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Energy from the gauge invariant observables

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Abstract

For a classical solution $|\Psi\rangle$ in Witten's cubic string field theory, the gauge invariant observable $\langle I|\mathcal{V}|\Psi\rangle$ is conjectured to be equal to the difference of the one-point functions of the closed string state corresponding to $\mathcal{V}$, between the trivial vacuum and the one described by $|\Psi\rangle$. For a static solution $|\Psi\rangle$, if $\mathcal{V}$ is taken to be $c\bar{c}\partial X^0\bar{\partial}X^0$, the gauge invariant observable is expected to be proportional to the energy of $|\Psi\rangle$. We prove this relation assuming that $|\Psi\rangle$ satisfies equation of motion and some regularity conditions. We discuss how this relation can be applied to various solutions obtained recently.

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1 Introduction

A great variety of analytic classical solutions have been found for Witten's cubic string field theory [1], since the discovery of the analytic tachyon vacuum solution by Schnabl [2] [1]. In order to study the physical properties of these solutions, important gauge invariant quantities to be calculated are the energy and the gauge invariant observables $\langle I|\mathcal{V}(i)|\Psi\rangle$ discovered in [4, 5]. The gauge invariant observable is conjectured to coincide with the difference of the one-point functions of an on-shell closed string state between the trivial vacuum and the one described by the solution $|\Psi\rangle$ [6, 7].

What we would like to show in this paper is that energy can be expressed by using a gauge invariant observable. Namely, for a static solution $|\Psi\rangle$ of the equation of motion, the gauge invariant observable with[2]

$$\mathcal{V} = \frac{2}{\pi i} c\bar{c}\partial X^0 \bar{\partial} X^0 \,, \tag{1.1}$$

is proportional to the energy:

$$E = \frac{1}{g^2} \langle I|\mathcal{V}(i)|\Psi\rangle \ . \tag{1.2}$$

Here g is the coupling constant of the string field theory. Naively such a gauge invariant observable is proportional to the expectation value of the energy momentum tensor $\langle T_{00}\rangle$ and thus the energy of the system. Usually, the energy is more difficult to calculate compared with the gauge invariant observables. For most of the solutions obtained so far, both the energy and the gauge invariant observable are calculated and it turns out that the results are consistent with (1.2).

In this paper, we will prove that (1.2) holds if $|\Psi\rangle$ satisfies the equation of motion and some regularity conditions. The state-operator correspondence of the worldsheet theory implies that the string field $|\Psi\rangle$ can be expressed as

$$\mathcal{O}_\Psi |0\rangle \ ,$$

where $|0\rangle$ is the SL(2,$\mathbb{R}$) invariant vacuum and $\mathcal{O}_\Psi$ can be expressed in terms of local operators on the upper half plane. We will first discuss the case in which $\mathcal{O}_\Psi$ consists of local operators located away from the curve $|\xi| = 1$, where ξ is the complex coordinate on the upper half plane. As we will see, the proof of (1.2) is relatively easy in such a case. However, most of the solutions obtained since [2] do not satisfy this condition because they involve non-local operators such as K, B. Fortunately our method of proof can be refined to be applicable to such cases. We discuss applications of our results to the solutions obtained recently.

This paper is organized as follows. In section 2, we give a proof of the relation (1.2), assuming $|\Psi\rangle$ can be expressed using local operators. In section 3, we take Okawa type solutions [8, 9, 10] as an example and explain how we should generalize our method of proof to deal with solutions involving non-local operators K, B. In section 4 we apply our results to other solutions discovered recently. Section 5 is devoted to discussions.

[1] For a review on these solutions, see [3].

[2] Throughout this paper, we assume that the variable X^0 is described by the free worldsheet theory with the Neumann boundary condition.

2 A proof of (1.2) for local $\mathcal{O}_\Psi$

In this section, we consider the case in which $\mathcal{O}_\Psi$ is made from local operators located away from $|\xi| = 1$. We also assume that $\mathcal{O}_\Psi$ does not involve X^0 variable.

2.1 Open string field theory in a weak gravitational background

In order to derive (1.2), we start from considering the following modification of the string field action,

$$S_h = -\frac{1}{g^2}\left[\frac{1}{2}\langle\Psi|Q|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi * \Psi\rangle + h\,\langle I|\mathcal{V}(i)|\Psi\rangle\right], \tag{2.1}$$

with $h \ll 1$. It has been shown in [11] that such a string field action describes string theory in a closed string background, for general on-shell $\mathcal{V}$. The vertex operator $\mathcal{V}$ in (1.1) is a linear combination of those for the constant graviton and dilaton. Therefore the action (2.1) should be the open string field theory in a constant metric and dilaton background.

By a general coordinate transformation, the constant metric can be turned into the original $\eta_{\mu\nu}$. Therefore we expect that we can somehow transform the string field action (2.1) into the original string field action with some rescaling of the coupling constant g. In order to do so, we notice that as an operator acting on $\mathcal{O}_\Psi|0\rangle$, $\mathcal{V}$ can be expressed in a BRST exact form

$$\mathcal{V}(i) = \{Q, \chi\}, \tag{2.2}$$

where

$$\begin{aligned}
\chi \;&\equiv\; \lim_{\delta\to 0}\left[\int_{P_1}\frac{d\xi}{2\pi i}j\left(\xi,\bar\xi\right) - \int_{\bar P_1}\frac{d\bar\xi}{2\pi i}\bar j\left(\xi,\bar\xi\right) + \frac{c(1)}{2\pi\delta}\right], \\
& j\left(\xi,\bar\xi\right) \equiv 4\partial X^0(\xi)\,\bar c\bar\partial X^0\left(\bar\xi\right), \\
& \bar j\left(\xi,\bar\xi\right) \equiv 4\bar\partial X^0\left(\bar\xi\right)c\partial X^0(\xi).
\end{aligned} \tag{2.3}$$

Here P_1 is the path depicted in Fig 1 and along the arcs of the circle $|\xi| = 1$. Because of our assumption, the presence of $\mathcal{O}_\Psi$ does not affect the operators defined on such contours. Since $j, \bar j$ diverge in the limit $\mathrm{Im}\xi \to 0$, we have introduced $\delta > 0$ to regularize the divergence. One can check that the limit on the right hand side of (2.3) is not singular.

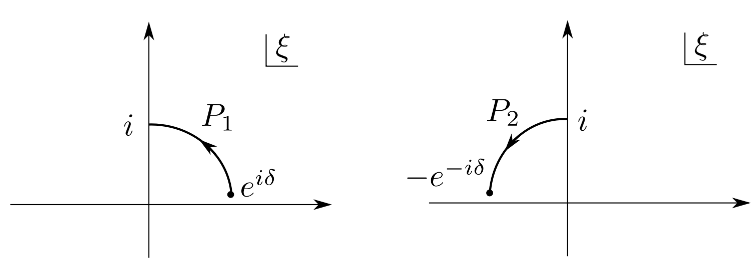

Figure 1: Contours P_1, P_2

(2.2) implies that in terms of the string field $|\Psi'\rangle$ defined as

$$|\Psi'\rangle \equiv |\Psi\rangle + h\chi |I\rangle \ , \tag{2.4}$$

the string field action S_h is expressed as

$$S_h = -\frac{1}{g^2}\left[\frac{1}{2}\langle\Psi'|Q'|\Psi'\rangle + \frac{1}{3}\langle\Psi'|\Psi' * \Psi'\rangle\right] + \mathcal{O}\left(h^2\right) \ , \tag{2.5}$$

with

$$Q' \ \equiv \ Q - h\left(\chi - \chi^\dagger\right) \ .$$

$\chi^\dagger$ denotes the BPZ conjugate of χ and

$$\begin{aligned}\chi - \chi^\dagger \ = \ \lim_{\delta\to 0}\Big[\int_{P_1+P_2}\frac{d\xi}{2\pi i}j\left(\xi,\bar\xi\right) - \int_{\bar P_1+\bar P_2}\frac{d\bar\xi}{2\pi i}\bar j\left(\xi,\bar\xi\right) \\ +\frac{c(1)}{2\pi\delta} - \frac{c(-1)}{2\pi\delta}\Big] \ ,\end{aligned}$$

where P_2 is the contour depicted in Fig. 1. We give the details of the definition of χ and the derivation of (2.2)(2.5) in appendix A.

Therefore the string field theory in the weak background is given by the cubic action with the modified BRST operator Q'. This string field theory is similar to the one considered in [12] as the open string field theory in the soft dilaton background. They have shown that the effect of such a background corresponds to a rescaling of the string coupling constant g. It is straightforward to generalize the techniques of [12] to our case. Let us define

$$\begin{aligned}\mathcal{G} \ &\equiv \ \lim_{\delta\to 0}\left[\int_{P_1+P_2}\frac{d\xi}{2\pi i}g_\xi\left(\xi,\bar\xi\right) - \int_{\bar P_1+\bar P_2}\frac{d\bar\xi}{2\pi i}g_{\bar\xi}\left(\xi,\bar\xi\right)\right] \ , \\ &g_\xi\left(\xi,\bar\xi\right) \equiv 2\left(X^0\left(\xi,\bar\xi\right) - X^0(i,-i)\right)\partial X^0(\xi) \ , \\ &g_{\bar\xi}\left(\xi,\bar\xi\right) \equiv 2\left(X^0\left(\xi,\bar\xi\right) - X^0(i,-i)\right)\bar\partial X^0\left(\bar\xi\right) \ .\end{aligned} \tag{2.6}$$

Because of the presence of $X^0(i,-i)$, $g_\xi, g_{\bar\xi}$ are well-defined operators on the worldsheet. Since $g_\xi, g_{\bar\xi}$ are singular at $\xi = i$, on the right hand side of (2.6) the integration contour is modified infinitesimally as in Figure 2. $g_\xi, g_{\bar\xi}$ are defined with the usual normal ordering prescription (C.2) and under a conformal transformation $\xi \to \xi'(\xi)$, g_ξ transforms as

$$g_{\xi'}\left(\xi',\bar\xi'\right) = \frac{\partial\xi}{\partial\xi'}g_\xi\left(\xi,\bar\xi\right) + \frac{1}{2}\partial_{\xi'}\ln\frac{\partial\xi}{\partial\xi'} \, . \tag{2.7}$$

It is straightforward to check that the limit on the right hand side of (2.6) is not singular.

Using (2.7) and the fact that $g_\xi, g_{\bar\xi}$ are singular at $\xi = i$, one can deduce the following identities:

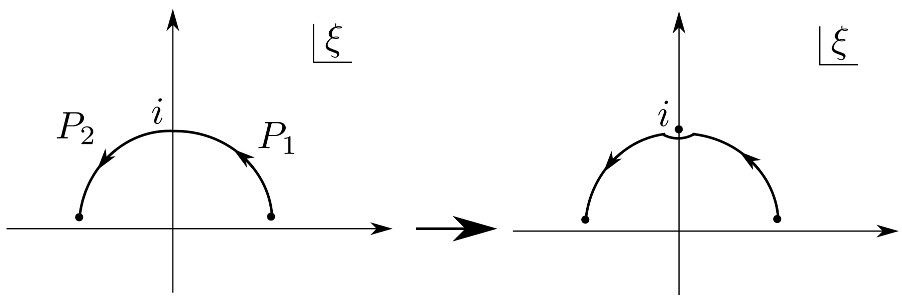

Figure 2: the contour to define $\mathcal{G}$

$$\langle \mathcal{G}\Psi_1|\Psi_2\rangle + \langle \Psi_1|\mathcal{G}\Psi_2\rangle = \langle \Psi_1|\Psi_2\rangle \ , \tag{2.8}$$

$$\langle \mathcal{G}\Psi_1|\Psi_2 * \Psi_3\rangle + \langle \Psi_1|\mathcal{G}\Psi_2 * \Psi_3\rangle + \langle \Psi_1|\Psi_2 * \mathcal{G}\Psi_3\rangle = \langle \Psi_1|\Psi_2 * \Psi_3\rangle \ . \tag{2.9}$$

As is explained in appendix A, it is also straightforward to get

$$[Q, \mathcal{G}] = \chi - \chi^\dagger \, . \tag{2.10}$$

Then, in terms of

$$|\Psi''\rangle \equiv (1 - h\mathcal{G})\, |\Psi'\rangle \ ,$$

S_h can be expressed as

$$S_h = -\frac{1+h}{g^2}\left[\frac{1}{2}\langle \Psi''|Q|\Psi''\rangle + \frac{1}{3}\langle \Psi''|\Psi'' * \Psi''\rangle\right] + \mathcal{O}\left(h^2\right) \ . \tag{2.11}$$

Thus S_h is proportional to the original string field theory action for the string field $|\Psi''\rangle$. By a field redefinition, the effect of the weak background is turned into a rescaling of the coupling constant g, due to the constant dilaton background. $\mathcal{G}$ can be regarded as the generator of general coordinate transformation.

2.2 Derivation of (1.2)

We can derive (1.2) from the two expressions (2.1)(2.11) of S_h. Suppose that $|\Psi\rangle$ is a static solution of the equation of motion

$$Q\,|\Psi\rangle + |\Psi * \Psi\rangle = 0 \, ,$$

and evaluate S_h using eqs.(2.1)(2.11). The right hand side of (2.1) can be expressed as

$$S_h = -E - \frac{h}{g^2}\langle I|\mathcal{V}\,(i)\,|\Psi\rangle \ ,$$

where $E = \frac{1}{g^2}\left[\frac{1}{2}\langle \Psi|Q|\Psi\rangle + \frac{1}{3}\langle \Psi|\Psi * \Psi\rangle\right]$ is the energy of the solution $|\Psi\rangle$. On the other hand, since $|\Psi''\rangle$ can be expressed as

$$|\Psi''\rangle = |\Psi\rangle + |\delta''\Psi\rangle \ ,$$

with $|\delta''\Psi\rangle \sim \mathcal{O}(h)$, the right hand side of (2.11) becomes

$$-\frac{1+h}{g^2}\left[\frac{1}{2}\langle\Psi|Q|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi*\Psi\rangle \right.$$
$$\left. + \langle\delta''\Psi|\left(Q|\Psi\rangle + |\Psi*\Psi\rangle\right)\right] + \mathcal{O}\left(h^2\right) .$$

Using the fact that $|\Psi\rangle$ is a solution of the equation of motion, one can see that (2.11) can be rewritten as

$$S_h = -(1+h)E + \mathcal{O}\left(h^2\right) .$$

Comparing these, we obtain

$$E = \frac{1}{g^2}\langle I|\mathcal{V}(i)|\Psi\rangle . \tag{2.12}$$

There is a more direct way to derive (2.12), which is essentially equivalent to the one above and will be used in the subsequent sections. From (2.8)(2.9), one can deduce

$$\begin{aligned} \frac{1}{3}\langle\Psi|\Psi*\Psi\rangle &= \langle\mathcal{G}\Psi|\Psi*\Psi\rangle , \\ \frac{1}{2}\langle\Psi|Q|\Psi\rangle &= \langle\mathcal{G}\Psi|Q|\Psi\rangle - \frac{1}{2}\langle\Psi|[Q,\mathcal{G}]|\Psi\rangle , \end{aligned} \tag{2.13}$$

and from (2.10) we get

$$[Q,\mathcal{G}]|\Psi\rangle = \left(\chi - \chi^\dagger\right)|\Psi\rangle . \tag{2.14}$$

Using these and the equation of motion, we obtain

$$\begin{aligned} E &= \frac{1}{g^2}\left[\frac{1}{2}\langle\Psi|Q|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi*\Psi\rangle\right] \\ &= \frac{1}{g^2}\left[\langle\mathcal{G}\Psi|\left\{Q|\Psi\rangle + |\Psi*\Psi\rangle\right\} - \frac{1}{2}\langle\Psi|[Q,\mathcal{G}]|\Psi\rangle\right] \\ &= -\frac{1}{2g^2}\langle\Psi|\left(\chi-\chi^\dagger\right)|\Psi\rangle \\ &= -\frac{1}{g^2}\langle I|\chi|\Psi*\Psi\rangle \\ &= \frac{1}{g^2}\langle I|\chi Q|\Psi\rangle \\ &= \frac{1}{g^2}\langle I|\mathcal{V}(i)|\Psi\rangle . \end{aligned} \tag{2.15}$$

Before closing this section, a few comments are in order:

- The vertex operator $\mathcal{V}$ is expressed in a BRST exact form (2.2), with χ being a completely legal operator. This fact may appear odd because it implies that all the amplitudes involving $\mathcal{V}$ vanish[3]. Actually (2.2) holds on the assumption that there exists

[3]This question was raised by M. Schnabl.

no operators around $\xi = 1$. In the derivation of (2.2) in appendix A, we use (A.5) which is valid only when such a condition is satisfied, which is the case in our setup. However, in calculating amplitudes, this is not guaranteed because of the existence of other vertex operators and (2.2) cannot be used in such a situation.

- It is also possible to use
$$\mathcal{V} = c\bar{c}\partial X^{\mu}\bar{\partial}X^{\nu}h_{\mu\nu}\,,$$
with $h^{\mu}_{\mu} = -1$ and derive (1.2), provided the variables X^{μ} are described by the free worldsheet theory with the Neumann boundary condition.

- Suppose that $|\Psi\rangle$ does not satisfy the equation of motion:
$$Q\,|\Psi\rangle + |\Psi * \Psi\rangle \equiv |\Gamma\rangle \neq 0\,. \tag{2.16}$$
It is easy to see that the relation (2.15) is modified as
$$E = \frac{1}{g^2}\,\langle I|\mathcal{V}\,(i)\,|\Psi\rangle - \frac{1}{g^2}\,\langle I|\,\chi\,|\Gamma\rangle + \frac{1}{g^2}\,\langle \mathcal{G}\Psi|\Gamma\rangle\;. \tag{2.17}$$

3 Derivation of (1.2) for Okawa type solutions

Most of the nontrivial solutions obtained so far are described by using operators K, B. These operators are given as integrations of T, b along the contours which intersects $P_1, \bar{P}_1, P_2, \bar{P}_2$ and do not commute with $g_{\xi}, g_{\bar{\xi}}$ to be used to define $\mathcal{G}$. In order to prove (1.2) for such $|\Psi\rangle$, we need to define the quantities which appear in the previous section in the presence of such operators. Moreover it is not so straightforward to prove (2.14) in such a setup.

In this section, as a prototype of such solutions, we consider the Okawa type solutions [8, 9, 10]
$$\Psi = F\,(K)\,c\frac{KB}{1-F\,(K)^2}cF\,(K)\;. \tag{3.1}$$
Here Ψ is expressed in terms of string fields K, B, c and the product of them is the star product[4]. Ψ gives a solution of the equation of motion if $F\,(K)\,, \frac{K}{1-F^2}$ are sufficiently regular functions of K. We will show that it is possible to define $\mathcal{G}$ which acts on such solutions and prove (2.13)(2.14) and derive (2.15).

It is assumed that $F\,(K)\,, \frac{K}{1-F^2}$ are given in a Laplace transformed form
$$\begin{aligned} F\,(K) &= \int_0^{\infty} dL e^{-LK} f\,(L)\;, \\ \frac{K}{1-F^2} &= \int_0^{\infty} dL e^{-LK} \tilde{f}\,(L)\;. \end{aligned}$$

[4]See [13, 14] for details.

Substituting these into (3.1), we obtain an expression of Ψ

$$\Psi = \int_0^\infty dL e^{-LK} \psi(L) \ , \tag{3.2}$$

where

$$\begin{aligned} \psi(L) &= \int dL_1 dL_2 dL_3 \delta(L - L_1 - L_2 - L_3) \\ &\quad \times c(L_2 + L_3) Bc(L_3) f(L_1) \tilde{f}(L_2) f(L_3) \ , \end{aligned} \tag{3.3}$$

and

$$c(z) = e^{zK} c e^{-zK} \ . \tag{3.4}$$

Ψ can be considered as the Laplace transform of ψ. We express (3.2) as

$$\Psi = \mathcal{L}\{\psi\} \ ,$$

where $\mathcal{L}$ denotes the operation of the Laplace transform. Then $\psi(L)$ is expressed as

$$\psi(L) = \mathcal{L}^{-1}\{\Psi\}(L) \ .$$

3.1 Definition of $\mathcal{G}$

Ψ is represented as a sum of wedge states with insertions $e^{-LK}\psi(L)$ as (3.2). In order to define $\mathcal{G}$ which acts on such Ψ, the contour to be used should depend on the length L of the wedge state. So we introduce

$$\begin{aligned} \mathcal{G}(L,\Lambda,\delta) &\equiv \lim_{z_0 \to i\infty} \left[\int_{P_{L,\Lambda,\delta}} \frac{dz}{2\pi i} g_z(z,\bar{z}) - \int_{\bar{P}_{L,\Lambda,\delta}} \frac{d\bar{z}}{2\pi i} g_{\bar{z}}(z,\bar{z}) \right] , \\ & g_z(z,\bar{z}) = 2\left(X^0(z,\bar{z}) - X^0(z_0,\bar{z}_0)\right)\partial X^0(z) \, , \\ & g_{\bar{z}}(z,\bar{z}) = 2\left(X^0(z,\bar{z}) - X^0(z_0,\bar{z}_0)\right)\bar{\partial} X^0(z) \, , \end{aligned}$$

and define $\mathcal{G}\Psi$ so that for any test state $|\phi\rangle = \phi(0)|0\rangle$, $\langle\phi|\mathcal{G}\Psi\rangle$ is given as

$$\langle\phi|\mathcal{G}\Psi\rangle = \lim_{(\Lambda,\delta)\to(\infty,0)} \int_0^\infty dL \left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi(0)\, e^{-\left(L+\frac{1}{2}\right)K} \mathcal{G}(L,\Lambda,\delta)\psi(L) \right\rangle_{C_{L+1}} \ . \tag{3.5}$$

Here $f(\xi) \equiv \frac{\pi}{2}\arctan\xi$, $\langle\cdots\rangle_{C_{L+1}}$ denotes the correlation function on the infinite cylinder C_{L+1} with circumference $L+1$ and $\phi(0)$ in the correlation function denotes the operator on C_{L+1} corresponding to $|\phi\rangle$, by abuse of notation. z which appears in the definition of $\mathcal{G}(L,\Lambda,\delta)$ is the complex coordinate on C_{L+1} such that $e^{-LK}\psi(L)$ corresponds to the region $0 \le \mathrm{Re} z \le L$. The contour $P_{L,\Lambda,\delta}$ is the one depicted in Figure (3), which consists of straight lines.

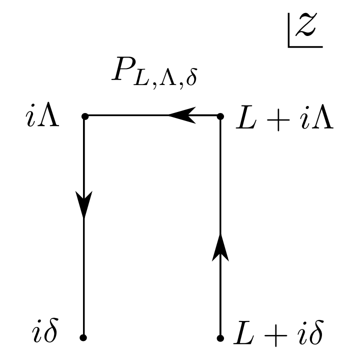

Figure 3: $P_{L,\Lambda,\delta}$

With $\mathcal{G}$ thus defined, we will prove the identity (2.9) assuming Ψ_i $(i=1,2,3)$ does not involve the X^0 variable. $\langle \mathcal{G}\Psi_1|\Psi_2 * \Psi_3\rangle$ is given as

$$\lim_{(\Lambda,\delta)\to(\infty,0)} \int_0^\infty dL_1 \int_0^\infty dL_2 \int_0^\infty dL_3 \\ \times \left\langle e^{(L_2+L_3)K} \mathcal{G}(L_1,\Lambda,\delta)\psi_1(L_1)e^{-L_2K}\psi_2(L_2)e^{-L_3K}\psi_3(L_3)\right\rangle_{C_{L_1+L_2+L_3}}, \tag{3.6}$$

in terms of the correlation function on the infinite cylinder $C_{L_1+L_2+L_3}$ with circumference $L_1+L_2+L_3$. Since ψ_i does not involve the X^0 variable, the correlation function on the right hand side of (3.6) is factorized as

$$\begin{aligned}&\left\langle e^{(L_2+L_3)K} \mathcal{G}(L_1,\Lambda,a)\psi_1(L_1)e^{-L_2K}\psi_2(L_2)e^{-L_3K}\psi_3(L_3)\right\rangle_{C_{L_1+L_2+L_3}} \\ &= \langle \mathcal{G}(L_1,\Lambda,a)\rangle^{X^0}_{C_{L_1+L_2+L_3}} \\ &\quad \times \left\langle e^{(L_2+L_3)K}\psi_1(L_1)e^{-L_2K}\psi_2(L_2)e^{-L_3K}\psi_3(L_3)\right\rangle_{C_{L_1+L_2+L_3}},\end{aligned} \tag{3.7}$$

where $\langle\cdots\rangle^{X^0}_{C_{L_1+L_2+L_3}}$ denotes the correlation function with respect to X^0 variable on $C_{L_1+L_2+L_3}$. The expectation value $\langle\mathcal{G}(L_1,\Lambda,\delta)\rangle^{X^0}_{C_{L_1+L_2+L_3}}$ can be calculated using (C.3). In the limit $\Lambda\to\infty, \delta\to 0$, we obtain

$$\lim_{(\Lambda,\delta)\to(\infty,0)} \langle\mathcal{G}(L_1,\Lambda,\delta)\rangle^{X^0}_{C_{L_1+L_2+L_3}} = \frac{L_1}{L_1+L_2+L_3}. \tag{3.8}$$

Therefore we get

$$\langle \mathcal{G}\Psi_1|\Psi_2 * \Psi_3\rangle = \int dL_1 dL_2 dL_3 \frac{L_1}{L_1+L_2+L_3} \\ \times \left\langle e^{(L_2+L_3)K}\psi_1(L_1)e^{-L_2K}\psi_2(L_2)e^{-L_3K}\psi_3(L_3)\right\rangle_{C_{L_1+L_2+L_3}}.$$

In the same way, we obtain

$$\begin{aligned}\langle\Psi_1|\mathcal{G}\Psi_2*\Psi_3\rangle &= \int dL_1dL_2dL_3\frac{L_2}{L_1+L_2+L_3}\\ &\quad\times\left\langle e^{(L_2+L_3)K}\psi_1(L_1)e^{-L_2K}\psi_2(L_2)e^{-L_3K}\psi_3(L_3)\right\rangle_{C_{L_1+L_2+L_3}},\\ \langle\Psi_1|\Psi_2*\mathcal{G}\Psi_3\rangle &= \int dL_1dL_2dL_3\frac{L_3}{L_1+L_2+L_3}\\ &\quad\times\left\langle e^{(L_2+L_3)K}\psi_1(L_1)e^{-L_2K}\psi_2(L_2)e^{-L_3K}\psi_3(L_3)\right\rangle_{C_{L_1+L_2+L_3}},\end{aligned}$$

and from these (2.9) is obvious. (2.8) can also be proved in a similar way.

3.2 (2.14) for Okawa type solutions

Since (2.8)(2.9) are satisfied, (2.13) can be deduced immediately. However, with the definition of $\mathcal{G}$ given in (3.5), proving (2.14) is not so straightforward. We elaborate on this here.

From the definition (3.5), we obtain

$$\begin{aligned}\langle\phi|\left[Q,\mathcal{G}\right]|\Psi\rangle &= \lim_{(\Lambda,\delta)\to(\infty,0)}\Bigg[\int_0^\infty dL\left\langle e^{\left(L+\frac{1}{2}\right)K}f\circ\phi\left(0\right)e^{-\left(L+\frac{1}{2}\right)K}Q\mathcal{G}(L,\Lambda,\delta)\psi\left(L\right)\right\rangle_{C_{L+1}}\\ &\qquad -\int_0^\infty dL\left\langle e^{\left(L+\frac{1}{2}\right)K}f\circ\phi\left(0\right)e^{-\left(L+\frac{1}{2}\right)K}\mathcal{G}(L,\Lambda,\delta)\mathcal{L}^{-1}\left\{Q\Psi\right\}\left(L\right)\right\rangle_{C_{L+1}}\Bigg]\\ &= \mathcal{A}_1+\mathcal{A}_2\,,\end{aligned}$$

where

$$\mathcal{A}_1 \equiv \lim_{(\Lambda,\delta)\to(\infty,0)}\int_0^\infty dL\left\langle e^{\left(L+\frac{1}{2}\right)K}f\circ\phi\left(0\right)e^{-\left(L+\frac{1}{2}\right)K}\left[Q,\mathcal{G}(L,\Lambda,\delta)\right]\psi\left(L\right)\right\rangle_{C_{L+1}}, \quad (3.9)$$

$$\begin{aligned}\mathcal{A}_2 &\equiv \lim_{(\Lambda,\delta)\to(\infty,0)}\int_0^\infty dL\Big\langle e^{\left(L+\frac{1}{2}\right)K}f\circ\phi\left(0\right)e^{-\left(L+\frac{1}{2}\right)K}\\ &\qquad\times\mathcal{G}(L,\Lambda,\delta)\left[Q\psi\left(L\right)-\mathcal{L}^{-1}\left\{Q\Psi\right\}\left(L\right)\right]\Big\rangle_{C_{L+1}}.\end{aligned} \quad (3.10)$$

Substituting (3.3) into (3.9), we obtain

$$\begin{aligned}\mathcal{A}_1 &= \int dL\int dL_1dL_2dL_3\delta\left(L-L_1-L_2-L_3\right)\\ &\quad\times f\left(L_1\right)\tilde{f}\left(L_2\right)f\left(L_3\right)\\ &\quad\times\left\langle e^{\left(L+\frac{1}{2}\right)K}f\circ\phi\left(0\right)e^{-\left(L+\frac{1}{2}\right)K}\left[Q,\mathcal{G}(L,\Lambda,\delta)\right]c\left(L_2+L_3\right)Bc\left(L_3\right)\right\rangle_{C_{L+1}}.\end{aligned} \quad (3.11)$$

The correlation function on the right hand side of (3.11) can be evaluated by plugging

$$
\begin{aligned}
[Q,\mathcal{G}(L,\Lambda,\delta)] &= \int_{P_{L,\Lambda,\delta}}\frac{dz}{2\pi i}4\partial X^0(z)\,\bar{c}\bar{\partial}X^0(\bar{z})-\int_{\bar{P}_{L,\Lambda,\delta}}\frac{d\bar{z}}{2\pi i}4\bar{\partial}X^0(\bar{z})\,c\partial X^0(z)\\
&\quad -2\left(c\partial X^0(i\infty)+\bar{c}\bar{\partial}X^0(-i\infty)\right)\left(\int_{P_{L,\Lambda,\delta}}\frac{dz}{2\pi i}\partial X^0(z)-\int_{\bar{P}_{L,\Lambda,\delta}}\frac{d\bar{z}}{2\pi i}\bar{\partial}X^0(\bar{z})\right)\\
&\quad +\int_{P_{L,\Lambda,\delta}}\frac{dz}{2\pi i}\frac{1}{2}\partial^2c-\int_{\bar{P}_{L,\Lambda,\delta}}\frac{d\bar{z}}{2\pi i}\frac{1}{2}\bar{\partial}^2\bar{c}\\
&\quad +\int_{P_{L,\Lambda,\delta}}dz\partial\kappa(z,\bar{z})+\int_{\bar{P}_{L,\Lambda,\delta}}d\bar{z}\bar{\partial}\kappa(z,\bar{z})\ , \qquad (3.12)\\
\kappa(z,\bar{z}) &\equiv \frac{1}{\pi i}\left(X^0(z,\bar{z})-X^0(i\infty,-i\infty)\right)\left(c\partial X^0(z)-\bar{c}\bar{\partial}X^0(\bar{z})\right)\ ,
\end{aligned}
$$

into it and rewriting the result in terms of the operator formalism. We need to take into account the fact that the correlation functions are defined with time ordering with respect to the time variable $\mathrm{Re}z$.

Since for $\mathrm{Im}z$, $\mathrm{Im}z'\sim\infty$,

$$
\begin{aligned}
\left\langle\partial X^0(z)\,\bar{\partial}X^0(\bar{z}')\right\rangle_{C_L} &\sim -2\left(\frac{\pi}{L}\right)^2\exp\left(\frac{2\pi i}{L}(z-\bar{z}')\right)\ ,\\
c(z) &\propto \exp\left(-\frac{2\pi i}{L}z\right)\ ,
\end{aligned}
$$

we can ignore the $\mathrm{Im}z=\Lambda$ part of the contours $P_{L,\Lambda,\delta},\bar{P}_{L,\Lambda,\delta}$ in the first and the second terms of (3.12), in the limit $\Lambda\to\infty$. One can see that the contributions from the terms on the second and the third lines of (3.12) vanish in the limit $\delta\to 0$, because of the boundary conditions of $X^0,c,\bar{c}$.

In calculating the contribution of the terms on the fourth line of (3.12), we need to be careful because the contours $P_{L,\Lambda,\delta},\bar{P}_{L,\Lambda,\delta}$ intersect the contour for $B=\int_{a-i\infty}^{a+i\infty}\frac{dz}{2\pi i}b$ as depicted in Fig. 4. We obtain

$$
\begin{aligned}
&\left\langle e^{\left(L+\frac{1}{2}\right)K}f\circ\phi(0)\,e^{-\left(L+\frac{1}{2}\right)K}\left(\int_{P_{L,\Lambda,\delta}}dz\partial\kappa(z,\bar{z})+\int_{\bar{P}_{L,\Lambda,\delta}}d\bar{z}\bar{\partial}\kappa(z,\bar{z})\right)c(L_2+L_3)\,Bc(L_3)\right\rangle_{C_{L+1}}\\
&\quad=-\mathrm{Tr}\Big[e^{-\frac{1}{2}K}f\circ\phi(0)\,e^{-\left(L+\frac{1}{2}\right)K}c(L_2+L_3)\,Bc(L_3)\,\kappa(i\delta,-i\delta)\\
&\qquad+e^{-\frac{1}{2}K}f\circ\phi(0)\,e^{-\left(L+\frac{1}{2}\right)K}\kappa(L_1+i\delta,L_1-i\delta)\,c(L_2+L_3)\,Bc(L_3)\\
&\qquad+e^{-\frac{1}{2}K}f\circ\phi(0)\,e^{-\left(L+\frac{1}{2}\right)K}c(L_2+L_3)\left\{B,\kappa(a+i\Lambda,a-i\Lambda)\right\}c(L_3)\Big]\ .
\end{aligned}
$$

Putting all these pieces together and taking the limit $\Lambda\to\infty$, we obtain

$$
\begin{aligned}
\mathcal{A}_1 &= \int dL\,\mathrm{Tr}\left[e^{-\frac{1}{2}K}f\circ\phi(0)\,e^{-\frac{1}{2}K}\left(\chi e^{-LK}\psi(L)+e^{-LK}\psi(L)\,\chi\right)\right]\\
&\quad+\int dL\frac{1}{L+1}\mathrm{Tr}\left[e^{-\frac{1}{2}K}f\circ\phi(0)\,e^{-\frac{1}{2}K}e^{-LK}\alpha(L)\right]\ , \qquad (3.13)
\end{aligned}
$$

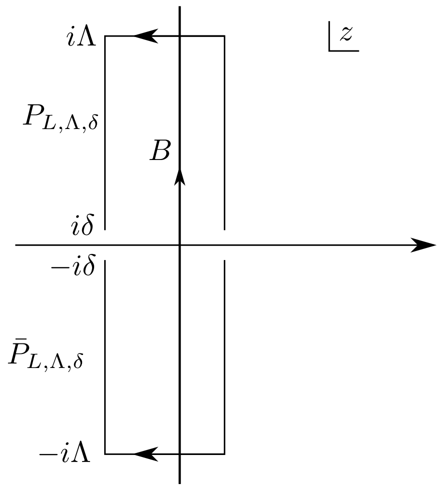

Figure 4: $P_{L,\Lambda,\delta}$ and B in $\mathcal{A}_1$

where $\alpha(L)$ is defined in (B.5) and χ here is given as

$$\chi = \lim_{(\Lambda,\delta)\to(\infty,0)} \left[\int_{i\delta}^{i\Lambda} \frac{dz}{2\pi i} 4\partial X^0(z)\, \bar{c}\bar{\partial} X^0(\bar{z}) - \int_{-i\delta}^{-i\Lambda} \frac{d\bar{z}}{2\pi i} 4\bar{\partial} X^0(\bar{z})\, c\partial X^0(z) + \frac{c(0)}{2\pi\delta} \right].$$

$\mathcal{A}_2$ is evaluated by substituting (B.7) into the right hand side of (3.10). Since $\mathcal{G}(0,\Lambda,\delta) = 0$,

$$\mathcal{A}_2 = \lim_{(\Lambda,\delta)\to(\infty,0)} \int_0^\infty dL \left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi(0)\, e^{-\left(L+\frac{1}{2}\right)K} \times \mathcal{G}(L,\Lambda,\delta) e^{LK} \partial_L \left(e^{-LK}\alpha(L)\right) \right\rangle_{C_{L+1}}. \quad (3.14)$$

The correlation function in the integrand can be rewritten as

$$\partial_t \left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi(0)\, e^{-\left(L+\frac{1}{2}\right)K} \mathcal{G}(L,\Lambda,\delta) e^{-tK} \alpha(L+t) \right\rangle_{C_{L+1}} \Bigg|_{t=0},$$

which can be evaluated in the limit $\Lambda\to\infty$, using (C.3) as

$$\partial_t \left[\frac{L}{L+t+1} \left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi(0)\, e^{-\left(L+\frac{1}{2}\right)K} \alpha(L+t) \right\rangle_{C_{L+t+1}} \right] \Bigg|_{t=0}$$
$$= \frac{L}{L+1} \left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi(0)\, e^{-\left(L+\frac{1}{2}\right)K} e^{LK} \partial_L \left(e^{-LK}\alpha(L)\right) \right\rangle_{C_{L+1}} - \frac{L}{(L+1)^2} \left\langle e^{\left(L+\frac{1}{2}\right)K} f\circ\phi(0)\, e^{-\left(L+\frac{1}{2}\right)K} \alpha(L) \right\rangle_{C_{L+1}}.$$

Substituting this into (3.14), we obtain

$$\begin{aligned}\mathcal{A}_2 &= \int_0^\infty dL \left\{ \frac{L}{L+1} \mathrm{Tr}\left[e^{-\frac{1}{2}K} f\circ\phi(0)\, e^{-\frac{1}{2}K} \partial_L \left(e^{-LK}\alpha(L)\right)\right] \right. \\ &\qquad \left. - \frac{L}{(L+1)^2} \mathrm{Tr}\left[e^{-\frac{1}{2}K} f\circ\phi(0)\, e^{-\frac{1}{2}K} e^{-LK}\alpha(L)\right] \right\} \\ &= -\int_0^\infty dL \frac{1}{L+1} \mathrm{Tr}\left[e^{-\frac{1}{2}K} f\circ\phi(0)\, e^{-\frac{1}{2}K} e^{-LK}\alpha(L)\right] .\end{aligned}$$

Putting these together, we get

$$\mathcal{A}_1 + \mathcal{A}_2 = \int dL \mathrm{Tr}\left[e^{-\frac{1}{2}K} f\circ\phi(0)\, e^{-\frac{1}{2}K} \left(\chi e^{-LK}\psi(L) + e^{-LK}\psi(L)\chi\right)\right] .$$

The right hand side coincides with $\langle\phi| \left(\chi - \chi^\dagger\right) |\Psi\rangle$ and thus we obtain

$$[Q, \mathcal{G}]\, |\Psi\rangle = \left(\chi - \chi^\dagger\right) |\Psi\rangle ,$$

in the setup in this section.

3.3 (1.2) for Okawa type solutions

With (2.14) established, it is straightforward to follow the procedure given in (2.15) and prove (1.2). In summary, we have proved (1.2) for Okawa type solutions Ψ assuming the following conditions:

- Ψ satisfies the equation of motion.
- $\alpha(\infty) = 0$ and $\alpha(0)$ is well-defined for $\alpha(L)$ defined in (B.5).

In addition to these, it is implicitly assumed that all the quantities which appear in the course of the calculations are finite[5]. Conditions other than the equation of motion are concerning the regularity of the solution. If the equation of motion is not satisfied, we obtain (2.17) with $|\Gamma\rangle$ given in (2.16).

4 Other solutions

We can use the method in the previous section and prove (1.2) for other types of solutions[6]. We will discuss BMT solution and Murata-Schnabl solution in the following.

[5] This is also assumed in section 2.

[6] Our results will not be useful for the marginal deformation solutions, for which it is trivial to calculate the energy, but may be relevant [15] in the context of the discussions in Ref. [16].

4.1 BMT solution

In [17], Bonora, Maccaferri, and Tolla construct solutions corresponding to relevant deformations of BCFT , called BMT solution[7]. They enlarged the K, B, c algebra by adding a relevant matter operator ϕ which satisfies

$$\begin{aligned}\lim_{s\to 0} s\phi\left(s\right)\phi\left(0\right) &= 0\,,\\ \left[c,\phi\right]=\left[B,\phi\right] &= 0\,,\\ Q\phi &= c\partial\phi+\partial c\delta\phi\,.\end{aligned}$$

The BMT solution is given as

$$\Psi = c\phi - \frac{1}{K+\phi}\left(\phi-\delta\phi\right)Bc\partial c\,. \tag{4.1}$$

In order to realize the lump solution, ϕ is usually taken to be the so-called Witten deformation

$$\phi\left(s\right) = u\left(\frac{1}{2}:X^2:\left(s\right)+\gamma-1+\ln\left(2\pi u\right)\right)\,,$$

or the cosine deformation

$$\phi(s) = u\left[-u^{-1/R^2}:\cos\left(\frac{1}{R}X\right):(s)+A(R)\right]\,.$$

Here X direction is noncompact for Witten deformation and a circle of radius $R>\sqrt{2}$ for the cosine deformation. $A\left(R\right)$ is a constant determined in [17].

If one tries to define the $\frac{1}{K+\phi}$ which appears in the BMT solution as

$$\frac{1}{K+\phi} \equiv \int_0^\infty dte^{-t(K+\phi)}\,,$$

via the Schwinger parametrization, the integral on the right hand side diverges because $\lim_{t\to\infty} e^{-t(K+\phi)}$ coincides with the deformed sliver state $\tilde{\Omega}^\infty$. One way to regularize the divergence is to replace $\frac{1}{K+\Phi}$ by $\frac{1}{K+\phi+\epsilon}$ with $1\gg\epsilon>0$ and consider

$$\Psi_\epsilon = c\phi - \frac{1}{K+\phi+\epsilon}\left(\phi-\delta\phi\right)Bc\partial c\,,$$

but Ψ_ϵ suffers from an anomaly in equation of motion [19]:

$$Q\Psi_\epsilon+\Psi_\epsilon^2 = \Gamma_\epsilon \equiv \frac{\epsilon}{K+\phi+\epsilon}\left(\phi-\delta\phi\right)c\partial c\,.$$

In [20], the authors propose a way to deal with the problem using the distribution theory.

[7] An earlier proposal for such solutions were made in [18]

It is quite easy to compute the gauge invariant observables for the BMT solution, but it is much more difficult to calculate the energy. Our method can be used to improve the situation a bit. In [21, 19], the authors define a solution

$$\Psi'_\epsilon = c\left(\phi+\epsilon\right) - \frac{1}{K+\phi+\epsilon}\left(\phi-\delta\phi+\epsilon\right)Bc\partial c\,,$$

as a possible regularization of the BMT solution, but it actually describes the tachyon vacuum. It is shown that if the energy of the solution Ψ'_ϵ is that of the tachyon vacuum, one can prove analytically that the energy of the BMT solution coincides with that of the lump solution [19, 21]. The gauge invariant observables of Ψ'_ϵ can be calculated analytically, which turn out to be equal to those of the tachyon vacuum but the energy is calculated only numerically [21, 22] in the case of the Witten deformation. We will use our method to calculate the energy of Ψ'_ϵ. It is quite straightforward to generalize the calculations in the previous section to Ψ'_ϵ, starting from the Laplace transformed form

$$\begin{aligned} \Psi'_\epsilon &= \int_0^\infty dL e^{-LK}\psi'_\epsilon\left(L\right)\,, \\ \psi'_\epsilon\left(L\right) &= \delta\left(L\right)c\left(\phi+\epsilon\right) - e^{-\epsilon L-\int_0^L ds\phi(s)}\left(\phi-\delta\phi+\epsilon\right)Bc\partial c\,, \end{aligned}$$

where

$$\phi\left(s\right) = e^{sK}\phi e^{-sK}\,,$$

and the operators are time ordered. For Ψ'_ϵ, one can obtain

$$\mathcal{L}^{-1}\left\{Q\Psi'_\epsilon\right\}\left(L\right) = Q\mathcal{L}^{-1}\left\{\Psi'_\epsilon\right\}\left(L\right) - e^{LK}\partial_L\left(e^{-LK}\alpha'_\epsilon\left(L\right)\right) - \delta\left(L\right)\alpha'_\epsilon\left(0\right)\,,$$

with

$$\alpha'_\epsilon\left(L\right) = e^{-\epsilon L-\int_0^L ds\phi(s)}\left(\phi-\delta\phi\right)c\partial c\,.$$

Since $\alpha'_\epsilon\left(\infty\right)=0$ and $\alpha'_\epsilon\left(0\right)$ is well-defined, all the manipulations in the previous section are valid provided that we do not encounter any divergences in the course of the calculations. In the case of the Witten deformation, there exist divergences coming from noncompactness of the direction corresponding to X and our method is not applicable. Ψ'_ϵ corresponding to the cosine deformation does not seem to have such a problem[8] and we can see that the energy coincides with that of the tachyon vacuum.

It may be possible to calculate the energy of Ψ_ϵ directly for the cosine deformation. Since Ψ_ϵ has an anomaly in equation of motion, we need to evaluate the second and the third terms of (2.17). In order to do so, we need to know the IR behavior of some correlation functions of ϕ.

[8] The partition function

$$g\left(uT\right) \equiv \mathrm{Tr}e^{-T(K+\phi)}\,,$$

can be calculated perturbatively [23] and is finite for $0 \leq uT < \infty$. The UV and IR behaviors of the correlation functions of ϕ's are harmless.

4.2 Murata-Schnabl solution

Murata and Schnabl [24, 25] propose that the Okawa type solution (3.1) with

$$\begin{aligned} G(K) &\equiv 1 - F^2(K) \\ &= \left(\frac{K+1}{K}\right)^{N-1}, \end{aligned} \tag{4.2}$$

corresponds to a configuration with N D-branes. Since the solution itself is singular for $N \neq 0, 1$, the authors need some regularization in calculating various quantities. In order to define the gauge invariant observables, they replace K by $K + \epsilon$ ($\epsilon \ll 1$) and consider

$$F(K+\epsilon)\, c \frac{B(K+\epsilon)}{1 - F^2(K+\epsilon)} cF(K+\epsilon)\ .$$

or its gauge equivalent

$$\Psi_\epsilon = F^2(K+\epsilon)\, cB \frac{K+\epsilon}{1 - F^2(K+\epsilon)} c\,.$$

The energy of the solution is calculated using a different way to regularize the divergence. They obtain the energy and the gauge invariant observables which coincide with those for N D-branes. It is necessary to find a more solid way to define the solution, and there are many attempts to rectify the situation [26, 27, 28, 29, 30].

As an application of our results, let us calculate the energy of Ψ_ϵ in this paper. Since Ψ_ϵ has an anomaly in equation of motion,

$$Q\Psi_\epsilon + \Psi_\epsilon^2 = \Gamma_\epsilon\,,$$

where

$$\begin{aligned} \Gamma_\epsilon &= \epsilon\,(1 - G_\epsilon(K))\, c \frac{K+\epsilon}{G_\epsilon(K)} c\,, \\ G_\epsilon(K) &\equiv G(K+\epsilon)\ , \end{aligned}$$

the relation we have is

$$E = \frac{1}{g^2}\left[\langle I|\mathcal{V}(i)\,|\Psi_\epsilon\rangle - \langle I|\,\chi\,|\Gamma_\epsilon\rangle + \langle \mathcal{G}\Psi_\epsilon|\Gamma_\epsilon\rangle\right]\ , \tag{4.3}$$

which can be proved as in the previous section. After some calculations, details of which are presented in appendix D, we obtain in the limit $\epsilon \to 0$

$$\begin{aligned} \langle I|\mathcal{V}(i)\,|\Psi_\epsilon\rangle &= \frac{N-1}{2\pi^2} \\ \langle I|\,\chi\,|\Gamma_\epsilon\rangle &\to R_N\,, && (4.4) \\ \langle \mathcal{G}\Psi_\epsilon|\Gamma_\epsilon\rangle &\to 0\,, && (4.5) \end{aligned}$$

where

$$R_N \equiv \begin{cases} -\frac{i}{8\pi^3}\sum_{k=0}^{N-2}\frac{N!}{k!(k+2)!(N-2-k)!}\left((2\pi i)^{k+2}-(-2\pi i)^{k+2}\right) & ,\ (N\geq 1)\ , \\ \frac{i}{8\pi^3}\sum_{k=0}^{-N-1}\frac{(1-N)!}{k!(k+2)!(-N-1-k)!}\left((2\pi i)^{k+2}-(-2\pi i)^{k+2}\right) & ,\ (N\leq 0)\ . \end{cases}$$

Therefore we get the energy

$$E=\frac{1}{g^2}\left(\frac{N-1}{2\pi^2}-R_N\right)\ .$$

This coincides with the desired value $\frac{N-1}{2\pi^2}$ for $N=-1,0,1,2$. Thus, for these N, the anomaly Γ_ϵ is harmless at least in the calculation of energy, although we do not know the reason why this is so for $N=-1,2$ [9].

5 Conclusion and discussion

In this paper, we present a way to show that the energy is proportional to a gauge invariant observable, which corresponds to the graviton one point function, for a classical solution in Witten's cubic open string field theory. We give a method which can be used to show this even for the solutions which involve K,B. Usually the gauge invariant observables are much easier to calculate compared with the energy. In a recent paper [31], it is found that the boundary states can also be constructed from the gauge invariant observables. Therefore now we possess a more efficient way to study the physical properties of solutions which have been or will be discovered.

Recently in [30] the authors propose several new types of solutions made from K,B,c. It seems that our method can be applied to these solutions and derive (1.2) if the solutions are sufficiently regular. One particularly interesting solution mentioned in [30] is the one due to Masuda, which is claimed to have the energy of the double brane configuration but the gauge invariant observables of the perturbative vacuum. It would be intriguing to check how our derivation of (1.2) fails for this solution.

Interrelationship between energy and the gauge invariant observable will be important in exploring various aspects of string fields. For example, in the case of the BMT solution, the calculation of gauge invariant observables reduces to the integral of total derivative. This implies that these gauge invariant observables may have some topological nature. On the other hand, in [27], the energy is interpreted to be the winding number in string field theory. Our results may shed some light on the study of the topological invariants of the space of string fields.

[9] $N=-1,2$ may be argued to be special in the following sense. Ψ_ε is gauge equivalent to

$$\epsilon\left(\frac{1}{G_\epsilon}-1\right)cBG_\epsilon c\,,$$

which is regular in the limit $\epsilon\to 0$ for $N=-1,2$. Another reason for $N=-1$ may be because there exists a regular solution [30].

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A Derivations of (2.2), (2.5) and (2.10)

Since the quantities which appear in section 2 involve unusual combinations of operators, some explanation is necessary about the definitions and the treatment of them. In this appendix, we present the details of the definition of $\chi, \mathcal{G}$ and the derivation of (2.2)(2.5)(2.10).

$\{Q, \chi\} = \mathcal{V}(i, -i)$

Introducing θ such that $\xi = e^{i\theta}$, the contour integral on the right hand side of (2.3) is expressed as

$$\begin{aligned}&\int_{P_1} \frac{d\xi}{2\pi i} j(\xi, \bar{\xi}) - \int_{\bar{P}_1} \frac{d\bar{\xi}}{2\pi i} \bar{j}(\xi, \bar{\xi}) \\ &\quad = \int_{\delta}^{\frac{\pi}{2}} \frac{d\theta}{2\pi i} i e^{i\theta} j\left(e^{i\theta}, e^{-i\theta}\right) - \int_{\delta}^{\frac{\pi}{2}} \frac{d\theta}{2\pi i} \left(-i e^{-i\theta}\right) \bar{j}\left(e^{i\theta}, e^{-i\theta}\right) . \end{aligned} \tag{A.1}$$

In calculating the BRST variation of this quantity, it is useful to notice

$$\frac{1}{2\pi i} j(\xi, \bar{\xi}) = \oint_{\xi} \frac{d\xi'}{2\pi i} b(\xi') \mathcal{V}(\xi, \bar{\xi}) , \tag{A.2}$$

$$-\frac{1}{2\pi i} \bar{j}(\xi, \bar{\xi}) = \oint_{\bar{\xi}} \frac{d\bar{\xi}'}{2\pi i} \bar{b}(\bar{\xi}') \mathcal{V}(\xi, \bar{\xi}) , \tag{A.3}$$

where $\mathcal{V}(\xi, \bar{\xi})$ is the vertex operator defined in (1.1). Since $\mathcal{V}$ is BRST invariant, it is straightforward to show

$$\begin{aligned}&\left\{Q, \int_{P_1} \frac{d\xi}{2\pi i} j(\xi, \bar{\xi}) - \int_{\bar{P}_1} \frac{d\bar{\xi}}{2\pi i} \bar{j}(\xi, \bar{\xi})\right\} \\ &\quad = \int_{\delta}^{\frac{\pi}{2}} d\theta \left(\frac{de^{i\theta}}{d\theta} \partial_{\xi} \mathcal{V}\left(e^{i\theta}, e^{-i\theta}\right) + \frac{de^{-i\theta}}{d\theta} \partial_{\bar{\xi}} \mathcal{V}\left(e^{i\theta}, e^{-i\theta}\right)\right) \\ &\quad = \mathcal{V}(i, -i) - \mathcal{V}\left(e^{i\delta}, e^{-i\delta}\right) . \end{aligned} \tag{A.4}$$

Assuming that there are no other operators around $\xi = 1$, the OPE's of $c, \bar{c}, X^0$ imply

$$\mathcal{V}\left(e^{i\delta}, e^{-i\delta}\right) = \frac{c\partial c(1)}{2\pi\delta} + \mathcal{O}(\delta) = \left\{Q, \frac{c(1)}{2\pi\delta}\right\} + \mathcal{O}(\delta) , \tag{A.5}$$

for $\delta \sim 0$. The assumption is valid in the setup of this paper. Using (A.5), we obtain

$$\{Q, \chi\} = \mathcal{V}(i, -i) \ .$$

It is possible to generalize our construction here to other closed string vertex operators. For any BRST invariant closed string vertex operator $\mathcal{V}\left(\xi, \bar{\xi}\right)$, one can define $j, \bar{j}$ as in (A.2)(A.3), and one can prove (A.4). If $\mathcal{V}\left(e^{i\delta}, e^{-i\delta}\right)$ can be expressed as

$$\mathcal{V}\left(e^{i\delta}, e^{-i\delta}\right) = \{Q, \mathcal{U}\} + \mathcal{O}(\delta) \ , \tag{A.6}$$

in the limit $\delta \to 0$ as in (A.5), we obtain $\mathcal{V}(i, -i) = \{Q, \chi\}$ with

$$\chi \equiv \lim_{\delta \to 0} \left[\int_{P_1} \frac{d\xi}{2\pi i} j\left(\xi, \bar{\xi}\right) - \int_{\bar{P}_1} \frac{d\bar{\xi}}{2\pi i} \bar{j}\left(\xi, \bar{\xi}\right) + \mathcal{U} \right] .$$

(A.6) holds if there exists no on-shell open string vertex operator V_o such that

$$\langle \mathcal{V} V_o \rangle_{\mathrm{disk}} \neq 0 \, .$$

(2.5)

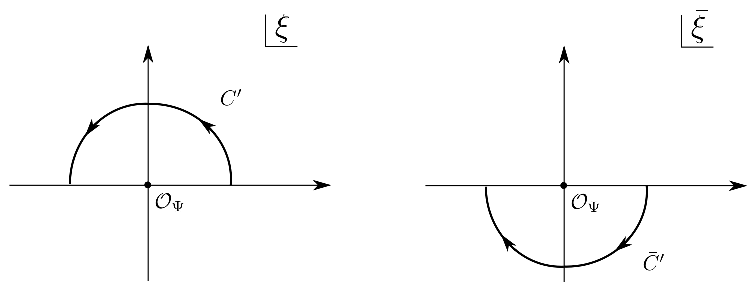

Figure 5: C'

Substituting (2.4) into (2.1), we obtain

$$\begin{aligned} S_h &= -\frac{1}{g^2} \left[\frac{1}{2} \langle \Psi' | Q | \Psi' \rangle + \frac{1}{3} \langle \Psi' | \Psi' * \Psi' \rangle + h \langle I | \mathcal{V}(i) | \Psi \rangle \right. \\ &\qquad \left. - h \langle I | \chi Q | \Psi \rangle - \frac{h}{2} \langle \Psi' | \left(\chi - \chi^\dagger \right) | \Psi' \rangle \right] , \end{aligned}$$

where we have used

$$\begin{aligned} \chi | I \rangle &= \chi^\dagger | I \rangle \ , \\ \langle \Psi | \chi | \Psi \rangle &= - \langle \Psi | \chi^\dagger | \Psi \rangle \ . \end{aligned}$$

Since $Q|I\rangle = 0$,

$$\langle I|\chi Q|\Psi\rangle = \langle I|\{Q,\chi\}|\Psi\rangle \ ,$$

and we may be able to use (2.2) to show (2.5). We should check if the Q in the open string field action yields the BRST variation of χ as an operator in the bulk. The BRST operator acting on a string field $|\Psi\rangle = \mathcal{O}_\Psi(0)|0\rangle$ is given as

$$Q|\Psi\rangle = \left(\int_{C'} \frac{d\xi}{2\pi i} J_{\rm B} - \int_{\bar{C}'} \frac{d\bar{\xi}}{2\pi i} \bar{J}_{\rm B}\right) \mathcal{O}_\Psi(0)|0\rangle \ ,$$

where $J_{\rm B}, \bar{J}_{\rm B}$ are the BRST current and $C', \bar{C}'$ are depicted in the figure 5. Since $J_{\rm B}(\xi) = \bar{J}_{\rm B}(\bar{\xi})$ for real ξ the contour integral can be expressed as

$$\oint_0 \frac{d\xi}{2\pi i} J_{\rm B} \,,$$

on the doubled Riemann surface. $(Q\chi(i,-i) + \chi(i,-i)Q)|\Psi\rangle$ in the open string field theory is given as

$$\left(\oint_{C''} \frac{d\xi}{2\pi i} J_{\rm B} - \oint_{\bar{C}''} \frac{d\bar{\xi}}{2\pi i} \bar{J}_{\rm B}\right) \chi(\xi,\bar{\xi})\, \mathcal{O}_\psi|0\rangle \ ,$$

where the contours $C'', \bar{C}''$ are the one which surrounds $P_1\bar{P}_1$ as depicted in figure 6. Hence the contour integral yields the BRST variation of χ and we obtain $\mathcal{V}(i,-i)|\Psi\rangle$.

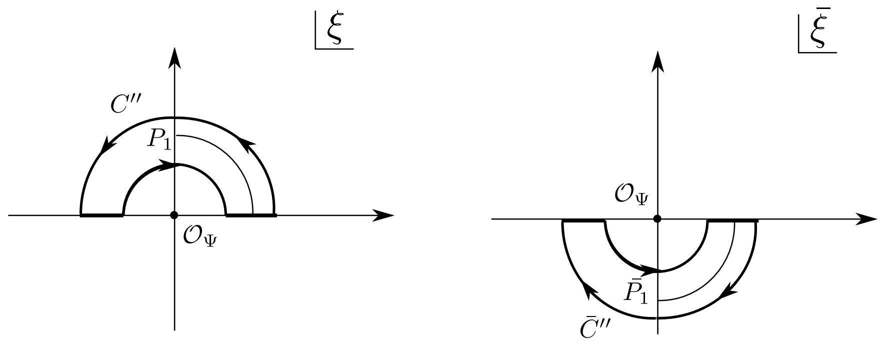

Figure 6: Contour which surrounds P_1

$\{Q,\mathcal{G}\} = \chi - \chi^\dagger$

The contour integral on the right hand side of (2.6) is defined in the same way as in (A.1). It is straightforward to calculate the BRST variations of $g_\xi, g_{\bar{\xi}}$ as

$$\begin{aligned}
\left[Q, g_\xi(\xi,\bar{\xi})\right] &= \frac{1}{2}\partial^2 c(\xi) + \partial_\xi\left(2\left(X^0(\xi,\bar{\xi}) - X^0(i,-i)\right)c\partial X^0(\xi)\right) \\
&\quad +2\bar{c}\bar{\partial}X^0\partial X^0(\xi,\bar{\xi}) - 2\left(c\partial X^0(i) + \bar{c}\bar{\partial}X^0(-i)\right)\partial X^0(\xi) \ , \\
\left[Q, g_{\bar{\xi}}(\xi,\bar{\xi})\right] &= \frac{1}{2}\bar{\partial}^2 \bar{c}(\bar{\xi}) + \partial_{\bar{\xi}}\left(2\left(X^0(\xi,\bar{\xi}) - X^0(i,-i)\right)\bar{c}\bar{\partial} X^0(\bar{\xi})\right) \\
&\quad +2c\partial X^0\bar{\partial}X^0(\xi,\bar{\xi}) - 2\left(c\partial X^0(i) + \bar{c}\bar{\partial}X^0(-i)\right)\bar{\partial}X^0(\bar{\xi}) \ ,
\end{aligned}$$

and we find $[Q,\mathcal{G}]$ is equal to

$$\lim_{\delta\to 0}\left[\frac{1}{4\pi i}\left(\partial c\left(-e^{-i\delta}\right)-\bar{\partial}c\left(-e^{i\delta}\right)-\partial c\left(e^{i\delta}\right)+\bar{\partial}\bar{c}\left(e^{-i\delta}\right)\right)\right.$$
$$+\frac{1}{2\pi i}\left(\int d\xi\partial_{\xi}+\int d\bar{\xi}\partial_{\bar{\xi}}\right)\left(2\left(X^{0}\left(\xi,\bar{\xi}\right)-X^{0}\left(i,-i\right)\right)c\partial X^{0}\left(\xi\right)\right)$$
$$-\frac{1}{2\pi i}\left(\int d\xi\partial_{\xi}+\int d\bar{\xi}\partial_{\bar{\xi}}\right)\left(2\left(X^{0}\left(\xi,\bar{\xi}\right)-X^{0}\left(i,-i\right)\right)\bar{c}\bar{\partial}X^{0}\left(\bar{\xi}\right)\right)$$
$$+\int_{P_1+P_2}\frac{d\xi}{2\pi i}4\partial X^{0}\bar{c}\bar{\partial}X^{0}\left(\xi,\bar{\xi}\right)-\int_{\bar{P}_1+\bar{P}_2}\frac{d\bar{\xi}}{2\pi i}4\bar{\partial}X^{0}c\partial X^{0}\left(\xi,\bar{\xi}\right)$$
$$\left.-2\left(c\partial X^{0}\left(i\right)+\bar{c}\bar{\partial}X^{0}\left(-i\right)\right)\left(\int_{P_1+P_2}\frac{d\xi}{2\pi i}\partial X^{0}\left(\xi\right)-\int_{\bar{P}_1+\bar{P}_2}\frac{d\bar{\xi}}{2\pi i}\bar{\partial}X^{0}\left(\bar{\xi}\right)\right)\right].$$

The terms on the first line cancel with each other in the limit $\delta\to 0$ because of the boundary conditions of $c,\bar{c}$. Those on the fifth vanish if $\mathcal{O}_{\Psi}$ does not involve X^{0}. The second and the third lines yield in the limit $\delta\to 0$

$$\frac{1}{\pi i}\left(X^{0}\left(\xi,\bar{\xi}\right)-X^{0}\left(i,-i\right)\right)\left(c\partial X^{0}\left(\xi\right)-\bar{c}\bar{\partial}X^{0}\left(\bar{\xi}\right)\right)\Big|_{\left(\xi,\bar{\xi}\right)=\left(e^{i\delta},e^{-i\delta}\right)}^{\left(-e^{-i\delta},-e^{i\delta}\right)}$$
$$\sim-\frac{c\left(-1\right)}{2\pi\delta}+\frac{c\left(1\right)}{2\pi\delta}.$$

Thus we get

$$[Q,\mathcal{G}]=\chi-\chi^{\dagger}.$$

B Laplace transformed form of the string field

We derive two formulas (B.1) (B.7) concerning the Laplace transform of the string field defined in section 3.

For two string fields A_1,A_2, which can be expressed as a sum of wedge states with insertions, it is easy to show

$$\mathcal{L}^{-1}\left\{A_1A_2\right\}\left(L\right)=\int_0^L dL'e^{L'K}\mathcal{L}^{-1}\left\{A_1\right\}\left(L-L'\right)e^{-L'K}\mathcal{L}^{-1}\left\{A_2\right\}\left(L'\right). \tag{B.1}$$

The right hand side can be regarded as an operator version of convolution.

For $\psi\left(L\right)$ in (3.3),

$$\begin{aligned}Q\psi\left(L\right) &= Q\mathcal{L}^{-1}\left\{\Psi\right\}\left(L\right)\\ &= \int dL_1dL_2dL_3\delta\left(L-L_1-L_2-L_3\right)\\ &\quad\times\left[c\partial c\left(L_2+L_3\right)Bc\left(L_3\right)-c\left(L_2+L_3\right)Kc\left(L_3\right)+c\left(L_2+L_3\right)Bc\partial c\left(L_3\right)\right]\\ &\quad\times f\left(L_1\right)\tilde{f}\left(L_2\right)f\left(L_3\right),\end{aligned} \tag{B.2}$$

which is not equal to

$$\begin{aligned}
\mathcal{L}^{-1}\{Q\Psi\}(L) &= \int dL_1 dL_2 dL_3 \delta(L - L_1 - L_2 - L_3) \\
&\quad \times \Bigg[\{\partial c(L_2+L_3) Bc(L_3) + c(L_2+L_3) Bc\partial c(L_3)\} \\
&\quad\quad \times f(L_1) \tilde{f}(L_2) f(L_3) \\
&\quad\quad - c(L_2+L_3) c(L_3) f(L_1) \mathcal{L}^{-1}\left\{\frac{K^2}{1-F^2}\right\}(L_2) f(L_3)\Bigg] . \quad \text{(B.3)}
\end{aligned}$$

Therefore the BRST transformation and $\mathcal{L}^{-1}$ do not commute with each other. Comparing (B.2) and (B.3), assuming $\alpha(0) = \alpha(\infty) = 0$, we obtain

$$\mathcal{L}^{-1}\{Q\Psi\}(L) = Q\mathcal{L}^{-1}\{\Psi\}(L) - e^{LK}\partial_L\left(e^{-LK}\alpha(L)\right) , \tag{B.4}$$

where

$$\alpha(L) \equiv \mathcal{L}^{-1}\left\{Fc\frac{K}{1-F^2}cF\right\}(L) . \tag{B.5}$$

We expect $\alpha(\infty) = 0$ for regular solutions. $\alpha(0)$ is related to the behavior of $F(K), \frac{K}{1-F^2}$ for $K \sim \infty$ and may not vanish even if Ψ is regular. For example, the Erler-Schnabl solution [13] has

$$\begin{aligned}
f(L) &= \frac{1}{\Gamma\left(\frac{1}{2}\right)} L^{-\frac{1}{2}} e^{-L} , \\
\alpha(L) &= e^{-L}\frac{1}{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}\int_0^L dL' (L-L')^{-\frac{1}{2}} L'^{-\frac{1}{2}} c\partial c(L') ,
\end{aligned}$$

and

$$\alpha(0) = c\partial c(0) ,$$

With $\alpha(0) \neq 0$, (B.4) cannot be valid for such solutions.

In order to get an identity similar to (B.4) for the solutions with $\alpha(\infty) = 0, \alpha(0) \neq 0$, we regularize Ψ and consider

$$\Psi_\eta \equiv F(K) e^{-\eta K} c \frac{BK}{1-F^2(K)} e^{-\eta K} cF(K) e^{-\eta K} ,$$

for $\eta > 0$. Ψ_η coincides with the original one in the limit $\eta \to 0$ and

$$\begin{aligned}
\mathcal{L}^{-1}\{\Psi_\eta\}(L) &= \int dL_1 dL_2 dL_3 \delta(L - L_1 - L_2 - L_3) \\
&\quad \times c(L_2+L_3) Bc(L_3) \mathcal{L}^{-1}\{F_\eta\}(L_1) \mathcal{L}^{-1}\left\{\tilde{F}_\eta\right\}(L_2) \mathcal{L}^{-1}\{F_\eta\}(L_3) ,
\end{aligned}$$

where

$$\begin{aligned} F_\eta(K) &\equiv F(K)\, e^{-\eta K}\,, \\ \tilde{F}_\eta(K) &\equiv \frac{K}{1-F^2(K)} e^{-\eta K}\,. \end{aligned}$$

$\mathcal{L}^{-1}\{F_\eta\}(L), \mathcal{L}^{-1}\left\{\tilde{F}_\eta\right\}(L)$ vanish for $L < \eta$ and we do not encounter any problem in deriving

$$\mathcal{L}^{-1}\{Q\Psi_\eta\}(L) = Q\mathcal{L}^{-1}\{\Psi_\eta\}(L) - e^{LK}\partial_L\left(e^{-LK}\alpha_\eta(L)\right)\,, \tag{B.6}$$

where

$$\alpha_\eta(L) \equiv \mathcal{L}^{-1}\left\{F_\eta c \tilde{F}_\eta c F_\eta\right\}(L)\,.$$

$\alpha_\eta(L) \sim \alpha(L)$ for $L \gg \eta$ and $\alpha_\eta(L) = 0$ for $L < 3\eta$. Therefore, in the limit $\eta \to 0$,

$$\partial\alpha_\eta(L) \to \partial\alpha(L) + \delta(L)\,\alpha(0)\,,$$

and (B.6) becomes

$$\mathcal{L}^{-1}\{Q\Psi\}(L) = Q\mathcal{L}^{-1}\{\Psi\}(L) - e^{LK}\partial_L\left(e^{-LK}\alpha(L)\right) - \delta(L)\,\alpha(0)\,, \tag{B.7}$$

which can be used for solutions with $\alpha(\infty) = 0, \alpha(0) \neq 0$, provided $\alpha(0)$ is well-defined. One can check that the Laplace transform of the right hand side yields $Q\Psi$.

C Correlation functions of X variables

In the calculations in section 3, we need the correlation functions of X variables, which are described by the free worldsheet theory with the Neumann boundary condition, on C_L. A conformal transformation which maps C_L to the upper half plane is given as

$$\begin{aligned} C_L &\to \text{UHP} \\ z &\to \xi = \tan\frac{\pi z}{L}\,. \end{aligned}$$

From the correlation functions

$$\begin{aligned} \langle\partial X^\mu(\xi)\,\partial X^\nu(\xi')\rangle_{\text{UHP}} &= \frac{-\frac{1}{2}\eta^{\mu\nu}}{(\xi-\xi')^2}\,, \\ \left\langle\partial X^\mu(\xi)\,\bar{\partial} X^\nu\left(\bar{\xi}'\right)\right\rangle_{\text{UHP}} &= \frac{-\frac{1}{2}\eta^{\mu\nu}}{\left(\xi-\bar{\xi}'\right)^2}\,, \end{aligned}$$

we can get

$$\begin{aligned} \langle\partial X^\mu(z)\,\partial X^\nu(z')\rangle_{C_L} &= -\frac{1}{2}\eta^{\mu\nu}\left(\frac{\pi}{L}\right)^2\frac{1}{\sin^2\frac{\pi(z-z')}{L}}\,, \\ \left\langle\partial X^\mu(z)\,\bar{\partial} X^\nu(\bar{z}')\right\rangle_{C_L} &= -\frac{1}{2}\eta^{\mu\nu}\left(\frac{\pi}{L}\right)^2\frac{1}{\sin^2\frac{\pi(z-\bar{z}')}{L}}\,. \end{aligned} \tag{C.1}$$

We are interested in the correlation function of the form $\langle(X^0(z,\bar{z})-X^0(z_0,\bar{z}_0))\,\partial X^0(z)\rangle_{C_L}$. Since the difference $X^0(z,\bar{z})-X^0(z_0,\bar{z}_0)$ for some $z_0,\bar{z}_0$ can be written as

$$X^0(z,\bar{z})-X^0(z_0,\bar{z}_0)=\int_{z_0}^{z}dz'\partial X^0(z')+\int_{\bar{z}_0}^{\bar{z}}d\bar{z}'\bar{\partial}X^0(\bar{z}')\ ,$$

using $\partial X^0,\bar{\partial}X^0$, the correlation function $\langle(X^0(z,\bar{z})-X^0(z_0,\bar{z}_0))\,\partial X^0(z)\rangle_{C_L}$ is well-defined. Here it is assumed that the operators are normal ordered as

$$:X^0\partial X^0:(z,\bar{z})\equiv\lim_{z'\to z}\left[X^0(z,\bar{z})\,\partial X^0(z')-\frac{1}{2}\frac{1}{z'-z}\right]\ . \tag{C.2}$$

From (C.1) we obtain

$$\begin{aligned}&\left\langle\left(X^0(z,\bar{z})-X^0(z_0,\bar{z}_0)\right)\partial X^0(z)\right\rangle_{C_L}\\&\qquad=\frac{\pi}{2L}\left[\cot\frac{\pi(z-\bar{z})}{L}-\cot\frac{\pi(z-z_0)}{L}-\cot\frac{\pi(z-\bar{z}_0)}{L}\right]\ .\end{aligned} \tag{C.3}$$

If one chooses the reference point z_0 to be $i\infty$, we get

$$\left\langle\left(X^0(z,\bar{z})-X^0(i\infty,-i\infty)\right)\partial X^0(z)\right\rangle_{C_L}=\frac{\pi}{2L}\cot\frac{\pi(z-\bar{z})}{L}\ .$$

D Derivation of (4.4)(4.5)

We would like to calculate the second and the third terms on the right hand side of (4.3) in the limit $\epsilon\to 0$. These can be calculated basically using the s-z trick [24, 25].

Using

$$\mathcal{L}^{-1}\{\Gamma_\epsilon\}(L)\ =\ \int_0^\infty dL_1dL_2\delta\left(L-\sum_i L_i\right)c(L_2)c(0)\mathcal{L}^{-1}\left\{F_\epsilon^2\right\}(L_1)\mathcal{L}^{-1}\left\{\frac{K+\epsilon}{G_\epsilon}\right\}(L_2)\,,$$

and

$$\begin{aligned}&\langle c(L_2)\,c(0)\,c(z)\rangle_{C_L}\\&\qquad=-\frac{1}{2}\left(\frac{L}{\pi}\right)^3\left[\left(\sin\left(\frac{\pi z}{L}\right)\right)^2\sin\frac{2\pi L_2}{L}-\left(\sin\left(\frac{\pi L_2}{L}\right)\right)^2\sin\frac{\pi z}{L}\right]\ ,\\&\left\langle c(L_2)\,c(0)\left(\int_{i\delta}^{i\Lambda}\frac{dz}{2\pi i}4\partial X^0(z)\,\bar{c}\bar{\partial}X^0(\bar{z})-\int_{-i\delta}^{-i\Lambda}\frac{d\bar{z}}{2\pi i}4\bar{\partial}X^0(\bar{z})\,c\partial X^0(z)\right)\right\rangle_{C_L}\\&\qquad\underset{(\delta,\Lambda)\to(0,\infty)}{\longrightarrow}\frac{1}{4\pi}\left(\frac{L}{\pi}\right)^2\sin\frac{2\pi L_2}{L}\,,\\&\langle c(L_2)\,c(0)\,\kappa(i\delta,-i\delta)\rangle_{C_L}\underset{\delta\to 0}{\longrightarrow}0\,,\end{aligned}$$

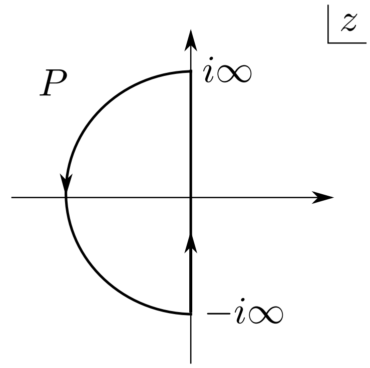

Figure 7: contour P

$\langle I|\,\chi\,|\Gamma_\epsilon\rangle$ becomes

$$\begin{aligned}\langle I|\,\chi\,|\Gamma_\epsilon\rangle &= \frac{-1}{4\pi^3}\epsilon\int_0^\infty ds s^2\int_0^\infty dL_1 dL_2\delta\left(s-\sum_i L_i\right)\\ &\qquad\times\mathcal{L}^{-1}\left\{G_\epsilon\right\}(L_1)\mathcal{L}^{-1}\left\{\frac{K+\epsilon}{G_\epsilon}\right\}(L_2)\sin\frac{2\pi}{s}L_2\\ &= \frac{-1}{4\pi^3}\epsilon\int_0^\infty ds s^2\int_0^\infty dL_1 dL_2\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}e^{\left(s-\sum_i L_i\right)z}\\ &\qquad\times\mathcal{L}^{-1}\left\{G_\epsilon\right\}(L_1)\mathcal{L}^{-1}\left\{\frac{K+\epsilon}{G_\epsilon}\right\}(L_2)\sin\frac{2\pi}{s}L_2\\ &= \frac{i}{8\pi^3}\epsilon\int_0^\infty ds s^2\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}e^{sz}G_\epsilon(z)\Delta\left(\frac{z+\epsilon}{G_\epsilon}\right)\\ &= \frac{i}{8\pi^3}\epsilon\int_0^\infty ds s^2\oint_P\frac{dz}{2\pi i}e^{sz}G_\epsilon(z)\Delta\left(\frac{z+\epsilon}{G_\epsilon}\right).\end{aligned}\tag{D.1}$$

Here P is contour on the z plane shown in figure 7 and Δ is defined as [24, 25]

$$\Delta F(z)=F\left(z-\frac{2\pi i}{s}\right)-F\left(z+\frac{2\pi i}{s}\right).$$

For the Murata-Schnabl solution (4.2), (D.1) is evaluated as

$$\begin{aligned}\langle I|\,\chi\,|\Gamma_\epsilon\rangle &= R_N+\mathcal{O}(\epsilon),\\ R_N &\equiv \begin{cases}-\frac{i}{8\pi^3}\sum_{k=0}^{N-2}\frac{N!}{k!(k+2)!(N-2-k)!}\left((2\pi i)^{k+2}-(-2\pi i)^{k+2}\right) & ,\ (N\geq 1)\ ,\\ \frac{i}{8\pi^3}\sum_{k=0}^{-N-1}\frac{(1-N)!}{k!(k+2)!(-N-1-k)!}\left((2\pi i)^{k+2}-(-2\pi i)^{k+2}\right) & ,\ (N\leq 0)\ ,\end{cases}\end{aligned}\tag{D.2}$$

for $\epsilon\ll 1$.

The third term on the right hand side of (4.3) becomes

$$\begin{aligned}
&\int dL_1 dL_2 \frac{L_1}{L_1+L_2} \langle e^{L_2 K}\mathcal{L}^{-1}\{\Psi_\epsilon\}(L_1)\, e^{-L_2 K}\mathcal{L}^{-1}\{\Gamma_\epsilon\}(L_2)\rangle_{C_{L_1+L_2}} \\
&\quad = \epsilon \int_0^\infty ds \prod_{i=1}^{4} dL_i \delta\left(s - \sum_{i=1}^{4} L_i\right) \frac{L_1+L_2}{s} \\
&\qquad \times \mathrm{Tr}\left[e^{-L_1 K}\mathcal{L}^{-1}\{G_\epsilon\}(L_1)\, cBe^{-L_2 K}\mathcal{L}^{-1}\left\{\frac{K+\epsilon}{G_\epsilon}\right\}(L_2)\, c \right. \\
&\qquad\qquad \left. \times\; e^{-L_3 K}\mathcal{L}^{-1}\{G_\epsilon\}(L_3)\, ce^{-L_4 K}\mathcal{L}^{-1}\left\{\frac{K+\epsilon}{G_\epsilon}\right\}(L_4)\, c\right] .
\end{aligned}$$

Using

$$L\mathcal{L}^{-1}\{f\}(L) = \mathcal{L}^{-1}\{\partial f\}(L) \ ,$$

and eq.(2.5) in [25], we obtain

$$\begin{aligned}
&\int dL_1 dL_2 \frac{L_1}{L_1+L_2} \langle e^{L_2 K}\mathcal{L}^{-1}\{\Psi_\epsilon\}(L_1)\, e^{-L_2 K}\mathcal{L}^{-1}\{\Gamma_\epsilon\}(L_2)\rangle_{C_{L_1+L_2}} \\
&\quad = \frac{i}{8\pi^3}\epsilon \int_0^\infty dss \oint_C \frac{dz}{2\pi i} e^{sz} \frac{1}{2i} \\
&\qquad \times \left\{\left[\frac{z+\epsilon}{G_\epsilon}, G_\epsilon, \frac{z+\epsilon}{G_\epsilon}, G'_\epsilon\right] + \left[\left(\frac{z+\epsilon}{G_\epsilon}\right)', G_\epsilon, \frac{z+\epsilon}{G_\epsilon}, G_\epsilon\right]\right\},
\end{aligned}$$

where

$$\begin{aligned}
[F_1, F_2, F_3, F_4] \equiv\; & [-F_1 \Delta F_2 F_3 F'_4 + F_1 \Delta (F_2 F'_3) F_4 + F_1 \Delta (F_2 F_3) F'_4 - F_1 F'_2 F_3 \Delta F_4 \\
& + F_1 F'_2 \Delta (F_3 F_4) + F_1 F_2 \Delta (F'_3 F_4) - F_1 \Delta (F_2 F'_3 F_4) - F_1 (F_2 \Delta F_3 F_4)'] .
\end{aligned}$$

The contribution of $\mathcal{O}(\epsilon^0)$ is given by the following replacements

$$\begin{aligned}
G'(z) &\rightarrow -(N-1)\, G(z) \ , \\
G''(z) &\rightarrow N(N-1)\frac{1}{z^2} G(z) \ , \\
\left(\frac{z}{G}\right)'(z) &\rightarrow NG^{-1}(z) \ ,
\end{aligned}$$

and one can see

$$\int dL_1 dL_2 \frac{L_1}{L_1+L_2} \langle e^{L_2 K}\mathcal{L}^{-1}\{\Psi_\epsilon\}(L_1)\, e^{-L_2 K}\mathcal{L}^{-1}\{\Gamma_\epsilon\}(L_2)\rangle_{C_{L_1+L_2}} \sim \mathcal{O}(\epsilon) \ .$$

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