ and $F(t,x,u) \in C^\infty(\boldsymbol{R}^{1+n} \times \boldsymbol{R}^m)$
- E^0, A_j^0 $(j = 1, \dots, n) : m \times m$ matrices which are real, symmetric and constant. Moreover E^0 is positive definite, the roots $\lambda_j(\xi)$ of the equation $H(\lambda) = \det(E^0\lambda - \sum_{j=1}^n A_j^0 \xi_j) = 0$ are non zero, real and distinct and the slowness surface $\{\xi; \lambda_j(\xi) = \pm 1\}$ are strictly convex.
- $E(u) - E^0 = O(|u|^p)$, $|u| \to 0$.
- $A_j(u) - A_j^0 = O(|u|^p)$, $|u| \to 0$.
- $\partial_t^j \partial_x^\alpha (F(t,x,u) - F(t,x,0)) = O(|u|^{p+1})$, umiformly in $(t,x) \in \boldsymbol{R}^{n+1}$ $|u| \to 0$, for each (j, α).

We get the following theorem under the above assumption.

THEOREM 0.2. *Let us assume that* $\varphi \in W^{l,1} \cap H^l$, $\|\varphi\|_{W^{l,1}} + \|\varphi\|_{H^l} \leq \delta$, *a positive integer* $p > \frac{n+1}{n-1}$ *and* $F(t,x,0) \in C^0((-\infty,\infty); H^l \cap W^{l,1}(\boldsymbol{R}^n))$ *satisfying*

$$\|F(t,\cdot,0)\|_{W^{l,1}} + \|F(t,\cdot,0)\|_{H^l} \leq \delta \langle t \rangle^{-\sigma} (\sigma > 1),$$

where δ *is a small positive constant,* $l \geq l_0 + 2$ *and* $l_0 = \left[\frac{n}{2}\right] + 1$. *Then the Cauchy problem* (0.3) *has a time global solution*

$$u(t) \in C^0((-\infty,\infty); H^l \cap W^{1,\infty}(\boldsymbol{R}^n)) \cap C^1((-\infty,\infty); H^{l-1} \cap L^\infty(\boldsymbol{R}^n)). \tag{0.4}$$

Moreover the solution of the Cauchy problem (0.3) *has the following decay estimate;*

$$\|u(t)\|_{W^{1,\infty}(\boldsymbol{R}^n)} \leq \exists C\delta \langle t \rangle^{-(n-1)/2}, \quad t \in \boldsymbol{R}^1, \tag{0.5}$$

where $\langle t \rangle = \sqrt{1 + |t|^2}$.

Furthermore we shall prove the existence of the scattering operator among the nonlinear equation (0.1) and the linear equation (0.2). The result is the following theorem.

THEOREM 0.3. *Assume that Assumption* 0.1 *is valid. Let* $l \geq 2(l_0+1)$ *and* $F(t,x,0) \equiv 0$. *For* $\forall \varphi_- \in W^{l,1} \cap H^l$ *and* $\|\varphi_-\|_{H^l} + \|\varphi_-\|_{W^{l,1}} \leq \delta \ll 1$, *there exists*

$$u(t) \in C^0((-\infty,\infty);H^l) \cap C^1((-\infty,\infty);H^{l-1}), \tag{0.6}$$

satisfying (0.3) *and*

$$\|u(t) - e^{itH_0}\varphi_-\|_{H^{l-1}} \leq C\langle t\rangle^{-(n-1)/2}, \quad (t \leq 0). \tag{0.7}$$

Moreover, there exists

$$\varphi_+ \in H^l, \tag{0.8}$$

such that

$$\|u(t) - e^{itH_0}\varphi_+\|_{H^{l-1}} \leq C\langle t\rangle^{-(n-1)/2}, \quad (t \geq 0), \tag{0.9}$$

where $p > \frac{n+1}{n-1}$, *the constant* C *depends only on* δ *and* p.

This paper is organized as follows: In the section 1 we shall prove the existence of the time global solution and derive the decay estimate of the solution for the Cauchy problem (0.3). Intertwining the equation (0.1) and a linear symmetric hyperbolic system with constsnt coefficients (0.2), the existence of scattering operator for the equation (0.1) is proved in the section 2.

1 Global Solutions and Decay Estimates

Let's consider the Cauchy problem (0.2). Multiplying the equation (0.2) by $(E^0)^{-1}$ from left-hand side, then we have

$$\begin{cases} \dfrac{\partial u^0}{\partial t} = iH_0 u^0, \\ u^0(0,x) = \varphi_0(x), \end{cases} \tag{1.1}$$

where $x \in \boldsymbol{R}^n$, $t \in \boldsymbol{R}^1$, $H_0 = (E^0)^{-1}\sum_{j=1}^n A_j^0 D_{x_j}$, $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$, $i = \sqrt{-1}$. E^0 is an $m \times m$ matrix which is real, symmetric, and positive definite, A_j^0 $(j = 1,\dots,n)$ are $m \times m$ matrices which are real, symmetric and constant, and $u^0 = u^0(t,x)$ is a real $m \times 1$ matrix. Let us assume that $\varphi_0(x) \in C_0^\infty(\boldsymbol{R}^n)$. For $\varphi_0(x)$, we see that the solution of the Cauchy problem (1.1) can be written in the following way:

$$u^0(t,x) = (e^{itH_0}\varphi_0)(x). \tag{1.2}$$

For (1.1), we obtain the following proposition.

PROPOSITION 1.1. *The solution* $e^{itH_0}\varphi_0$ *of* (1.1) *satisfies*

$$\|e^{itH_0}\varphi_0\|_{W^{l,\infty}(\boldsymbol{R}^n)} \le C\langle t\rangle^{-(n-1)/2}\|\varphi_0\|_{W^{l+l_0,1}(\boldsymbol{R}^n)}, \tag{1.3}$$

where $\varphi_0(x) \in W^{l_0,1}(\boldsymbol{R}^n) \cap H^l(\boldsymbol{R}^n)$, $l_0 = [n/2]+1$ *and* l *is a non negative integer.*

This result will be used to prove the existence of time global solution for the nonlinear case. The proof of this proposition is just the same one as Theorem 0.1 in [1].

Using Proposition 1.1, we can solve the Cauchy problem (0.3). We assume that $E(u)$, $A_j(u)$, $F(u)$ satisfy Assumption 0.1. The existence theorem of local solution is as follows.

PROPOSITION 1.2. *Let* $\varphi \in H^s$, $s \in \boldsymbol{N}$, $s > \frac{n}{2}+1$, $g_1 := \kappa_s\|\varphi\|_{H^s}$ *and* $g_2 > g_1$ *arbitrary but fixed. Then there is a* $T > 0$ *such that there exists a unique classical solution* $u \in C^1([0,T]\times\boldsymbol{R}^n)$ *of the Cauchy problem* (0.3) *with*

$$\sup_{(t,x)\in[0,T]\times\boldsymbol{R}^n} |u(t,x)| \le g_2 \tag{1.4}$$

and

$$u \in C^0([0,T];H^s) \cap C^1([0,T];H^{s-1}), \tag{1.5}$$

where T *is a function of* $\|\varphi\|_{H^s}$ *and* g_2.

(PROOF OF PROPOSITION 1.2). See R. Racke [3].

In order to have the decay estimate of the solution of (0.1), we use the following Gagliardo-Nirenberg type inequalities.

LEMMA 1.3. (1) *For the functions* $u_1, u_2, \dots u_m$, *if* $|\alpha| = \alpha^1 + \cdots + \alpha^m$, *then*

$$\|\partial_x^{\alpha^1}u_1\partial_x^{\alpha^2}u_2\cdots\partial_x^{\alpha^m}u_m\|_{L^2(\boldsymbol{R}^n)} \le C\sum_{k=1}^m\left(\prod_{i\ne k}\|u_i\|_{L^\infty(\boldsymbol{R}^n)}\right)\|u_k\|_{H^{|\alpha|}} \tag{1.6}$$

(2) *For the functions* $u_1, u_2, \dots u_m$, *if* $|\alpha| = \alpha^1 + \cdots + \alpha^m$, *then*

$$\|\partial_x^{\alpha^1} u_1 \partial_x^{\alpha^2} u_2 \cdots \partial_x^{\alpha^m} u_m\|_{L^1(\boldsymbol{R}^n)} \leq C \sum_{j \neq k} \left(\prod_{i \neq j,k} \|u_i\|_{L^\infty(\boldsymbol{R}^n)} \right) \|u_j\|_{H^{|\alpha|}} \|u_k\|_{H^{|\alpha|}} \tag{1.7}$$

(3) *Let us assume* $f(x,u) \in C^\infty(\boldsymbol{R}^m; B^\infty(\boldsymbol{R}^n))$ *and* $f(u) = O(|u|^p)$, $|u| \to 0$, *where an integer* $p > 1$. *Then*

$$\|f(u)\|_{H^l} \leq C \|u\|_{L^\infty(\boldsymbol{R}^n)}^{p-1} \|u\|_{H^l}, \tag{1.8}$$

where C *depends only on* f *and* $\|u\|_{L^\infty(\boldsymbol{R}^n)}$.

(4) *Let us assume* $f(x,u) \in C^\infty(\boldsymbol{R}^m; B^\infty(\boldsymbol{R}^n))$ *and* $f(u) = O(|u|^p)$, $|u| \to 0$, *where an integer* $p > 2$. *Then*

$$\|f(u)\|_{W^{l,1}} \leq C \|u\|_{L^\infty(\boldsymbol{R}^n)}^{p-2} \|u\|_{H^l}^2, \tag{1.9}$$

where C *depends only on* f *and* $\|u\|_{L^\infty(\boldsymbol{R}^n)}$.

(Proof of Lemma 1.3). See M. E. Taylor [4].

(Proof of Theorem 0.2). Let us introduce E^1 and A_j^1 as follows:

$$\begin{cases} E^1(u) = E(u) - E^0 \\ A_j^1(u) = A_j(u) - A_j^0. \end{cases} \tag{1.10}$$

Then the equation (0.1) can be written in the following way:

$$E^0 \frac{\partial u}{\partial t} = \sum_{j=1}^{n} A_j^0 \frac{\partial u}{\partial x_j} + F_1(u), \tag{1.11}$$

where

$$\begin{aligned} F_1(u) = F_1(t,x,u) &= F(t,x,u) - E^1(u) \frac{\partial u}{\partial t} + \sum_{j=1}^{n} A_j^1(u) \frac{\partial u}{\partial x_j} \\ &= F(u) - E^1(u) E(u)^{-1} \left(\sum_{j=1}^{n} A_j(u) \frac{\partial u}{\partial x_j} + F(t,x,u) \right) + \sum_{j=1}^{n} A_j^1(u) \frac{\partial u}{\partial x_j}. \end{aligned} \tag{1.12}$$

Moreover, using the new function $\tilde{F}_1(u)$ defined by

$$\tilde{F}_1(u) = (E^0)^{-1} F_1(u), \tag{1.13}$$

(1.11) can be written in the following way

$$\frac{\partial u}{\partial t} = iH_0(D)u + \tilde{F}_1(u). \tag{1.14}$$

Applying the representation formula for (1.14), we obtain

$$u(t) = e^{itH_0}\varphi + \int_0^t e^{i(t-\tau)H_0}\tilde{F}_1(u(\tau))\,d\tau. \tag{1.15}$$

Now let us introduce the following norm;

$$M(t) = M(u(t)) = \sup_{0\le\tau\le t}\{\langle\tau\rangle^{(n-1)/2}\|u(\tau)\|_{W^{l/2,\infty}(\boldsymbol{R}^n)} + \|u(\tau)\|_{H^l}\}. \tag{1.16}$$

Then, from Proposition 1.1, we deduce that

$$\|u(t)\|_{W^{l/2,\infty}} \le \frac{C\|\varphi\|_{W^{l_0+l/2,1}}}{\langle t\rangle^{(n-1)/2}} + C\int_0^t \frac{\|\tilde{F}_1(u)\|_{W^{l_0+l/2,1}}}{\langle t-\tau\rangle^{(n-1)/2}}\,d\tau. \tag{1.17}$$

Taking account of the condition on $F(t,x,0)$, from (1.12) and Lemma 1.3 (4) we have

$$\begin{aligned}
\|\tilde{F}_1(u)\|_{W^{l_0+l/2,1}} &\le C\left(\|F(u)\|_{W^{l_0+l/2,1}} + \left\|E^1(u)E(u)^{-1}\sum_{j=1}^n A_j(u)\frac{\partial u}{\partial x_j}\right\|_{W^{l_0+l/2,1}}\right.\\
&\qquad \left. + \|E^1(u)E(u)^{-1}F(u)\|_{W^{l_0+l/2,1}} + \left\|\sum_{j=1}^n A_j^1(u)\frac{\partial u}{\partial x_j}\right\|_{W^{l_0+l/2,1}}\right)\\
&\le C(\|u\|_{L^\infty}^{p-1}\|u\|_{H^{l_0+l/2}}^2 + \|u\|_{L^\infty}^{p-1}\|u\|_{H^{l_0+1+l/2}}^2 + \|u\|_{L^\infty}^{2p-1}\|u\|_{H^{l_0+l/2}}^2)\\
&\quad + C\|F(t,\cdot,0)\|_{W^{l_0+l/2,1}}\\
&\le C(\|u\|_{L^\infty}^{p-1}\|u\|_{H^{l_0+1+l/2}}^2 + \|u\|_{L^\infty}^{2p-1}\|u\|_{H^{l_0+1+l/2}}^2 + \delta\langle t\rangle^{-\sigma}),
\end{aligned} \tag{1.18}$$

where $C = C(M(t))$. From (1.18) and the definition of $M(t)$, we compute

$$\|\tilde{F}_1(u)\|_{W^{l_0+l/2,1}} \le C(\|u\|_{L^\infty}^{p-1}\|u\|_{H^{l_0+1+l/2}}^2 + \|u\|_{L^\infty}^{2p-1}\|u\|_{H^{l_0+1+l/2}}^2 + \delta\langle t\rangle^{-\sigma}) \tag{1.19}$$

$$\begin{aligned}
&\le C(\langle\tau\rangle^{-((n-1)/2)(p-1)}M(t)^{p+1}\\
&\qquad + \langle\tau\rangle^{-((n-1)/2)(2p-1)}M(t)^{2p+1} + \delta\langle t\rangle^{-\sigma}),
\end{aligned} \tag{1.20}$$

because of $l_0 + 1 + \frac{l}{2} \le l$. Applying (1.19) to (1.17), we can obtain the following estimate.

$$\|u(t)\|_{W^{l/2,\infty}(R^n)} \le C\delta\langle t\rangle^{-(n-1)/2} + CM(t)^{p+1}\int_0^t \langle t-\tau\rangle^{-(n-1)/2}\langle\tau\rangle^{-((n-1)/2)(p-1)}\,d\tau$$

$$+ CM(t)^{2p+1}\int_0^t \langle t-\tau\rangle^{-(n-1)/2}\langle\tau\rangle^{-((n-1)/2)(2p-1)}\,d\tau$$

$$+ C\delta\int_0^t \langle t-\tau\rangle^{-(n-1)/2}\langle\tau\rangle^{-\sigma}\,d\tau, \tag{1.21}$$

where $C = C(M(t))$. We caluculate the integrals in the right-hand side of (1.21) as follows;

$$\int_0^t \langle t-\tau\rangle^{-(n-1)/2}(\langle\tau\rangle^{-((n-1)/2)(p-1)} + \langle\tau\rangle^{-\sigma})\,d\tau \le C\langle t\rangle^{-(n-1)/2}, \tag{1.22}$$

where $p > \frac{n+1}{n-1}$. Thus we can obtain

$$\|u(t)\|_{W^{l/2,\infty}(R^n)} \le C\langle t\rangle^{-(n-1)/2}(\delta + M(t)^{p+1} + M(t)^{2p+1}). \tag{1.23}$$

Hence, it follows that

$$\langle t\rangle^{(n-1)/2}\|u(t)\|_{W^{l/2,\infty}(R^n)} \le C(\delta + M(t)^{p+1} + M(t)^{2p+1}). \tag{1.24}$$

Operating D_x^α to the equation (0.1), we have

$$E(u)\frac{\partial D_x^\alpha u}{\partial t} = \sum_{j=1}^n A_j(u)\frac{\partial D_x^\alpha u}{\partial x_j} + F_\alpha(u), \tag{1.25}$$

where

$$F_\alpha(u) = D_x^\alpha F(u) - \sum_{\alpha'<\alpha}\binom{\alpha}{\alpha'}D_x^{\alpha-\alpha'}E(u)D_x^{\alpha'}\frac{\partial u}{\partial t}$$

$$+ \sum_{j=1}^n\sum_{\alpha'<\alpha}\binom{\alpha}{\alpha'}D_x^{\alpha-\alpha'}A_j(u)D_x^{\alpha'}\frac{\partial u}{\partial x_j}. \tag{1.26}$$

Putting $u_\alpha = D_x^\alpha u$, $|\alpha| \le l$, we have

$$E(u)\frac{\partial u_\alpha}{\partial t} = \sum_{j=1}^n A_j(u)\frac{\partial u_\alpha}{\partial x_j} + F_\alpha(u). \tag{1.27}$$

Now we compute

$$\frac{d}{dt}(E(u)u_\alpha, u_\alpha)_{L^2(\boldsymbol{R}^n)} = \left(\nabla_u E(u)E(u)^{-1}\left(\sum_{j=1}^n A_j(u)\frac{\partial u}{\partial x_j} + F(u)\right)u_\alpha, u_\alpha\right)_{L^2(\boldsymbol{R}^n)}$$

$$- 2\,\mathbf{Re}\left(u_\alpha, \sum_{j=1}^n \frac{\partial A_j(u)}{\partial x_j} u_\alpha\right)_{L^2(\boldsymbol{R}^n)} + 2\,\mathbf{Re}(F_\alpha, u_\alpha)_{L^2(\boldsymbol{R}^n)}. \tag{1.28}$$

Therefore it follows that

$$\frac{d}{dt}(E(u)u_\alpha, u_\alpha)^{1/2}_{L^2(\boldsymbol{R}^n)} \le \frac{1}{2}c(t)(E(u)u_\alpha, u_\alpha)^{1/2}_{L^2(\boldsymbol{R}^n)} + \frac{1}{2}\|E(u)^{-1/2}F_\alpha\|_{L^2(\boldsymbol{R}^n)}, \tag{1.29}$$

and

$$c(t) = \sup_x \left|\nabla_u E(u)E(u)^{-1}\left(\sum_{j=1}^n A_j(u)\frac{\partial u}{\partial x_j} + F(u)\right)E(u)^{-1}\right|$$

$$+ 2\sup_x \left|E(u)^{-1}\sum_{j=1}^n \frac{\partial A_j(u)}{\partial x_j}\right|. \tag{1.30}$$

Applying the Gronwall type inequality to (1.29), we get

$$(E(u)u_\alpha, u_\alpha)^{1/2}_{L^2(\boldsymbol{R}^n)} \le (E(u)u_\alpha(0), u_\alpha(0))^{1/2}_{L^2(\boldsymbol{R}^n)} \exp\left(\int_0^t c(\tau)\,d\tau\right)$$

$$+ \int_0^t \|E(u)^{-1/2}F_\alpha\|_{L^2(\boldsymbol{R}^n)} \exp\left(\int_\tau^t c(s)\,ds\right) d\tau. \tag{1.31}$$

Using the equation (0.1) and Lemma 1.3, we see after a simple calculation that

$$\|E(u)^{-1/2}F_\alpha(u)\|_{L^2(\boldsymbol{R}^n)} \le C\|F_\alpha(u)\|_{L^2(\boldsymbol{R}^n)}$$

$$\le C\|D_x^\alpha F(u)\|_{L^2(\boldsymbol{R}^n)} + C\sum_{\alpha'<\alpha}\left\|D_x^{\alpha-\alpha'}E(u)D_x^{\alpha'}\frac{\partial u}{\partial t}\right\|_{L^2(\boldsymbol{R}^n)}$$

$$+ C\sum_{j=1}^n\sum_{\alpha'<\alpha}\left\|D_x^{\alpha-\alpha'}A_j(u)D_x^{\alpha'}\frac{\partial u}{\partial x_j}\right\|_{L^2(\boldsymbol{R}^n)}$$

$$\le C(\|u\|_{H^l}\|u\|^p_{W^{l/2,\infty}} + \|F(t,\cdot,0)\|_{H^l})$$

$$\le C(\|u\|^p_{W^{l/2,\infty}(\boldsymbol{R}^n)}\|u\|_{H^l} + \delta\langle t\rangle^{-\sigma}). \tag{1.32}$$

Combining (1.31) and (1.32), we have

$$(E(u)u_\alpha, u_\alpha)^{1/2}_{L^2(R^n)} \le (E(u)u_\alpha(0), u_\alpha(0))^{1/2}_{L^2(R^n)} \exp\left(\int_0^t c(\tau)\, d\tau\right)$$
$$+ C\int_0^t \|u(\tau)\|^p_{W^{l/2,\infty}(R^n)}\|u(\tau)\|_{H^l} \exp\left(\int_\tau^t c(s)\, ds\right) d\tau. \quad (1.33)$$

Taking the sum with respect to α up to l and using the equivalence of norms, we obtain

$$\|u\|_{H^l} \le C\|u(0)\|_{H^l} \exp\left(\int_0^t c(\tau)\, d\tau\right)$$
$$+ C\int_0^t \|u\|^p_{W^{l/2,\infty}(R^n)}\|u\|_{H^l} \exp\left(\int_\tau^t c(s)\, ds\right) d\tau. \quad (1.34)$$

Moreover from (1.10) and Lemma 1.3 we deduce

$$c(t) = \sup_x \left|\nabla_u E(u)E(u)^{-1}\left(\sum_{j=1}^n A_j(u)\frac{\partial u}{\partial x_j} + F(u)\right)E(u)^{-1}\right|$$
$$+ 2\sup_x\left|E(u)^{-1}\sum_{j=1}^n \frac{\partial A_j(u)}{\partial x_j}\right|$$
$$\le C(\|u\|^{p-1}_{L^\infty(R^n)}\|u\|_{W^{1,\infty}(R^n)} + \|u\|^{2p}_{W^{1,\infty}(R^n)} + \|F(t,\cdot,0)\|_{L^\infty})$$
$$\le C(\|u\|^p_{W^{1,\infty}(R^n)} + \|u\|^{2p}_{W^{1,\infty}(R^n)} + \delta\langle t\rangle^{-\sigma}). \quad (1.35)$$

From the definition of $M(t)$ we have

$$\int_0^t c(\tau)\, d\tau \le C\int_0^t (\|u\|^p_{W^{1,\infty}(R^n)} + \|u\|^{2p}_{W^{1,\infty}(R^n)} + \delta\langle \tau\rangle^{-\sigma})\, d\tau$$
$$\le C\left(M(t)^p\int_0^t \langle\tau\rangle^{-((n-1)/2)p}\, d\tau + M(t)^{2p}\int_0^t\langle\tau\rangle^{-(n-1)p}\, d\tau + \delta\right)$$
$$\le C(M(t)^p + M(t)^{2p} + \delta). \quad (1.36)$$

where $\frac{n-1}{2}p > 1$, $(n-1)p > 1$ and $\sigma > 1$. Consequently we find that

$$\|u\|_{H^l} \le C\|u(0)\|_{H^l} \exp\left(\int_0^t c(\tau)\, d\tau\right)$$
$$+ C\int_0^t \|u\|^p_{W^{l/2,\infty}(R^n)}\|u\|_{H^l} \exp\left(\int_\tau^t c(s)\, ds\right) d\tau,$$
$$\le C\delta \exp(CM(t)^p + CM(t)^{2p})$$

$$+ CM(t)^{p+1} \exp(CM(t)^p + CM(t)^{2p}) \int_0^t \langle\tau\rangle^{-((n-1)/2)p}\, d\tau$$

$$\le C\delta \exp(CM(t)^p + CM(t)^{2p})$$

$$+ CM(t)^{p+1} \exp(CM(t)^p + CM(t)^{2p}). \tag{1.37}$$

Combining (1.24) and (1.37) and taking the supremum on $0 \le \tau \le t$, we obtain

$$M(t) \le C(\delta + M(t)^{p+1} + M(t)^{2p+1}) + C\delta \exp(CM(t)^p + CM(t)^{2p})$$

$$+ CM(t)^{p+1} \exp(CM(t)^p + CM(t)^{2p}). \tag{1.38}$$

Set $M(t) = x$. Then inequality (1.38) can be written in the following way

$$x \le C(\delta + x^{p+1} + x^{2p+1} + \delta e^{Cx^p + Cx^{2p}} + x^{p+1} e^{Cx^p + Cx^{2p}}). \tag{1.39}$$

Finally, from $M(0) \le \delta \ll 1$, we can see $x \le C\delta$. This means that

$$M(t) \le C\delta. \tag{1.40}$$

This completes the proof of Theorem 0.2. Q.E.D.

2 Scattering for the Nonlinear Symmetric Hyperbolic Systems

In this chapter we shall prove Theorem 0.3. First we construct a solution satisfying

$$\begin{cases} E(u)\dfrac{\partial u}{\partial t} = \displaystyle\sum_{j=1}^n A_j(u)\frac{\partial u}{\partial x_j} + F(u), & x \in \boldsymbol{R}^n,\ t \le 0, \\ \displaystyle\lim_{t\to-\infty} (u(t) - u^0(t)) = 0 \quad \text{in } H^l, \end{cases} \tag{2.1}$$

where we denote

$$u^0(t) = e^{itH_0}\varphi_-. \tag{2.2}$$

Now we set $w(t) = u(t) - u^0(t)$, then we have

$$\begin{aligned} E(u)\frac{d}{dt}w(t) &= E(u)\frac{d}{dt}(u(t) - u^0(t)) \\ &= \sum_{j=1}^n A_j(u)\frac{\partial}{\partial x_j}u(t) + F(u) - iE(u)H_0u^0(t) \\ &= \sum_{j=1}^n A_j(u)\frac{\partial}{\partial x_j}w(t) + \sum_{j=1}^n (A_j(u) - E(u)(E^0)^{-1}A_j(0))\frac{\partial}{\partial x_j}u^0(t) + F(u). \end{aligned} \tag{2.3}$$

Therefore, the following equation is equivalent to (2.1):

$$\begin{cases} E((w(t)+u^0(t))\dfrac{d}{dt}w(t) = \displaystyle\sum_{j=1}^{n} A_j(w(t)+u^0(t))\dfrac{\partial}{\partial x_j}w(t) + \tilde{F}(w(t)+u^0(t)), \\ \displaystyle\lim_{t\to-\infty} w(t) = 0 \quad \text{in } H^l, \end{cases} \tag{2.4}$$

where

$$\tilde{F}(w(t)+u^0(t)) = \sum_{j=1}^{n}\{A_j(w(t)+u^0(t)) - E(w(t)+u^0(t))(E^0)^{-1}A_j(0)\}\frac{\partial}{\partial x_j}u^0(t) + F(w(t)+u^0(t)). \tag{2.5}$$

In order to solve (2.4), we consider the linearlized equation of (2.4). We define the function space X_δ^l as follows

$$X_\delta^l = \{w(t) \in C^0((-\infty,0];H^l);\ \|w(t)\|_{H^l} \le \delta\langle t\rangle^{-(n-1)/2}, \forall t \le 0\}. \tag{2.6}$$

Let $v \in X_\delta^l$, and we put $A_v(t) = E(v(t)+u^0(t))^{-1}\sum_{j=1}^{n} A_j(v(t)+u^0(t))\frac{\partial}{\partial x_j}$. Let consider the following linear problem

$$\begin{cases} \dfrac{d}{dt}w(t) = A_v(t)w(t) + f(t), \\ \displaystyle\lim_{t\to-\infty} w(t) = 0 \quad \text{in } H^l, \end{cases} \tag{2.7}$$

where $f \in L^1((-\infty,0);H^l)$. In order to solve (2.7) we shall consider the following

$$\begin{cases} \dfrac{d}{dt}u(t) = A_v(t)u(t), \quad t \in \boldsymbol{R}^1,\ x \in \boldsymbol{R}^n, \\ u(\tau) = u_0, \end{cases} \tag{2.8}$$

where $u_0 \in H^l$ and τ is a fixed number arbitary chosen. We can represent the solution of (2.8) as

$$u(t) = E_v(t,\tau)u_0, \tag{2.9}$$

where $E_v(t,\tau)$ is the fundamental solution which depends on v, that is, $E_v(t,\tau)$ is a solution for

$$\begin{cases} \dfrac{d}{dt}E_v(t,\tau) = A_v(t)E_v(t,\tau), \\ E_v(\tau,\tau) = I, \end{cases} \tag{2.10}$$

where I is the identity matrix, and the fundamental solution $E_v(t,\tau)$ satisfies the following property

$$E_v(t,\tau)E_v(\tau,r) = E_v(t,r), \quad \forall t,\tau,r. \tag{2.11}$$

Then we can obtain the following proposition.

PROPOSITION 2.1. *Assume $p > \frac{n+1}{n-1}$ and $v \in X_\delta^l$. There is a positive constant C such that for $u_0 \in H^l$, $l \geq 2(l_0+1)$, $(E_v(t,\tau)u_0)(x)$ is a solution of* (2.8) *and satisfies for any $t,\tau \in \boldsymbol{R}^1$*

$$\|E_v(t,\tau)u_0\|_{H^l} \leq e^{C\delta^p}\|u_0\|_{H^l}. \tag{2.12}$$

(PROOF OF PROPOSITION 2.1). Operating D_x^α to the equation (2.8), we have

$$\frac{d}{dt}D_x^\alpha u(t) = A_v(t)D_x^\alpha u(t) + f_\alpha(t), \tag{2.13}$$

where

$$f_\alpha(t) = \sum_{\alpha'<\alpha}\binom{\alpha}{\alpha'}D_x^{\alpha-\alpha'}A_v(t)D_x^{\alpha'}u(t) + D_x^\alpha f(t). \tag{2.14}$$

Since $\|u(t)\|_{H^l}^2$ is equivalent to $\sum_{|\alpha|\leq l}\|\sqrt{E(v+u^0)}D_x^\alpha u(t)\|_{L^2(\boldsymbol{R}^n)}^2$, we compute

$$\frac{d}{dt}\sum_{|\alpha|\leq l}(E(v+u^0)D_x^\alpha u(t), D_x^\alpha u(t))_{L^2(\boldsymbol{R}^n)}$$

$$= 2\sum_{|\alpha|\leq l}\mathbf{Re}\left(E\frac{d}{dt}D_x^\alpha u(t), D_x^\alpha u(t)\right)_{L^2(\boldsymbol{R}^n)} + \left(\frac{dE}{dt}D_x^\alpha u(t), D_x^\alpha u(t)\right)_{L^2(\boldsymbol{R}^n)}$$

$$= 2\sum_{|\alpha|\leq l}\mathbf{Re}(EA_v(t)D_x^\alpha u(t), D_x^\alpha u(t))_{L^2(\boldsymbol{R}^n)} + 2\sum_{|\alpha|\leq l}\mathbf{Re}(Ef_\alpha(t), D_x^\alpha u(t))_{L^2(\boldsymbol{R}^n)}$$

$$+ \left(\frac{dE}{dt}D_x^\alpha u(t), D_x^\alpha u(t)\right)_{L^2(\boldsymbol{R}^n)}. \tag{2.15}$$

Since

$$2\sum_{|\alpha|\leq l}\mathbf{Re}(EA_v(t)D_x^\alpha u(t), D_x^\alpha u(t))_{L^2(\boldsymbol{R}^n)}$$

$$= 2\sum_{|\alpha|\leq l}\sum_{j=1}^n\mathbf{Re}\left(D_x^\alpha u(t), \left(\frac{\partial}{\partial x_j}A_j(v(t)+u^0(t))\right)D_x^\alpha u(t)\right)_{L^2(\boldsymbol{R}^n)}$$

$$\leq 2a(t)\|u(t)\|_{H^l}^2, \tag{2.16}$$

and

$$\left|\left(\frac{dE}{dt}D_x^\alpha u(t), D_x^\alpha u(t)\right)_{L^2(R^n)}\right| \leq C\|v(t)+u^0(t)\|_{L^\infty}^{p-1}\|v_t(t)+u_{0t}(t)\|_{L\infty}\|u\|_{H^l}$$

$$\leq C\langle t\rangle^{-((n-1)/2)p}\|u\|_{H^l}$$

where $a(t)=\sup_{x\in R^n}\sum_{j=1}^n\left|\frac{\partial}{\partial x_j}(A_j(v(t)+u^0(t)))\right|$ and

$$\sum_{|\alpha|\leq l}\mathbf{Re}(Ef_\alpha(t), D_x^\alpha u(t))_{L^2(R^n)} \leq \sum_{|\alpha|\leq l}|(Ef_\alpha(t), D_x^\alpha u(t))_{L^2(R^n)}|$$

$$\leq C\sum_{|\alpha|\leq l}\|f_\alpha(t)\|_{L^2(R^n)}\|u(t)\|_{H^l}, \tag{2.17}$$

we have

$$\frac{d}{dt}\|u(t)\|_{H^l} \leq (a(t)+C\langle t\rangle^{-((n-1)/2)p})\|u(t)\|_{H^l}+\frac{1}{2}\sum_{|\alpha|\leq l}\|f_\alpha(t)\|_{L^2(R^n)}. \tag{2.18}$$

On the other hand from (2.14) we obtain

$$\sum_{|\alpha|\leq l}\|f_\alpha(t)\|_{L^2(R^n)} \leq C\sum_{|\alpha|\leq l}\sum_{\alpha'<\alpha}\|D_x^{\alpha-\alpha'}A_v(t)D_x^{\alpha'}u(t)\|_{L^2(R^n)}$$

$$\leq C\sum_{|\alpha|\leq l}\sum_{\alpha'\leq l/2}\sum_{j=1}^n\left\|D_x^{\alpha-\alpha'}A_j^1(v(t)+u^0(t))\frac{\partial}{\partial x_j}D_x^{\alpha'}u(t)\right\|_{L^2(R^n)}$$

$$+C\sum_{|\alpha|\leq l}\sum_{l/2\leq\alpha'\leq l-1}\sum_{j=1}^n\left\|D_x^{\alpha-\alpha'}A_j^1(v(t)+u^0(t))\frac{\partial}{\partial x_j}D_x^{\alpha'}u(t)\right\|_{L^2(R^n)}$$

$$=I+II. \tag{2.19}$$

By using Lemma 1.3, Assumption 0.1 and Sobolev's lemma, we caluculate

$$I\leq C\sum_{|\alpha|\leq l}\sum_{\alpha'\leq l/2}\sum_{j=1}^n\left\|\frac{\partial}{\partial x_j}D_x^{\alpha'}u(t)\right\|_{L^\infty(R^n)}\|D_x^{\alpha-\alpha'}(A_j^1(v(t)+u^0(t)))\|_{L^2(R^n)}$$

$$\leq C\|u(t)\|_{W^{l/2+1,\infty}}\|v(t)+u^0(t)\|_{L^\infty(R^n)}^{p-1}\|v(t)+u^0(t)\|_{H^l}$$

$$\leq C\|u(t)\|_{H^{l/2+1+l_0}}\|v(t)+u^0(t)\|_{L^\infty(R^n)}^{p-1}\|v(t)+u^0(t)\|_{H^l}$$

$$\leq C\|u(t)\|_{H^l}\|v(t)+u^0(t)\|_{L^\infty(R^n)}^{p-1}\|v(t)+u^0(t)\|_{H^l}$$

$$\leq C\delta^p\langle t\rangle^{-((n-1)/2)(p-1)}\|u(t)\|_{H^l}, \tag{2.20}$$

where $l \geq 2(l_0+1)$, $l_0 = [\frac{n}{2}]+1$, and

$$II \leq C \sum_{|\alpha| \leq l} \sum_{l/2 \leq \alpha' \leq l-1} \sum_{j=1}^{n} \|D_x^{\alpha-\alpha'} A_j^1 (v(t)+u^0(t))\|_{L^\infty(\boldsymbol{R}^n)} \left\| \frac{\partial}{\partial x_j} D_x^{\alpha'} u(t) \right\|_{L^2(\boldsymbol{R}^n)},$$

$$\leq C\|v(t)+u_0(t)\|_{W^{l/2,\infty}(\boldsymbol{R}^n)}^p \|u(t)\|_{H^l},$$

$$\leq C\delta^p \langle t \rangle^{-((n-1)/2)(p-1)} \|u(t)\|_{H^l}. \tag{2.21}$$

Hence we have

$$\frac{d}{dt}\|u(t)\|_{H^l} \leq b(t)\|u(t)\|_{H^l}, \tag{2.22}$$

where $b(t) = a(t) + C\delta^p \langle t \rangle^{-((n-1)/2)(p-1)}$. Therefore we can see that

$$\frac{d}{dt}\left\{ \exp\left(-\int_0^t b(s)\, ds \right) \|u(t)\|_{H^l} \right\} \leq 0. \tag{2.23}$$

Integrating (2.23) over $[\tau, t]$, where $\tau \in (-\infty, t]$, we have

$$\|u(t)\|_{H^l} \leq \exp\left(\int_\tau^t b(s)\, ds \right) \|u(\tau)\|_{H^l}. \tag{2.24}$$

Next we will estimate $b(t) = a(t) + C\delta^p \langle t \rangle^{-\frac{n-1}{2}(p-1)}$.

$$a(t) \leq \sum_{j=1}^{n} \left\| \nabla_u A_j (v(t)+u^0(t)) \frac{\partial}{\partial x_j}(v(t)+u^0(t)) \right\|_{L^\infty(\boldsymbol{R}^n)}$$

$$\leq C\|v(t)+u^0(t)\|_{L^\infty(\boldsymbol{R}^n)}^{p-1} \left\| \frac{\partial}{\partial x_j}(v(t)+u^0(t)) \right\|_{L^\infty(\boldsymbol{R}^n)}$$

$$\leq C(\|v(t)\|_{H^{l_0+1}} + \|u^0(t)\|_{W^{1,\infty}(\boldsymbol{R}^n)})^p$$

$$\leq C\delta^p \langle t \rangle^{-((n-1)/2)(p-1)}, \tag{2.25}$$

hence we get

$$\int_\tau^t b(s)\, ds \leq C\delta^p \int_\tau^t \langle s \rangle^{-((n-1)/2)(p-1)}\, ds$$

$$\leq C\delta^p \int_{-\infty}^{0} \langle s \rangle^{-((n-1)/2)(p-1)}\, ds$$

$$\leq C\delta^p, \tag{2.26}$$

where $\frac{n-1}{2}(p-1) > 1$. Consequently, we have from (2.24)

$$\|u(t)\|_{H^l} \le e^{C\delta^p} \|u(\tau)\|_{H^l}. \tag{2.27}$$

Thus we can obtain (2.12). Q.E.D.

Now we shall turn to the investigation of (2.7). Applying $E(0,t)$ to the equation (2.7), we have

$$E_v(0,t)\frac{d}{dt}w(t) = E_v(0,t)A_v(t)w(t) + E_v(0,t)f(t), \tag{2.28}$$

and

$$\begin{aligned} \frac{d}{dt}(E_v(0,t)w(t)) &= \left(\frac{d}{dt}E_v(0,t)\right)w(t) + E_v(0,t)\frac{d}{dt}w(t) \\ &= \left(\frac{d}{dt}E_v(0,t)\right)w(t) + E_v(0,t)A_v(t)w(t) + E_v(0,t)f(t). \end{aligned} \tag{2.29}$$

By using the relation

$$E_v(t,0)E_v(0,t) = I, \tag{2.30}$$

we have

$$\left(\frac{d}{dt}E_v(t,0)\right)E_v(0,t) + E_v(t,0)\frac{d}{dt}E_v(0,t) = 0. \tag{2.31}$$

Combining the equation (2.10) and (2.31), we have

$$\begin{aligned} E_v(t,0)\frac{d}{dt}E_v(0,t) &= -\left(\frac{d}{dt}E_v(t,0)\right)E_v(0,t), \\ &= -A_v(t). \end{aligned} \tag{2.32}$$

Hence we get

$$\frac{d}{dt}E_v(0,t) = -E_v(0,t)A_v(t). \tag{2.33}$$

Insert (2.33) into (2.29), we have

$$\frac{d}{dt}(E_v(0,t)w(t)) = E_v(0,t)f(t). \tag{2.34}$$

If $w(t) \to 0$ as $t \to -\infty$ in H^l, then we can see that

$$E(0,t)w(t) = \int_{-\infty}^{t} E_v(0,s)f(s)\,ds, \tag{2.35}$$

where $f \in L^1((-\infty,\infty);H^l)$. That is

$$w(t) = \int_{-\infty}^{t} E_v(t,s)f(s)\,ds. \tag{2.36}$$

By using Proposition 2.1, we can obtain

$$\begin{aligned}\|w(t)\|_{H^l} &\le \int_{-\infty}^{t} \|E_v(t,s)f(s)\|_{H^l}\,ds \\ &\le e^{C\delta^p}\int_{-\infty}^{t} \|f(s)\|_{H^l}\,ds.\end{aligned} \tag{2.37}$$

Since $f(s) \in L^1((-\infty,0);H^l)$, then we can see that $\|w(t)\|_{H^l} \to 0$ as $t \to -\infty$. Consequently, we have the following proposition.

Proposition 2.2. *Let $v \in X_\delta^l$. Assume that $f(t) \in L^1((-\infty,\infty);H^l)$. Then*

$$w(t) = \int_{-\infty}^{t} E_v(t,\tau)f(\tau)\,d\tau \tag{2.38}$$

is a solution of (2.7) *and satisfies the following estimate;*

$$\|w(t)\|_{H^l} \le e^{C\delta^p}\int_{-\infty}^{t} \|f(s)\|_{H^l}\,ds. \tag{2.39}$$

Therefore $w(t) \to 0$ in H^l as $t \to -\infty$.

In order to solve (2.4), we consider the following linear equation

$$\begin{cases} \dfrac{d}{dt}w(t) = A_v(t)w(t) + \tilde{F}(v(t) + u^0(t)), \\ \lim\limits_{t\to-\infty} w(t) = 0 \quad \text{in } H^l. \end{cases} \tag{2.40}$$

By using the representation fomula of Proposition 2.2, we have

$$w(t) = \int_{-\infty}^{t} E_v(t,s)\tilde{F}(v(s) + u^0(s))\,ds. \tag{2.41}$$

Now we put the right side as

$$w(t) = \Psi(v)(t), \tag{2.42}$$

for $v \in X_\delta^l$ and define for $v, \tilde{v} \in X_\delta^l$,

$$S(t) = \Psi(v)(t) - \Psi(\tilde{v})(t). \tag{2.43}$$

From (2.40), we have

$$\frac{d}{dt} S(t) = A_v(t) S(t) + G(t), \tag{2.44}$$

where

$$\begin{aligned} G(t) &= \sum_{j=1}^{n} \{\tilde{A}_j(v(t) + u^0(t)) - \tilde{A}_j(\tilde{v}(t) + u^0(t))\} \frac{\partial}{\partial x_j} \Psi(\tilde{v})(t) \\ &\quad + \tilde{F}(v(t) + u^0(t)) - \tilde{F}(\tilde{v}(t) + u^0(t)) \\ &= B(t)(v(t) - \tilde{v}(t)), \end{aligned} \tag{2.45}$$

$\tilde{A}_j(v(t) + u^0(t)) = E(v(t) + u^0(t))^{-1} A_j(v(t) + u^0(t))$ and

$$\begin{aligned} B(t) &= \sum_{j=1}^{n} \int_0^1 \nabla A_j(\tilde{v}(t) + u^0(t) + \theta(v(t) - \tilde{v}(t)))\, d\theta \frac{\partial}{\partial x_j} \Psi(\tilde{v})(t) \\ &\quad + \int_0^1 \nabla \tilde{F}(\tilde{v}(t) + u^0(t) + \theta(v(t) - \tilde{v}(t)))\, d\theta. \end{aligned} \tag{2.46}$$

Moreover, putting $y(t) = v(t) - \tilde{v}(t)$ and taking a norm in H^{l-1}, from Lemma 1.3 (1) we obtain

$$\begin{aligned} \|G(t)\|_{H^{l-1}} &= \|B(t) y(t)\|_{H^{l-1}} \\ &\leq C \|B(t)\|_{H^{l-1}} \|y(t)\|_{H^{l-1}}, \end{aligned} \tag{2.47}$$

where $l - 1 \geq l_0$. Since

$$\begin{aligned} &\left\| \sum_{j=1}^{n} \int_0^1 \nabla A_j(\tilde{v}(t) + u^0(t) + \theta(v(t) - \tilde{v}(t)))\, d\theta \frac{\partial}{\partial x_j} \Psi(\tilde{v})(t) \right\|_{H^{l-1}} \\ &\quad \leq C \sum_{j=1}^{n} \int_0^1 \|\nabla A_j(\tilde{v}(t) + u^0(t) + \theta(v(t) - \tilde{v}(t)))\|_{L^\infty}\, d\theta \left\| \frac{\partial}{\partial x_j} \Psi(\tilde{v}(t)) \right\|_{H^{l-1}} \\ &\quad \leq C(\|v(t)\|_{H^{l_0}} + \|\tilde{v}(t)\|_{H^{l_0}} + \|u^0(t)\|_{L^\infty})^{p-1} (\|\Psi(\tilde{v})(t)\|_{H^l} + \|u^0(t)\|_{H^l}) \\ &\quad \leq C \delta^p \langle t \rangle^{-((n-1)/2)(p-1)}, \end{aligned} \tag{2.48}$$

and

$$\left\| \int_0^1 \nabla \tilde{F}(\tilde{v}(t) + u^0(t) + \theta(v(t) - \tilde{v}(t)))\, d\theta \right\|_{H^{l-1}}$$

$$\leq \int_0^1 \|(\tilde{v}(t) + u^0(t) + \theta(v(t) - \tilde{v}(t)))\|_{L^\infty}^{p-1} \|(\tilde{v}(t) + u^0(t) + \theta(v(t) - \tilde{v}(t)))\|_{H^{l-1}}\, d\theta$$

$$\leq C\delta^p \langle t \rangle^{-((n-1)/2)(p-1)}, \tag{2.49}$$

we have

$$\|B(t)\|_{H^{l-1}} \leq C\delta^p \langle t \rangle^{-((n-1)/2)(p-1)}. \tag{2.50}$$

Thus, from $\Psi(\tilde{v})(t) \in X_\delta^l$, we can obtain from (2.47)

$$\|G(t)\|_{H^{l-1}} \leq C\delta \langle t \rangle^{-((n-1)/2)(p-1)} \|v(t) - \tilde{v}(t)\|_{H^{l-1}}. \tag{2.51}$$

From (2.44), we have

$$S(t) = \int_{-\infty}^t E_v(t,\tau) G(\tau)\, d\tau. \tag{2.52}$$

Thus, by using (2.51) and Proposition 2.2, we can see

$$\begin{aligned} \|S(t)\|_{H^{l-1}} &\leq \int_{-\infty}^t \|E_v(t,\tau)G(\tau)\|_{H^{l-1}}\, d\tau \\ &\leq e^{C\delta^p} \int_{-\infty}^t \|G(\tau)\|_{H^{l-1}}\, d\tau \\ &\leq e^{C\delta^p} \int_{-\infty}^t C\delta^p \langle \tau \rangle^{-((n-1)/2)(p-1)} \|v(\tau) - \tilde{v}(\tau)\|_{H^{l-1}}\, d\tau \\ &\leq Ce^{C\delta^p} \delta^p \langle t \rangle^{-((n-1)/2)(p-1)+1} \sup_{-\infty<\tau\leq t} \|v(\tau) - \tilde{v}(\tau)\|_{H^{l-1}}, \end{aligned} \tag{2.53}$$

where $\frac{n-1}{2}p > 1$. We assume that $Ce^{C\delta^p}\delta^p < \frac{1}{2}$, then we obtain for $t \geq 0$

$$\sup_{-\infty<\tau\leq t} \|\Psi(v)(\tau) - \Psi(\tilde{v})(\tau)\|_{H^{l-1}} \leq \frac{1}{2} \langle t \rangle^{-(n-1)/2} \sup_{-\infty<\tau\leq t} \|v(\tau) - \tilde{v}(\tau)\|_{H^{l-1}}. \tag{2.54}$$

Thus we get the following proposition.

PROPOSITION 2.3. *Let $w(t) = \Psi(v)(t)$ be a solution of (2.40) in X_δ^l and $l \geq 2(l_0 + 1)$. Then there exists a positive constant δ such that Ψ is a mapping*

$$\Psi : X_\delta^l \to X_\delta^l \tag{2.55}$$

and satisfies (2.54).

The inequality (2.54) implies that Ψ is a contraction mapping and we can see the existence of the fixed point of Ψ. In fact, we put

$$\begin{cases} w_0 = 0, \\ w_1 = \Psi(0), \\ \quad \vdots \\ w_k = \Psi(w_{k-1}), \end{cases} \tag{2.56}$$

where $w_k \in X_\delta^l$, $\forall k$, then, from Proposition 2.3, we can see

$$w = \lim_{k\to\infty} w_k \quad \text{in } H^{l-1}, \tag{2.57}$$

and from the definition of X_δ^l,

$$\|w_k(t)\|_{H^l} \le \delta \langle t \rangle^{-(n-1)/2}, \quad \forall k. \tag{2.58}$$

Hence we obtain that $\{w_k(t)\}$ involves a subsequence $\{w_{k_j}(t)\}$, which weakly converge in H^l. Therefore we have $w(t) \in H^l$ satiafying

$$\|w(t)\|_{H^l} \le \lim_{j\to\infty} \sup \|w_{k_j}(t)\|_{H^l} \le \delta \langle t \rangle^{-(n-1)/2}, \quad t \le 0. \tag{2.59}$$

Now we can define $u_-(t) = w(t) + e^{itH_0}\varphi_-$. The above inequality yields that

$$\|u_-(t) - e^{itH_0}\varphi_-\|_{H^l} \le C\delta \langle t \rangle^{-(n-1)/2} \quad t \le 0. \tag{2.60}$$

and that $u_-(t)$ satisfies the equation (0.1) in $t \le 0$. Next we want to extend $u_-(t)$ to $[0, \infty)$ as follows. To do so, we shall find a solution $u_+(t) \in C^0([0, +\infty); H^l)$ such that

$$\begin{cases} E(u_+)\dfrac{\partial u_+}{\partial t} = \displaystyle\sum_{j=1}^n A_j(u_+)\frac{\partial u_+}{\partial x_j} + F(u_+), \\ u_+(0) = u_-(0), \end{cases} \tag{2.61}$$

but we can not know if the initial data $u_-(0)$ is in $W^{l,1}(\boldsymbol{R}^n)$. So, instead of the equation (2.61) we consider the equation (2.65) below with zero initial data. Let define $u_0^+(t) = e^{itH_0}\psi$ for $t \ge 0$, where

$$\psi = \varphi_- + \int_{-\infty}^{0} e^{-i\tau H_0}(A_w^1(\tau) + \tilde{F}(w(\tau) + u^0(\tau))\, d\tau \tag{2.62}$$

and $w(t)$ is the solution of (2.40). Then from (2.4) we can see that $u_0^+(0) = \psi = w(0) + \varphi_- = u_-(0)$. Moreover noting that from Lemma 1.2 (4)

$$\begin{aligned} &\|(A_w^1(t) + \tilde{F}(w(t) + u^0(t))\|_{W^{\tilde{l}+1,1}} \\ &\quad \le \|A^1(\cdot\,, w + u_0)(w + u_0)(t) + F(w(t) + u^0(t)\|_{W^{\tilde{l}+1,1}} \\ &\quad \le C\|(w + u_0)(t)\|_{W^{1,\infty}}^{p-1}\|w + u_0\|_{H^{\tilde{l}+1}}^2 \\ &\quad \le C\delta^2\langle t\rangle^{-((n-1)/2)(p-1)}, \end{aligned} \tag{2.63}$$

we can estimate

$$\begin{aligned} \|u_0^+(t)\|_{W^{1,\infty}} &\le C\Big(\langle t\rangle^{-(n-1)/2}\|\varphi_-\|_{W^{\tilde{l}+1,1}} \\ &\quad + \int_{-\infty}^{t} \langle t-\tau\rangle^{-(n-1)/2}\|A_w^1(\tau) + \tilde{F}(w(\tau) + u^0(\tau))\|_{W^{\tilde{l}+1,1}}\, d\tau\Big) \\ &\le C\delta\langle t\rangle^{-(n-1)/2}. \end{aligned} \tag{2.64}$$

We define $u_+(t) = w^+(t) + u_0^+(t)$ for $t \ge 0$, where $w^+(t)$ satisfies

$$\begin{cases} \dfrac{d}{dt}w^+(t) = A(x, w^+(t) + u_0^+(t))w^+(t) + A^1(x, w^+(t) + u_0^+(t))u_0^+(t), \\ \qquad\qquad t > 0 \\ w^+(0) = 0 \end{cases} \tag{2.65}$$

which is equivalent to the equation (2.61). We can show the existence of the solution $w^+(t)$ of the equation (2.65) applying Theorem 0.3 to the equation (2.65), because $\varphi = 0$ and $f(t) = A^1(x, u_0^+(t))u_0^+(t)$ satisfies the condition of Theorem 0.3, if we take $\sigma = \frac{n-1}{2}(p-1) > 1$. Moreover $u_+(t) = w^+(t) + u_0^+(t)$ satisfies (2.61). Therefore $u_+(t)$ is an extension to $t > 0$ of $u_-(t)$ and it follows from Proposition 5.4 with $f = 0$ that there is $C > 0$ such that

$$\|u_+(t)\|_{H^l} \le C\|u_+(0)\|_{H^l} = C\|\psi\|_{H^l} \le C\|\varphi_-\|_{H^{l+1}} \le C\delta, \quad t > 0 \tag{2.66}$$

and

$$\|u_+(t)\|_{W^{1,\infty}} \le C\langle t\rangle^{-(n-1)/2}\delta, \quad t > 0. \tag{2.67}$$

Hence we can define

$$u(t) = \begin{cases} u_+(t), & t \geq 0, \\ u_-(t), & t \leq 0, \end{cases} \tag{2.68}$$

satisfying

$$u(t) \in C^0((-\infty, +\infty); H^l) \cap C^1((-\infty, +\infty); H^{l-1}), \tag{2.69}$$

and

$$E(u)\frac{\partial u}{\partial t} = \sum_{j=1}^{n} A_j(u)\frac{\partial u}{\partial x_j} + F(u), \quad -\infty < t < +\infty. \tag{2.70}$$

Next, we seek $\varphi_+ \in H^l$ satisfying

$$\begin{cases} E(u)\dfrac{\partial u}{\partial t} = \displaystyle\sum_{j=1}^{n} A_j(u)\frac{\partial u}{\partial x_j} + F(u), \\ \displaystyle\lim_{t\to+\infty}(u(t) - e^{itH_0}\varphi_+) = 0. \end{cases} \tag{2.71}$$

From (1.10), we rewrite the equation (2.71) as follows

$$E^0\frac{\partial u}{\partial t} = \sum_{j=1}^{n} A_j^0\frac{\partial u}{\partial x_j} + F_1(t), \tag{2.72}$$

where

$$F_1(t) = F(u) - E^1(u)E(u)^{-1}\left(\sum_{j=1}^{n} A_j(u)\frac{\partial u}{\partial x_j} + F(u)\right) + \sum_{j=1}^{n} A_j^1(u)\frac{\partial u}{\partial x_j}. \tag{2.73}$$

Multiplying the equation (2.72) by $(E^0)^{-1}$ from left-hand side, we have

$$\frac{\partial u}{\partial t} = (E^0)^{-1}\sum_{j=1}^{n} A_j^0\frac{\partial u}{\partial x_j} + \tilde{F}_1(t), \tag{2.74}$$

where $\tilde{F}_1(t) = (E^0)^{-1}F_1(t)$. Now we put

$$\tilde{u}(t) = e^{-itH_0}u(t), \tag{2.75}$$

then we obtain

$$\frac{d}{dt}\tilde{u}(t) = e^{-itH_0}\tilde{F}_1(t). \tag{2.76}$$

Integrating the equation (2.76) over $t' \le \tau \le t$, we have

$$\tilde{u}(t) - \tilde{u}(t') = \int_{t'}^{t} e^{-i\tau H_0} \tilde{F}_1(\tau)\, d\tau. \tag{2.77}$$

Taking a norm in H^{l-1}, we find that

$$\|\tilde{u}(t) - \tilde{u}(t')\|_{H^{l-1}} \le \int_{t'}^{t} \|\tilde{F}_1(\tau)\|_{H^{l-1}}\, d\tau. \tag{2.78}$$

Combining (1.10), (2.73) and Lemma 1.3 (1), we deduce that

$$\begin{aligned}\|\tilde{F}_1(t)\|_{H^{l-1}} \le C\Bigg(&\|F(u)\|_{H^{l-1}} + \sum_{j=1}^{n} \left\| E^1(u)E(u)^{-1}A_j(u)\frac{\partial u}{\partial x_j}\right\|_{H^{l-1}} \\ &+ \|E^1(u)E(u)^{-1}F(u)\|_{H^{l-1}} + \sum_{j=1}^{n}\left\| A_j^1(u)\frac{\partial u}{\partial x_j}\right\|_{H^{l-1}}\Bigg), \\ \le C(&\|u\|_{L^\infty}^{2p}\|u\|_{H^l} + \|u\|_{L^\infty}^{p}\|u\|_{H^l}). \end{aligned} \tag{2.79}$$

Hence from (2.78) and (2.79) we see

$$\begin{aligned}\|\tilde{u}(t) - \tilde{u}(t')\|_{H^{l-1}} &\le \int_{t'}^{t} \|\tilde{F}_1(\tau)\|_{H^{l-1}}\, d\tau, \\ &\le C\int_{t'}^{t} \|u\|_{L^\infty}^{p}\|u\|_{H^l}(1 + \|u\|_{L^\infty}^{p})\, d\tau, \\ &\le C(\langle t\rangle^{-((n-1)/2)(p+1)} + \langle t'\rangle^{-((n-1)/2)(p+1)}) \quad \forall t, t' \ge 0, \end{aligned} \tag{2.80}$$

where $\frac{n-1}{2}p > 1$. Hence we find that $\{\tilde{u}(t)\}_{t\ge 0}$ is a Cauchy sequence in H^{l-1} and that there exists $\varphi_+ \in H^{l-1}$ such that

$$\|u(t) - e^{itH_0}\varphi_+\|_{H^{l-1}} \le C\langle t\rangle^{-(n-1)/2}, \quad \forall t \ge 0. \tag{2.81}$$

On the other hand, since $\{u(t)\}_{t\ge 0}$ is bounded in H^l, then $\{u(t)\}_{t\ge 0}$ involve $\{u(t_j)\}_{j=1}^{\infty}$ which weakly converges to φ_+ in H^l, that is, $u(t_j) \to \varphi_+$ (strong) in H^{l-1}, and $u(t_j) \to \varphi_+$ (weak) in H^l. Therefore we can see $\varphi_+ \in H^l$. Thus we complete the proof of Theorem 0.3. Q.E.D.

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