

## A CLASS OF MULTIVALENT FUNCTIONS

By

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### 1. Introduction.

Let  $A(p)$  be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in N = 1, 2, 3, \dots)$$

which are analytic in  $U = \{z \mid |z| < 1\}$ .

A function  $f(z) \in A(p)$  is said to be  $p$ -valently starlike iff

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{in } U.$$

We denote by  $S(p)$  the subclass of  $A(p)$  consisting of functions which are  $p$ -valently starlike in  $U$ . Further, a function in  $A(p)$  is said to be  $p$ -valently convex iff

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \quad \text{in } U.$$

Also we denote by  $C(p)$  the subclass of  $A(p)$  consisting of all  $p$ -valently convex functions in  $U$ .

MacGregor [2] investigated the class of functions which are analytic in  $U$ ,  $f(0) = f'(0) - 1 = 0$  and satisfy the condition

$$|f'(z) - 1| < 1 \quad \text{in } U.$$

Let  $F$  denote the class of functions which satisfy the above conditions.

MacGregor [2, Theorem 6] obtained the following result:

**THEOREM A.** *If  $f(z) \in F$ , then  $f(z)$  is starlike in  $|z| < \sqrt{4/4} \doteq 0.894$ .*

Nunokawa [4] and Nunokawa, Fukui, Owa, Saitoh and Sekine [6] improved Theorem A. Mocanu [3] showed that there is a function  $f(z) \in A(1)$  which is a member of  $F$  but not starlike in  $|z| < 1$ .

**THEOREM B.** *If  $f(z) \in F$ , then  $f(z)$  is starlike in  $|z| < r_1 < 1$ , where  $r_1$  is the*

root of the equation

$$\log(9-4r^2+4r^3-r^4)-\log 9(1-r^2)+\sin^{-1}r=\pi,$$

that is  $r_1 \doteq 0.934$ .

A proof of Theorem B can be found in [6, Corollary].

## 2. Main theorem.

In this paper, we need the following lemmata.

LEMMA 1. *Let  $w(z)$  be analytic in the unit disk  $U$ , with  $w(0)=0$ . If  $|w(z)|$  attains its maximum value on the circle  $|z|=r$  at a point  $z_0$ , then we can write*

$$z_0 w'(z_0) = k w(z_0)$$

where  $k$  is a real number and  $k \geq 1$ .

A proof can be found in [1].

Applying Lemma 1, we can obtain the following lemma.

LEMMA 2. *Let  $p(z)$  be analytic in  $U$ ,  $p(0)=1$  and suppose that*

$$(1) \quad |p(z) + zp'(z) - 1| < \sqrt{2} \quad \text{in } U.$$

Then we have

$$|p(z) - 1| < \frac{\sqrt{2}}{2} \quad \text{in } U$$

and

$$|\arg p(z)| < \frac{\pi}{4} \quad \text{in } U.$$

PROOF. Putting

$$p(z) = 1 + \frac{\sqrt{2}}{2} w(z),$$

then  $w(z)$  is analytic in  $U$  and  $w(0)=0$ . If there exists a point  $z_0 \in U$  such that

$$\max_{|z| \leq |z_0} |w(z)| = |w(z_0)| \geq 1,$$

then from Lemma 1, we have

$$z_0 w'(z_0) = k w(z_0), \quad (k \geq 1).$$

Then we have

$$\begin{aligned} |p(z_0) + z_0 p'(z_0) - 1| &= \frac{\sqrt{2}}{2} |w(z_0) + k w(z_0)| \\ &= \frac{\sqrt{2}}{2} |w(z_0)|(1+k) \geq \frac{\sqrt{2}}{2} (1+k) \geq \sqrt{2}. \end{aligned}$$

This contradicts (1). Therefore we have

$$|w(z)| < 1 \quad \text{in } U.$$

This shows that

$$|p(z) - 1| < \frac{\sqrt{2}}{2} \quad \text{in } U.$$

and therefore we have

$$|\arg p(z)| < \frac{\pi}{4} \quad \text{in } U.$$

This completes our proof.

Applying the same method as in the proof of [5, Lemma 6 and Theorem 5], we can easily obtain the following lemma.

LEMMA 3. Let  $p \geq 2$ . If  $f(z) \in A(p)$  satisfies the condition

$$\operatorname{Re} \frac{z f^{(p-1)}(z)}{f^{(p-2)}(z)} > 0 \quad \text{in } U,$$

then we have

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > 0 \quad \text{in } U.$$

MAIN THEOREM. Let  $p \geq 2$ . If  $f(z) \in A(p)$  satisfies the condition

$$(2) \quad |f^{(p)}(z) - p!| < \sqrt{2}(p!) \quad \text{in } U,$$

then  $f(z)$  is  $p$ -valently starlike in  $U$ .

PROOF. Let us put

$$(3) \quad p(z) = \frac{f^{(p-1)}(z)}{p! z}, \quad (p(0) = 1).$$

Then we have

$$p(z) + z p'(z) - 1 = \frac{f^{(p)}(z)}{p!} - 1,$$

and from the hypothesis (2), we have

$$|p(z) + z p'(z) - 1| < \sqrt{2} \quad \text{in } U.$$

Then, from Lemma 2 and (3), we have

$$(4) \quad \left| \arg p(z) \right| = \left| \arg \frac{f^{(p-1)}(z)}{p!z} \right| = \left| \arg \frac{f^{(p-1)}(z)}{z} \right| < \frac{\pi}{4} \quad \text{in } U.$$

Applying the same idea as in the proof of [7, Theorem 1] and integrating on the line segment from 0 to  $z$ , we have

$$(5) \quad \begin{aligned} \frac{f^{(p-2)}(z)}{z^2} &= \frac{1}{z^2} \int_0^z f^{(p-1)}(t) dt \\ &= \frac{1}{r^2} \int_0^r \frac{f^{(p-1)}(t)}{t} \rho d\rho \end{aligned}$$

where  $z = re^{i\theta}$ ,  $0 < r < 1$ ,  $t = \rho e^{i\theta}$  and  $0 \leq \rho \leq r$ .

From (4), we have

$$(6) \quad \left| \arg \frac{f^{(p-1)}(t)}{t} \rho \right| = \left| \arg \frac{f^{(p-1)}(t)}{t} \right| < \frac{\pi}{4} \quad \text{in } U.$$

Applying the same idea as in the proof of [8, Lemma 1] and since  $s = f^{(p-1)}(t)/t$  lies in the convex sector  $\{|\arg s| < \pi/4\}$ , then from (5) and (6), the same is true of its integral mean value of (5).

Therefore, we have

$$(7) \quad \begin{aligned} \left| \arg \frac{f^{(p-2)}(z)}{z^2} \right| &= \left| \arg \frac{1}{r^2} \int_0^r \frac{f^{(p-1)}(t)}{t} \rho d\rho \right| \\ &= \left| \arg \int_0^r \frac{f^{(p-1)}(t)}{t} \rho d\rho \right| < \frac{\pi}{4} \quad \text{in } U. \end{aligned}$$

From (4) and (7), we have

$$\begin{aligned} \left| \arg \frac{z f^{(p-1)}(z)}{f^{(p-2)}(z)} \right| &= \left| \arg \frac{f^{(p-1)}(z)}{z} \frac{z^2}{f^{(p-2)}(z)} \right| \\ &\leq \left| \arg \frac{f^{(p-1)}(z)}{z} \right| + \left| \arg \frac{f^{(p-1)}(z)}{z^2} \right| < \frac{\pi}{2} \quad \text{in } U. \end{aligned}$$

This shows that

$$(8) \quad \operatorname{Re} \frac{z f^{(p-1)}(z)}{f^{(p-2)}(z)} > 0 \quad \text{in } U.$$

From Lemma 3 and (8), we have

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > 0 \quad \text{in } U.$$

This completes our proof.

From the main theorem, we easily have the following corollary.

**COROLLARY 1.** *Let  $p \geq 2$ . If  $f(z) \in A(p)$  satisfies the condition*

$$(9) \quad |zf^{(p+1)}(z) + pf^{(p)}(z) - p(p!)| < \sqrt{2}p(p!) \quad \text{in } U,$$

then  $f(z)$  is  $p$ -valently convex in  $U$ .

PROOF. Putting

$$g(z) = \frac{zf'(z)}{p},$$

then  $g(z)$  is a function of  $A(p)$ .

From the hypothesis (9) and the main theorem, we have that  $g(z)$  is  $p$ -valently starlike in  $U$ . Therefore,  $f(z)$  is  $p$ -valently convex in  $U$ .

This completes our proof.

COROLLARY 2. Let  $p \geq 2$ . If  $f(z) \in A(p)$  satisfies the condition

$$(10) \quad |f^{(p+1)}(z)| < \sqrt{2}(p!) \quad \text{in } U,$$

then,  $f(z)$  is  $p$ -valently starlike in  $U$ .

PROOF. By an easy calculation and from (10), we have

$$\begin{aligned} |f^{(p)}(z) - p!| &= \left| \int_0^z f^{(p+1)}(t) dt \right| \\ &\leq \int_0^r |f^{(p+1)}(t)| d\rho < \sqrt{2}(p!)|z| < \sqrt{2}(p!) \end{aligned}$$

where  $|z| = r < 1$  and  $0 \leq |t| = \rho < r$ .

From the main theorem,  $f(z)$  is  $p$ -valently starlike in  $U$ .

This completes our proof.

REMARK 1. It is easily confirmed that the function

$$f(z) = z^p + \frac{p! e^{-\alpha}}{\alpha^p} \sqrt{2} \left\{ e^{\alpha z} - \sum_{k=0}^p \frac{(\alpha z)^k}{k!} \right\}$$

satisfies the conditions (2) and (10), therefore  $f(z)$  is  $p$ -valently starlike in  $U$ .

On the other hand, the function

$$g(z) = z^p + \int_0^z \frac{p! e^{-\alpha}}{t\alpha^p} \left( e^{\alpha t} - \sum_{k=0}^p \frac{(\alpha t)^k}{k!} \right) dt$$

satisfies the condition (9), therefore  $g(z)$  is  $p$ -valently convex in  $U$ .

REMARK 2. To prove the main theorem, we have to obtain

$$\operatorname{Re} \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} > 0 \quad \text{in } U.$$

Therefore we have to suppose  $p \geq 2$ .

On the other hand, it is easily confirmed that the function

$$f(z) = z + \frac{\sqrt{2}}{2} z^2 \in A(1)$$

satisfies the condition

$$|f'(z) - 1| < \sqrt{2} \quad \text{in } U,$$

but  $f(z)$  is not starlike in  $U$ .

This shows that the main theorem does not hold good for the case  $p=1$ .

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