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ASYMPTOTIC THEORY FOR ESTIMATION OF LOCATION IN NON-REGULAR CASES, I: ORDER OF CONVERGENCE OF CONSISTENT ESTIMATORS*

By Masafumi AKAHIRA
University of Electro-Communications

1. Introduction

Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of independent identically distributed (i.i.d.) random variables. We assume that a parameter space Θ is an open set in a Euclidean p-space with a norm $||\cdot||$. In the textbook discussion of an asymptotic theory, it is usually shown that the asymptotically best (in some sense or other) estimator $\{T_n^*\}$ has the asymptotic distribution of order $\sqrt{n}$, in the sense that the distribution of $\sqrt{n}(T_n^*-\theta)$ converges to some probability law (in most cases normal). There were the sporadic examples that the distribution of $n(T_n^*-\theta)$ or $\sqrt{n \log n}(T_n^*-\theta)$ converges to some law (Woodroofe [7]) when X_i's are i.i.d. random variables with an uniform distribution or a truncated distribution. The purpose of this paper is to give a systematic treatment for the problem whether for a given sequence $\{c_n\}$, $c_n(T_n^*-\theta)$ converges to some law, and what is the possible bound for such a sequence. In a location parameter case it will be shown that such a bound is explicitely given, and the above mentioned are too special cases of our result. The asymptotic distribution of $c_n(T_n^*-\theta)$ and the bound for it will be discussed in the subsequent paper (Akahira [1]). Also some results in terms of the asympotic distributions of estimators are given in Takeuchi [6].

Suppose that $\{T_n\}$ is a (sequence of) consistent estimator(s). $\{T_n\}$ is defined to be consistent with order $\{c_n\}$, where $\{c_n\}$ is an increasing sequence of positive numbers (c_n tending to infinity) if for every $\varepsilon>0$ and every ϑ of Θ, there exist a sufficiently small positive number δ and a sufficiently large positive number L satisfying the following:

$$\varlimsup_{n\to\infty} \sup_{\theta:||\theta-\vartheta||<\delta} P_\theta^{(n)}(\{c_n||T_n-\theta||\geq L\})<\varepsilon$$

A necessary condition for the existence of such an estimator is established, and the bounds of the order of consistency of estimators are obtained. As a special example, a location parameter case is discussed when the density function of $x-\theta$ satisfies the following:

Assumption (A).
$$f(x)>0 \quad \text{for} \quad a<x<b,$$
$$f(x)=0 \quad \text{for} \quad x\leq a,\ x\geq b,$$

Assumption (B). $f(x)$ is twice continuously differentiable in the interval (a, b) and

$$\lim_{x\to a+0}(x-a)^{1-\alpha} \quad f(x)=A',$$
$$\lim_{x\to b-0}(b-x)^{1-\beta} \quad f(x)=B',$$

where both α and β are positive constants satisfying $\alpha\leq\beta<\infty$, and A' and B' are positive finite numbers.

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Assumption (C). $A''=\lim_{x\to a+0}(x-a)^{2-\alpha}|f'(x)|$ and $B''=\lim_{x\to b-0}(b-x)^{2-\beta}|f'(x)|$ are finite. For $\alpha\geq 2$, $f''(x)$ is bounded.

It is shown that the bound of $\{c_n\}$ is given by $c_n=n^{1/\alpha}$ if $0<\alpha<2$, $c_n=(n\log n)^{1/2}$ if $\alpha=2$, $c_n=n^{1/2}$ if $\alpha>2$, and the existence of estimators with such order of consistency is established.

2. Notations and Definitions

Let $\mathfrak{X}$ be an abstract sample space whose generic point is denoted by x, $\mathfrak{B}$ a σ-field of subsets of $\mathfrak{X}$, and let Θ be a parameter space, which is assumed to be an open set in a Euclidean p-space R^p with a norm denoted $||\cdot||$. We consider a sequence of classes of probability measure $\{P_{\theta i}:\theta\in\Theta\}$ $(i=1, 2, \cdots)$ each defined over $(\mathfrak{X}, \mathfrak{B})$. We shall denote by $(\mathfrak{X}^{(n)}, \mathfrak{B}^{(n)})$ the n-fold direct products of $(\mathfrak{X}, \mathfrak{B})$ and the corresponding product measures by $P_\theta^{(n)}=P_{\theta 1}X\cdots XP_{\theta n}$. For each $n=1, 2, \cdots$, the points of $\mathfrak{X}^{(n)}$ will be denoted by $\tilde{x}_n=(x_1, \cdots, x_n)$ and the corresponding random variable by $\tilde{X}_n$ with the probability distribution $P_\theta^{(n)}$. An estimator of θ is defined to be a sequence $\{T_n:n=1, 2, \cdots\}$ of $\mathfrak{B}^{(n)}$-measurable function T_n on $\mathfrak{X}^{(n)}$ into Θ $(n=1, 2, \cdots)$.

Definition 2.1. An estimator $\{T_n:n=1, 2, \cdots\}$ is called (weakly) consistent if for every $\varepsilon>0$ and every θ of Θ

$$\lim_{n\to\infty}P_\theta^{(n)}(\{||T_n-\theta||>\varepsilon\})=0.$$

Definition 2.2. For an increasing sequence of positive numbers $\{c_n\}$ (c_n tending to infinity) an estimator $\{T_n:n=1, 2, \cdots\}$ is called consistent with order $\{c_n\}$ (or $\{c_n\}$-consistent for short) if for every $\varepsilon>0$ and every ϑ of Θ, there exist a sufficiently small positive number δ and a sufficiently large positive number L satisfying the following:

$$\varlimsup_{n\to\infty}\sup_{\theta:||\theta-\vartheta||<\delta}P_\theta^{(n)}(\{c_n||T_n-\theta||\geq L\})<\varepsilon. \tag{2.1}$$

It is easily seen that if $\{T_n\}$ is a $\{c_n\}$-consistent estimator, then $\{T_n\}$ is a consistent estimator. Order $\{c_n\}$ is called to be greater than order $\{c_n'\}$ if $\lim_{n\to\infty}c_n'/c_n=0$. For any two points θ and θ' in Θ, there exists a σ-finite measure μ_n such that $P_\theta^{(n)}$ and $P_{\theta'}^{(n)}$ are absolutely continuous with respect to μ_n. Further for any points θ and θ' in Θ we define

$$\begin{aligned} d_n(\theta, \theta') &= \int_{\mathfrak{X}^{(n)}}\left|\frac{dP_\theta^{(n)}}{d\mu_n}-\frac{dP_{\theta'}^{(n)}}{d\mu_n}\right|d\mu_n \\ &=2\sup_{B\in\mathfrak{B}^{(n)}}|P_\theta^{(n)}(B)-P_{\theta'}^{(n)}(B)|. \end{aligned} \tag{2.2}$$

It is easily seen that for each n, d_n is a metric on Θ independent of μ_n.

3. Necessary Conditions for Existences of Consistent Estimators

In this section we shall obtain the necessary conditions for the existences of a consistent estimator and a $\{c_n\}$-consistent estimator.

The following is already known. (e.g. Hoeffding and Wolfowitz [4]).

THEOREM 3.1. If there exists a consistent estimator, then for any two disjoint points θ_1 and θ_2 in Θ,

$$\lim_{n\to\infty}d_n(\theta_1, \theta_2)=2.$$

The proof is omitted.

The following theorem shows that the necessary condition for the existence of a consistent estimator is that the limit of the Kullback information is infinite.

THEOREM 3.2. Suppose that for each n, $\{x_n : dP_\theta^{(n)}/d\mu_n>0\}$ does not depend on θ. If there exists a consistent estimator, then the following holds: for any two disjoint points θ_1 and θ_2

$$\lim_{n\to\infty} I_n(\theta_1, \theta_2)=\infty,$$

where $I_n(\theta_1, \theta_2)=\int_{\mathfrak{X}^{(n)}}(dP_{\theta_1}^{(n)}/d\mu_n)\log(dP_{\theta_1}^{(n)}/dP_{\theta_2}^{(n)})d\mu_n$.

PROOF. We denote a consistent estimator by $\{T_n : n=1, 2, \cdots\}$. Let $0<\delta<\frac{1}{2}$. Putting $Y_n=dP_{\theta_2}^{(n)}/dP_{\theta_1}^{(n)}$, we have from Theorem 3.1 for sufficiently large n,

$$\begin{aligned} E_{\theta_1}^{(n)}(|Y_n-1|)&=\int_{\mathfrak{X}^{(n)}}|Y_n-1|dP_{\theta_1}^{(n)} \\ &=d(P_{\theta_1}^{(n)}, P_{\theta_2}^{(n)}) \\ &\geq 2-2\delta. \end{aligned} \tag{3.1}$$

Putting $Y_n^+=\max\{Y_n-1, 0\}$ and $Y_n^-=\max\{1-Y_n, 0\}$, we have for each $n=1, 2, \cdots$,

$$\begin{aligned} E_{\theta_1}^{(n)}(Y_n^+)-E_{\theta_1}^{(n)}(Y_n^-)&=\int_{\mathfrak{X}^{(n)}}\left\{\frac{dP_{\theta_1}^{(n)}}{d\mu_n}-\frac{dP_{\theta_2}^{(n)}}{d\mu_n}\right\}d\mu_n \\ &=0 \end{aligned}$$

and for sufficiently large n,

$$E_{\theta_1}^{(n)}(Y_n^+)+E_{\theta_1}^{(n)}(Y_n^-)=E_{\theta_1}^{(n)}(|Y_n-1|)\geq 2-2\delta.$$

Hence we obtain for sufficiently large n,

$$E_{\theta_1}^{(n)}(Y_n^+)=E_{\theta_1}^{(n)}(Y_n^-)\geq 1-\delta. \tag{3.2}$$

Since $0\leq Y_n^-\leq 1$ and (3.2) hold, for sufficiently large n,

$$\begin{aligned} 1-\delta\leq E_{\theta_1}^{(n)}(Y_n^-)&=\int_{\{Y_n^-\geq 1-2\delta\}}Y_n^- dP_{\theta_1}^{(n)}(x)+\int_{\{Y_n^-<1-2\delta\}}Y_n^- dP_{\theta_1}^{(n)}(x_n) \\ &\leq P_{\theta_1}^{(n)}(\{Y_n^-\geq 1-2\delta\})+(1-2\delta)P_{\theta_1}^{(n)}(\{Y_n^-\leq 1-2\delta\}) \\ &=2\delta P_{\theta_1}^{(n)}(\{Y_n^-\geq 1-2\delta\})+1-2\delta. \end{aligned}$$

Hence we have for sufficiently large n,

$$P_{\theta_1}^{(n)}(\{Y_n\geq 1-2\delta\})\geq\frac{1}{2}. \tag{3.3}$$

It follows from (3.3) that for sufficiently large n,

$$\begin{aligned} I_n(\theta_1, \theta_2)&=E_{\theta_1}^{(n)}(-\log Y_n) \\ &=E_{\theta_1}^{(n)}[-\log(1+Y_n^+)]-E_{\theta_1}^{(n)}[\log(1-Y_n^-)] \\ &\geq -E_{\theta_1}^{(n)}(Y_n^+)-\frac{1}{2}\log 2\delta \\ &\geq -1-\frac{1}{2}\log 2\delta. \end{aligned}$$

Therefore we have

$$\lim_{n\to\infty} I_n(\theta_1, \theta_2)=\infty.$$

Thus we complete the proof.

THEOREM 3.3 If there exists a $\{c_n\}$-consistent estimator, then for every $\varepsilon>0$ and every $\theta\in\Theta$ there is a positive number t such that

$$\underline{\lim}_{n\to\infty}\, d_n(\theta, \theta\pm tc_n^{-1}\mathbf{1})\geq 2-\varepsilon$$

where $\mathbf{1}=(1, \cdots, 1)'$.

PROOF. Suppose that $\{T_n : n=1, 2, \cdots\}$ be a $\{c_n\}$-consistent estimator. It follows from the definition of a $\{c_n\}$-consistent estimator that for every $\varepsilon>0$ and every θ of Θ, there exist positive numbers δ and L such that

$$\overline{\lim_{n\to\infty}} \sup_{\vartheta:||\vartheta-\theta||<\delta} P_{\vartheta}{}^{(n)}(\{c_n||T_n-\vartheta||\geq L\})<\varepsilon/4.$$

Let $t>2L$ be fixed. Since there exists n_0 such that for any $n>n_0$,

$$c_n>c_{n_0}>t/\delta,$$

$$\sup_{\vartheta:||\vartheta-\theta||<tc_{n_0}^{-1}} P_{\vartheta}{}^{(n)}(\{c_n||T_n-\vartheta||\geq L\})<\varepsilon/4,$$

it follows that

$$\overline{\lim_{n\to\infty}} P_{\theta+tc_n^{-1}1}{}^{(n)}(c_n||T_n-\theta-tc_n^{-1}\mathbf{1}||\geq L\})<\varepsilon/4, \tag{3.4}$$

$$\overline{\lim_{n\to\infty}} P_{\theta}{}^{(n)}(\{c_n||T_n-\theta||\geq L\})<\varepsilon/4. \tag{3.5}$$

From (3.4) we have

$$\overline{\lim_{n\to\infty}} P_{\theta+tc_n^{-1}1}{}^{(n)}(\{c_n||T_n-\theta-tc_n^{-1}\mathbf{1}||\geq t-L\})<\varepsilon/4. \tag{3.6}$$

Since the following holds:

$$\begin{aligned} d_n(\theta,\ \theta+tc_n^{-1}\mathbf{1})&=2\sup_{B\in\mathfrak{B}(n)}|P_{\theta+tc_n^{-1}}{}^{(n)}{}_1(B)-P_{\theta}{}^{(n)}(B)| \\ &\geq 2|P_{\theta+tc_n^{-1}}{}^{(n)}{}_1(\{c_n||T_n-\theta||\geq L\})-P_{\theta}{}^{(n)}(\{c_n||T_n-\theta||\geq L\})| \end{aligned} \tag{3.7}$$

for all n, it is sufficient to show that the inferior limit of the last term of (3.7) is not smaller than $2-\varepsilon$. Because we have

$$\{\tilde{x}_n: c_n||T_n(\tilde{x})-\theta-tc_n^{-1}\mathbf{1}||<t-L\}\subset\{\tilde{x}_n: c_n||T_n(\tilde{x}_n)-\theta||\geq L\}$$

for all n,

$$\begin{aligned} &\underline{\lim_{n\to\infty}} P_{\theta+tc_n^{-1}1}{}^{(n)}(\{c_n||T_n-\theta-tc_n^{-1}\mathbf{1}||<t-L\}) \\ &\leq\underline{\lim_{n\to\infty}} P_{\theta+tc_n^{-1}1}{}^{(n)}(\{c_n||T_n-\theta||\geq L\}). \end{aligned} \tag{3.8}$$

It follows from (3.6) and (3.8) that

$$1-\frac{\varepsilon}{4}\leq\underline{\lim_{n\to\infty}} P_{\theta+tc_n^{-1}1}{}^{(n)}(\{c_n||T_n-\theta||\geq L\}). \tag{3.9}$$

From (3.5) and (3.9) we obtain

$$2-\varepsilon\leq\underline{\lim_{n\to\infty}} 2|P_{\theta+tc_n^{-1}1}{}^{(n)}(\{c_n||T_n-\theta||\geq L\})-P_{\theta}{}^{(n)}(\{c_n||T_n-\theta||\geq L\})|.$$

Therefore we have

$$\underline{\lim_{n\to\infty}} d_n(\theta,\ \theta+tc_n^{-1}\mathbf{1})\geq 2-\varepsilon.$$

Similarly we also obtain

$$\underline{\lim_{n\to\infty}} d_n(\theta,\ \theta-tc_n^{-1}\mathbf{1})\geq 2-\varepsilon.$$

Thus we complete the proof.

4. Order of Convergence of $\{C_n\}$-Consistent Estimators for Location Parameter Cases

Before discussing order of convergence of $\{c_n\}$-consistent estimators in detail, we shall give a definition and lemmas.

Definition 4.1. (Generalized from Gnedenko and Kolmogorov [3]). For each $\theta\in\Theta$, the sums

$$Y_n(\theta)=X_1(\theta)+X_2(\theta)+\cdots+X_n(\theta)$$

of positive independent random variables $X_1(\theta)$, $X_2(\theta)$, $\cdots$, $X_n(\theta)$, $\cdots$ are said to be uniformly relatively stable for constants $B_n(\theta)$ if there exist positive constants $B_n(\theta)$ such that for any $\varepsilon>0$, $P_{\theta}{}^{(n)}\left(\left\{\left|\frac{Y_n(\theta)}{B_n(\theta)}-1\right|>\varepsilon\right\}\right)\to 0$ as $n\to\infty$ uniformly in any compact subset of Θ.

In the subsequent lemmas, we use the notation that for each k and each $\theta\in\Theta$, $F_{\theta k}(x)$ is the distribution function of $X_k(\theta)$.

LEMMA 4.1. (Gnedenko and Kolmogorov [3]). For each $\theta\in\Theta$, let $X_1(\theta)$, $X_2(\theta)$, $\cdots$, $X_n(\theta)$, $\cdots$ be a sequence of positive independent random variables. The sums

$$Y_n(\theta)=X_1(\theta)+X_2(\theta)+\cdots+X_n(\theta)$$

are uniformly relatively stable for constants $B_n(\theta)$, if there exists a sequence of positive constants $B_1(\theta)$, $B_2(\theta)$, $\cdots$, $B_n(\theta)$, $\cdots$ such that for any $\varepsilon>0$,

$$\sum_{k=1}^{n}\int_{\varepsilon B_n(\theta)}^{\infty} dF_{\theta k}(x)\to 0 \tag{4.1}$$

as $n\to\infty$ uniformly in any compact subset of Θ,

$$\frac{1}{B_n(\theta)}\sum_{k=1}^{n}\int_0^{\varepsilon B_n(\theta)} x dF_{\theta k}(x)\to 1 \tag{4.2}$$

as $n\to\infty$ uniformly in any compact subset of Θ.

The following lemma is a generalization of Lindeberg's condition (see Gnedenko and Kolmogorov [3]).

LEMMA 4.2. For each $\theta\in\Theta$, let $X_1(\theta)$, $X_2(\theta)$, $\cdots$, $X_n(\theta)$, $\cdots$ be a sequence of independent random variables.

The distribution laws of the sums

$$Y_n(\theta)=\frac{X_1(\theta)+X_2(\theta)+\cdots+X_n(\theta)}{B_n(\theta)}$$

converges to the normal law

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy$$

uniformly in any compact subset of Θ, if there exists a sequence of constants $B_n(\theta)$ such that $\lim_{n\to\infty} B_n(\theta)=\infty$ uniformly in any compact subset of Θ and for any $\varepsilon>0$,

$$\sum_{k=1}^{n}\int_{\{|x|>\varepsilon B_n(\theta)\}} dF_{\theta k}(x)\to 0 \tag{4.3}$$

as $n\to\infty$ uniformly in any compact subset of Θ, and

$$\frac{1}{\{B_n(\theta)\}^2}\sum_{k=1}^{n}\left\{\int_{\{|x|<\varepsilon B_n(\theta)\}} x^2 dF_{\theta k}(x)-\left(\int_{\{|x|<\varepsilon B_n(\theta)\}} x dF_{\theta k}(x)\right)^2\right\}\to 1 \tag{4.4}$$

as $n\to\infty$ uniformly in any compact subsets of Θ.

Now we assume that X_i's are identically distributed i.e. $P_{\theta i}=P_\theta$ $(i=1, 2, \cdots)$.

We suppose that every $P_\theta(\cdot)(\theta\in\Theta)$ is absolutely continuous with respect to a σ-finite measure μ. We denote the density $dP_\theta/d\mu$ by $f(\cdot:\theta)$ and by $A(\theta)\subset\mathfrak{X}$ the set of points in the space of $\mathfrak{X}$ for which $f(x:\theta)>0$. Then we may write $f(x:\theta)=\chi_{A(\theta)}(x)f(x:\theta)$, where $\chi_{A(\theta)}(\cdot)$ denotes the indicator of $A(\theta)$.

LEMMA 4.3. If

$$\int_{\overset{n}{\underset{i=1}{X}} A(\theta_1)\cap \overset{n}{\underset{i=1}{X}} A(\theta_2)}\left\{\prod_{i=1}^{n}\frac{f(x_i:\theta_1)}{f(x_i:\theta_2)}-1\right\}^2\prod_{i=1}^{n} f(x_i:\theta_2)d\mu^{(n)}<\infty,$$

then for any two points θ_1 and θ_2 in Θ and each $n=1, 2, \cdots$,

$$d(P_{\theta_1}{}^{(n)},\ P_{\theta_2}{}^{(n)}) \leq [1-\{P_{\theta_1}(A(\theta_2))\}^n]+[1-\{P_{\theta_2}(A(\theta_1))\}^n]$$
$$+\left[\left\{\int_{A(\theta_1)\cap A(\theta_2)} f^2(x:\theta_1)/f(x:\theta_2)d\mu\right\}^n\right.$$
$$\left.-2\{P_{\theta_1}(A(\theta_2))\}^n+\{P_{\theta_2}(A(\theta_1))\}^n\right]^{1/2}. \tag{4.5}$$

PROOF. Since for any two points θ_1 and θ_2 in Θ and each $n=1, 2, \cdots$,

$$\frac{dP_{\theta_j}{}^{(n)}}{d\mu^{(n)}}=\prod_{i=1}^n f(x_i:\theta_j)=\prod_{i=1}^n \chi_{A(\theta_j)}f(x_i:\theta_j)=\chi_{\mathop{X}_{i=1}^{n} A(\theta_j)}(\tilde{x}_n)\prod_{i=1}^n f(x_i:\theta_j) \quad (j=1,\ 2)$$

from (2.1) we have

$$d(P_{\theta_1}{}^{(n)},\ P_{\theta_2}{}^{(n)})$$
$$=\int_{\mathfrak{X}^{(n)}}\left|\chi_{\mathop{X}_{i=1}^{n} A(\theta_1)}(\tilde{x}_n)\prod_{i=1}^n f(x_i:\theta_1)-\chi_{\mathop{X}_{i=1}^{n} A(\theta_2)}(\tilde{x}_n)\prod_{i=1}^n f(x_i:\theta_2)\right|d\mu^{(n)}$$
$$=\int_{\mathop{X}_{i=1}^{n} A(\theta_1)-\mathop{X}_{i=1}^{n} A(\theta_2)}\prod_{i=1}^n f(x_i:\theta_1)d\mu^{(n)}+\int_{\mathop{X}_{i=1}^{n} A(\theta_2)-\mathop{X}_{i=1}^{n} A(\theta_1)}\prod_{i=1}^n f(x_i:\theta_2)d\mu^{(n)}$$
$$+\int_{\mathop{X}_{i=1}^{n} A(\theta_1)\cap \mathop{X}_{i=1}^{n} A(\theta_2)}\left|\prod_{i=1}^n f(x_i:\theta_1)-\prod_{i=1}^n f(x_i:\theta_2)\right|d\mu^{(n)}$$
$$=1-\int_{\mathop{X}_{i=1}^{n}(A(\theta_1)\cap A(\theta_2))}\prod_{i=1}^n f(x_i:\theta_1)d\mu^{(n)}+1-\int_{\mathop{X}_{i=1}^{n}(A(\theta_1)\cap A(\theta_2))}\prod_{i=1}^n f(x_i:\theta_2)d\mu^{(n)}$$
$$+\int_{\mathop{X}_{i=1}^{n} A(\theta_1)\cap \mathop{X}_{i=1}^{n} A(\theta_2)}\left|\prod_{i=1}^n f(x_i:\theta_1)-\prod_{i=1}^n f(x_i:\theta_2)\right|d\mu^{(n)}$$
$$=[1-\{P_{\theta_1}(A(\theta_2))\}^n]+[1-\{P_{\theta_2}(A(\theta_1))\}^n]$$
$$+\int_{\mathop{X}_{i=1}^{n} A(\theta_1)\cap \mathop{X}_{i=1}^{n} A(\theta_2)}\left|\prod_{i=1}^n f(x_i:\theta_1)-\prod_{i=1}^n f(x_i:\theta_2)\right|d\mu^{(n)}.$$

Further it follows form the assumption and the Schwarz's inequality that

$$\int_{\mathop{X}_{i=1}^{n} A(\theta_1)\cap \mathop{X}_{i=1}^{n} A(\theta_2)}\left|\prod_{i=1}^n f(x_i:\theta_1)-\prod_{i=1}^n f(x_i:\theta_2)\right|d\mu^{(n)}$$
$$=\int_{\mathop{X}_{i=1}^{n} A(\theta_1)\cap \mathop{X}_{i=1}^{n} A(\theta_2)}\left|\prod_{i=1}^n \frac{f(x_i:\theta_1)}{f(x_i:\theta_2)}-1\right|\prod_{i=1}^n f(x_i:\theta_2)d\mu^{(n)}$$
$$\leq\left[\int_{\mathop{X}_{i=1}^{n} A(\theta_1)\cap \mathop{X}_{i=1}^{n} A(\theta_2)}\left\{\prod_{i=1}^n \frac{f(x_i:\theta_1)}{f(x_i:\theta_2)}-1\right\}^2\prod_{i=1}^n f(x_i:\theta_2)d\mu^{(n)}\right]^{1/2}$$
$$=\left[\left\{\int_{A(\theta_1)\cap A(\theta_2)} f^2(x:\theta_1)/f(x:\theta_2)d\mu\right\}^2-2\{P_{\theta_1}(A(\theta_2))\}^n+\{P_{\theta_2}(A(\theta_1))\}^n\right]^{1/2}$$

Thus we complete the proof.

In order to use afterwards (4.5), we write

$$L(\theta_1,\ \theta_2)=1-\{P_{\theta_1}(A(\theta_2))\}^n$$
$$R(\theta_1,\ \theta_2)=1-\{P_{\theta_2}(A(\theta_1))\}^n$$
$$M(\theta_1,\ \theta_2)=\left[\left\{\int_{A(\theta_1)\cap A(\theta_2)} f^2(x:\theta_1)/f(x:\theta_2)d\mu\right\}^n-2\{P_{\theta_1}(A(\theta_2))\}^n+(P_{\theta_2}(A(\theta_1)))\}^n\right]^{1/2}$$

and we shall note that

$$M(\theta_1,\ \theta_2)\geq\int_{\underset{i=1}{\overset{n}{X}} A(\theta_1)\cap \underset{i=1}{\overset{n}{X}} A(\theta_2)}\left|\prod_{i=1}^{n} f(x_i:\theta_1)-\prod_{i=1}^{n} f(x_i:\theta_2)\right|d\mu^{(n)}.$$

Let $\mathfrak{X}=R^1$. Now we suppose that every $P_\theta(\cdot)(\theta\in\Theta)$ is absolutely continuous with respect to a Lebesgue measure m. Then we denote the density dP/dm by $f(\cdot:\theta)$ and suppose $f(x:\theta)=f(x-\theta)$. For the lemmas and theorems in sections 4 and 5 we make the following assumptions.

Assumption (A). $\quad f(x)>0$ for $a<x<b$,
$\quad f(x)=0$ for $x\leq a,\ x\geq b$.

Assumption (B). $f(x)$ is twice continuously differentiable in the interval $(a,\ b)$ and

$$\lim_{x\to a+0}(x-a)^{1-\alpha}f(x)=A'$$

$$\lim_{x\to b-0}(b-x)^{1-\beta}f(x)=B'$$

where both α and β are positive constants satisfying $\alpha\leq\beta<\infty$, and A' and B' are positive finite numbers.

Assumption (C). $A''=\lim_{x\to a+0}(x-a)^{2-\alpha}|f'(x)|$ and $B''=\lim_{x\to b-0}(b-x)^{2-\beta}|f'(x)|$ are finite. For $\alpha\geq 2$, $f''(x)$ is bounded.

For example we see that the beta distributions $Be\ (\alpha,\ \beta)$ $(0<\alpha\leq\beta\leq 2$ or $3<\alpha\leq\beta<\infty)$ satisfy Assumptions (A), (B) and (C).

Lemma 4.4. Suppose that a density function f satisfies Assumptions (A), (B) and (C). If $\alpha=2$, then the following hold: for any $\varepsilon>0$,

$$n\int_{\{x:\,\varepsilon c_1(n\log n)<-(\partial^2/\partial\theta^2)\log f(x-\theta)\}} f(x-\theta)dx\to 0 \tag{4.6}$$

as $n\to\infty$ uniformly in θ of Θ,

$$\frac{1}{c_1 n\log n}\int_{\{x:0<-(\partial^2/\partial\theta^2)\log f(x-\theta)<\varepsilon c_1 n\log n\}}\{-(\partial^2/\partial\theta^2)\log f(x-\theta)\}f(x-\theta)dx\to 1 \tag{4.7}$$

as $n\to\infty$ uniformly in θ of Θ, where $c_1=\dfrac{1}{2}\left(\dfrac{A''^2}{A'}+\dfrac{B''^2}{B'}\right)$ if $\beta=2$, $c_1=\dfrac{A''^2}{2A'}$ if $\beta>2$;

Proof. It follows from Assumptions (A), (B) and (C) that there exist n_0 and η_n such that for all $n\geq n_0$,

$$0<x-a<\eta_n,\quad 0<b-x<\eta_n, \tag{4.8}$$

implies

$$A'-\frac{1}{n}<(x-a)^{-1}f(x)<A'+\frac{1}{n},\ A''-\frac{1}{n}<|f'(x)|<A''+\frac{1}{n},$$

$$B'-\frac{1}{n}<(b-x)^{1-\beta}f(x)<B'+\frac{1}{n},\ B''-\frac{1}{n}<(b-x)^{2-\beta}|f'(x)|<B''+\frac{1}{n}. \tag{4.9}$$

Let $A_{-n}=A'-\dfrac{1}{n}$, $A_n=A'+\dfrac{1}{n}$, $B_{-n}=B'-\dfrac{1}{n}$, $B_n=B'+\dfrac{1}{n}$, $A_{-n}''=A''-\dfrac{1}{n}$, $A_n''=A''+\dfrac{1}{n}$, $B_{-n}''=B''-\dfrac{1}{n}$ and $B_n''=B''+\dfrac{1}{n}$.

Putting

$$I_{1n}=\int_{\left\{x:\,\varepsilon c_1 n\log n<-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2\right\}\cap(a,\ a+\eta_{n_0})} f(x)dx,$$

$$I_{2n}=\int_{\left\{x:\,\varepsilon c_1 n\log n<-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2\right\}\cap[a+\eta_n\ ,\,b-\eta_{n_0}]} f(x)dx,$$

$$I_{3n}=\int_{\left\{x:\, \varepsilon c_1 n\log n<-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2\right\}\cap(b-\eta_{n_0},\, b)} f(x)dx$$

we have

$$\begin{aligned}
&n\int_{\left\{x:\, \varepsilon c_1 n\log n<-\frac{\partial^2}{\partial\theta^2}\log f(x-\theta)\right\}} f(x-\theta)dx\\
&=n\int_{\left\{x:\, \varepsilon c_1 n\log n<-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2\right\}} f(x)dx\\
&=n(I_{1n}+I_{2n}+I_{3n}).
\end{aligned}\tag{4.10}$$

Since $f'(x)$ and $f''(x)$ are bounded,

$$\lim_{n\to\infty} nI_{2n}=0. \tag{4.11}$$

Since $f''(x)$ is bounded, from (4.8) and (4.9) we have for sufficiently large n,

$$\begin{aligned}
nI_{1n}&=n\int_{\left\{x:\, \varepsilon c_1 n\log n<-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2\right\}\cap(a,\, a+\eta_{n_0})} f(x)dx\\
&\leqq n\int_{\left\{x:\, \varepsilon c_1 n\log n<\left\{\frac{A''_n}{A_{-n}(x-a)}\right\}^2\right\}} f(x)dx+O(n^{-1}(\log n)^{-2})\\
&=n\int_a^{a+\frac{A''_n}{A_{-n}}(\varepsilon c_1 n\log n)^{-\frac{1}{2}}} A_n(x-a)dx+O(n^{-1}(\log n)^{-2})\\
&=\frac{n}{2}A_n\frac{A_n''}{A_{-n}}(\varepsilon c_1 n\log n)^{-1}+O(n^{-1}(\log n)^{-2}).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\, nI_{1n}\\
&\leq\overline{\lim_{n\to\infty}}\,\frac{n}{2}A_n\frac{A_n''}{A_{-n}}(\varepsilon c_1 n\log n)^{-1}\\
&=\frac{1}{2}A''/(\varepsilon c_1)^{-1}\lim_{n\to\infty}(\log n)^{-1}\\
&=0.
\end{aligned}\tag{4.12}$$

Repeating a similar argument, we have

$$\lim_{n\to\infty} nI_{3n}=0. \tag{4.13}$$

It follows from (4.10), (4.11) and (4.12) that (4.6) holds.

Putting

$$\begin{aligned}
I_{1n}'&=\int_{\left\{x:\, 0<-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2<\varepsilon c_1 n\log n\right\}\cap(a,\, a+\eta_{n_0})}\left[-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2\right]f(x)dx,\\
I_{2n}'&=\int_{\left\{x:\, 0<-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2<\varepsilon c_1 n\log n\right\}\cap[a+\eta_{n_0},\, b-\eta_{n_0}]}\left[-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2\right]f(x)dx,\\
I_{3n}'&=\int_{\left\{x:\, 0<-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2<\varepsilon c_1 n\log n\right\}\cap(b-\eta_{n_0},\, b)}\left[-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2\right]f(x)dx.
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{n}{c_1 n\log n}\int_{\left\{x:\, 0<-\frac{\partial^2}{\partial\theta^2}\log f(x-\theta)<\varepsilon c_1 n\log n\right\}}\left\{-\frac{\partial^2}{\partial\theta^2}\log f(x-\theta)\right\}f(x-\theta)dx\\
&=\frac{1}{c_1\log n}\int_{\left\{x:\, 0<-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2<\varepsilon c_1 n\log n\right\}}\left[-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2\right]f(x)dx\\
&=\frac{1}{c_1\log n}(I_{1n}'+I_{2n}'+I_{3n}').
\end{aligned}\tag{4.14}$$

Since $f'(x)$ and $f''(x)$ are bounded,

$$\lim_{n\to\infty} \frac{1}{c_1 \log n} I_{2n}' = 0. \tag{4.15}$$

Since $f''(x)$ is bounded, from (4.8) and (4.9) we have for sufficiently large n,

$$\begin{aligned}
&\frac{1}{c_1 \log n} I_{1n}' \\
&= \frac{1}{c_1 \log n} \int_{\left\{x:0<-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2<\varepsilon c_1 n \log n\right\} \cap (a,\, a+\eta_{n_0})} \left[-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2\right] f(x)dx \\
&\leq \frac{1}{c_1 \log n} \int_{\left\{x:\frac{A_{-n}''^2}{A_n^2}\cdot\frac{1}{(x-a)^2}<\varepsilon c_1 n \log n\right\} \cap (a,\, a+\eta_{n_0})} \left[-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2\right] f(x)dx \\
&\leq \frac{1}{c_1 \log n} \int_{\frac{A_{-n}''}{A_n}(\varepsilon c_1 n \log n)^{-1/2}}^{\eta_{n_0}} \frac{A''^2_n}{A_{-n}} \cdot \frac{1}{x} dx + O(n^{-1}) \\
&= \frac{1}{c_1 \log n} \left\{ \frac{A_n''^2}{A_{-n}} \left(\log \eta_{n_0} - \log \frac{A_{-n}''}{A_n} + \frac{1}{2} \log \varepsilon c_1 + \frac{1}{2} \log n + \frac{1}{2} \log \log n \right) \right\} \\
&\quad + O(n^{-1}).
\end{aligned}$$

Hence we obtain

$$\overline{\lim_{n\to\infty}} \frac{1}{c_1 \log n} I_{1n}' \leq \frac{A''^2}{2c_1 A'}. \tag{4.16}$$

Further we have

$$\begin{aligned}
&\underline{\lim_{n\to\infty}} \frac{1}{c_1 \log n} I_{1n}' \\
&\geq \underline{\lim_{n\to\infty}} \frac{1}{c_1 \log n} \int_{\left\{x:\frac{A_n''^2}{A_{-n}^2(x-a)^2}<\varepsilon c_1 n \log n\right\} \cap (a,\, a+\eta_{n_0})} \left[-\frac{f''(x)}{f(x)}+\left\{\frac{f'(x)}{f(x)}\right\}^2\right] f(x)dx \\
&\geq \underline{\lim_{n\to\infty}} \frac{1}{c_1 \log n} \int_{\frac{A_n''}{A_{-n}}(\varepsilon c_1 n \log n)^{1/2}}^{\eta_{n_0}} \frac{A_{-n}''^2}{A_n} \cdot \frac{1}{x} dx \\
&= \underline{\lim_{n\to\infty}} \frac{1}{c_1 \log n} \left\{ \frac{A_{-n}''^2}{A_n} \left(\log \eta_{n_0} - \log \frac{A_n''}{A_{-n}} + \frac{1}{2} \log \varepsilon c_1 + \frac{1}{2} \log n \right.\right. \\
&\quad \left.\left. + \frac{1}{2} \log \log n \right) \right\} \\
&\geq \frac{A''^2}{2c_1 A'}.
\end{aligned} \tag{4.17}$$

It follows from (4.16) and (4.17) that

$$\lim_{n\to\infty} \frac{1}{c_1 \log n} I_{1n}' = \frac{A''^2}{2c_1 A'}. \tag{4.18}$$

Repeating a similar argument, we have

$$\lim_{n\to\infty} \frac{1}{c_1 \log n} I_{3n}' = \begin{cases} \dfrac{B''^2}{2c_1 B'} & \text{for } \beta=2, \\ 0 & \text{for } \beta>2. \end{cases} \tag{4.19}$$

Hence it follows from (4.15), (4.18) and $c_1 = \frac{1}{2}\left(\frac{A''^2}{A'} + \frac{B''^2}{B'}\right)$ if $\beta=2$, $c_1 = \frac{A''^2}{2A'}$ if $\beta>2$ that (4.7) holds.

Thus the proof is completed.

LEMMA 4.5. Suppose that a density function f satisfies Assumptions (A), (B) and (C). If $\alpha=2$, then the following hold: for any $\varepsilon>0$,

$$n \int_{\left\{x:\left|\frac{\partial}{\partial\theta}\log f(x-\theta)\right|>\varepsilon c_2 (n \log n)^{1/2}\right\}} f(x-\theta)dx \to 0$$

as $n \to \infty$ uniformly in θ of Θ, and

$$\frac{1}{c_2^2 \log n}\left[\int_{\left\{x:\left|\frac{\partial}{\partial\theta}\log f(x-\theta)\right|<\varepsilon c_2(n\log n)^{1/2}\right\}}\left\{\frac{\partial}{\partial\theta}\log f(x-\theta)\right\}^2 f(x-\theta)dx - \left\{\int_{\left\{x:\left|\frac{\partial}{\partial\theta}\log f(x-\theta)\right|<\varepsilon c_2(n\log n)^{1/2}\right\}}\left(\frac{\partial}{\partial\theta}\log f(x-\theta)\right)f(x-\theta)dx\right\}^2\right]\to 1$$

as $n\to\infty$ uniformly in θ of Θ where $c_2=\left\{\frac{1}{2}\left(\frac{A''^2}{A'}+\frac{B''^2}{B'}\right)\right\}^{\frac{1}{2}}$ if $\beta=2$, $c_2=\frac{A''}{\sqrt{2A'}}$ if $\beta>2$.

The proof is omitted because it is given by the same way as that of lemma 4.4.

The following lemma is already given in Takeuchi [5].

LEMMA 4.6. Suppose that a density function f satisfies Assumptions (A), (B) and (C). If $\alpha>2$, then

$$\int_a^b \frac{\{f'(x)\}^2}{f(x)}dx<\infty.$$

PROOF. Since $f(a)=0$ and $\lim_{x\to a+0} f(x)=0$, by the second mean value theorem in a neighborhood of a, we have

$$\frac{\{f'(x)\}^2}{f(x)}=\frac{2f'(\xi)f''(\xi)}{f'(\xi)}=2f''(\xi)$$

for $a<\xi<x$. Since $f''(x)$ is continuous and bounded, $\{f(x)\}^2/f(x)$ is bounded in the neighborhood of a, and also that of b, and so is the integral. Thus the proof is completed.

In the following theorem we shall show that there exist consistent estimators with different orders according to α of density functions in a family satisfying Assumptions (A), (B) and (C).

THEOREM 4.1. Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent identically distributed random variables with a density function satisfying Assumptions (A), (B) and (C). For each α there exists a consistent estimator with the order given by Table 1 respectively, where M.L.E. is the maximun likelihood estimator of θ, the existence of which is guaranteed since f is continuous and bounded.

Table 1.

α	order c_n	$\{c_n\}$-consistent estimator
$0<\alpha<2$	$n^{1/\alpha}$	$\{\min_{1\le i\le n} X_i+\max_{1\le i\le n} X_i-(a+b)\}/2$
$\alpha=2$	$(n\log n)^{1/2}$	M.L.E.
$\alpha>2$	$n^{1/2}$	M.L.E.

PROOF. 1) $0<\alpha<2$. Let $T_n(\tilde{X}_n)=\{\min_{1\le i\le n} X_i+\max_{1\le i\le n} X_i-(a+b)\}/2$. It follows from Assumptions (A) and (B) that there are positive constants C and γ such that

$$C\le(x-a)^{1-\alpha}f(x) \quad \text{for all} \quad x\in(a,\ a+\gamma)$$
$$C\le(b-x)^{1-\beta}f(x) \quad \text{for all} \quad x\in(b-\gamma,\ b).$$

Then we shall show that $\{T_n: n=1, 2, \cdots\}$ is a $\{n^{1/\alpha}\}$-consistent estimator.
It is sufficient to know that for every $\varepsilon>0$, we can choose L satisfying

$$L>\max\left\{\frac{1}{2}\left(\frac{\alpha}{C}\log\frac{2}{\varepsilon}\right)^{1/\alpha},\ 0\right\}.$$

Indeed, since the following holds: for each $n=1, 2, \cdots,$

$$\{\tilde{x}_n : T_n(\tilde{x}_n)-\theta>Ln^{-1/\alpha} \text{ and } \max_{1\le i\le n} x_i\le b+\theta\}$$
$$\subset\{\tilde{x}_n : a+\theta+2Ln^{-1/\alpha}<x_i\le b+\theta(i=1, 2, \cdots, n)\},$$

we have for each $n=1, 2, \cdots,$

$$\begin{aligned}
&P_\theta^{(n)}(\{T_n-\theta>Ln^{-1/\alpha}\})\\
&=P_\theta^{(n)}(\{T_n-\theta>Ln^{-1/\alpha} \text{ and } \max_{1\le i\le n} x_i\le b+\theta\})\\
&\le P_\theta^{(n)}(\{a+\theta+2Ln^{-1/\alpha}<x_i\le b+\theta \ (i=1, 2, \cdots)\})\\
&=\left\{\int_{a+\theta+2Ln^{-1/\alpha}}^{b+\theta} f(x-\theta)dx\right\}^n\\
&=\left\{\int_{a+2Ln^{-1/\alpha}}^{b} f(x)dx\right\}^n\\
&=\left\{1-\int_a^{a+2Ln^{-1/\alpha}} f(x)dx\right\}^n.
\end{aligned} \tag{4.20}$$

Similarly we also obtain for each $n=1, 2, \cdots,$

$$\begin{aligned}
&P_\theta^{(n)}(\{T_n-\theta<-Ln^{-1/\alpha}\})\\
&=\left\{\int_a^{b-2Ln^{-1/\alpha}} f(x)dx\right\}^n\\
&=\left\{1-\int_{b-2Ln^{-1/\alpha}}^{b} f(x)dx\right\}^n.
\end{aligned} \tag{4.21}$$

It follows from (4.20) and (4.21) for that each $n=1, 2, \cdots,$

$$\begin{aligned}
&P_\theta^{(n)}(\{|T_n-\theta|>Ln^{-1/\alpha}\})\\
&\le\left\{1-\int_a^{a+2Ln^{-1/\alpha}} f(x)dx\right\}^n+\left\{1-\int_{b-2Ln^{-1/\alpha}}^{b} f(x)dx\right\}^n.
\end{aligned}$$

Hence we have uniformly in θ of Θ,

$$\begin{aligned}
&\overline{\lim}\, P_\theta^{(n)}(\{|T_n-\theta|>Ln^{-1/\alpha}\})\\
&\le\lim_{n\to\infty}\left\{1-\int_a^{a+2Ln^{-1/\alpha}} f(x)dx\right\}^n+\lim_{n\to\infty}\left\{1-\int_{b-2Ln^{-1/\alpha}}^{b} f(x)dx\right\}^n\\
&\le 2\exp\left\{-\frac{C(2L)^\alpha}{\alpha}\right\}\\
&<\varepsilon.
\end{aligned}$$

Therefore it is seen that $\{T_n\}$ is $\{n^{1/\alpha}\}$-consistent.

2) $\alpha=2$. Since the M.L.E. is a consistent estimator (Wald [6]) and it is a root of

$$\sum_{i=1}^n \frac{\partial}{\partial\theta}\log f(x_i-\theta)=0, \tag{4.22}$$

there exist at least a consistent solution of (4.22). We denote it by T_n^*.

Let $A_n=(n\log n)^{1/2}$ and put $L_n(\theta, \tilde{x}_n)=\prod_{i=1}^n f(x_i-\theta)$ for $\theta+a<x_i<\theta+b$ $(i=1, 2, \cdots, n)$.

Using the mean value theorem, we have

$$-\frac{1}{c^2A_n^2}\left[\frac{\partial^2}{\partial\theta^2}\log L_n\right]_{\theta=\theta_n^*}cA_n(T_n^*-\theta)=\frac{1}{cA_n}\left[\frac{\partial}{\partial\theta}\log L_n\right]_{\theta=\theta}, \tag{4.23}$$

where $|\theta-\theta_n^*|\le|\theta-T_n^*|$ and $c=\left\{\frac{1}{2}\left(\frac{A''^2}{A'}+\frac{B''^2}{B'}\right)\right\}^{1/2}$ if $\beta=2$, $c=\frac{A''}{\sqrt{2A'}}$ if $\beta>2$.

$-(\partial^2/\partial\theta^2)\log L_n$ is the sums of positive i.i.d. random variables $-(\partial^2/\partial\theta^2)\log f(X_1-\theta)$, $-(\partial^2/\partial\theta^2)\log f(X_2-\theta)$, $\cdots$, $-(\partial^2/\partial\theta^2)\log f(X_n-\theta)$. If $c^2A_n^2$ is taken as $B_n(\theta)$ in lemma 4.1, then it follows from lemma 4.4. that the conditions (4.1) and (4.2) hold. From lemma 4.1 we conclude that $-(\partial^2/\partial\theta^2)\log L_n$ is uniformly relatively stable for constant $c^2A_n^2$. Since

T_n^* is uniformly consistent in any compact subset of Θ (Wald [6]), θ_n^* converges in probability to θ uniformly in any compact subset of Θ. Furthermore since $(\partial^2/\partial\theta^2)\log L_n(\theta, \tilde{x}_n)$ is uniformly continuous is any compact subset of Θ, it is seen that $(-1/c^2A_n^2)[(\partial^2/\partial\theta^2)\log L_n]_{\theta=\theta_n^*}$ converges in probability to 1 uniformly in any compact subset of Θ.

$(\partial/\partial\theta)\log L_n$ is the sums of i.i.d. random variables $f_\theta(X_1-\theta)/f(X_1-\theta)$, $f_\theta(X_2-\theta)/f(X_2-\theta)$, $\cdots$, $f_\theta(X_n-\theta)/f(X_n-\theta)$, where $f_\theta(X-\theta)=(\partial/\partial\theta)f(X-\theta)$. If cA_n is taken as $B_n(\theta)$ in lemma 4.2, then it follows from lemma 4.5 that conditions (4.3) and (4.4) are satisfied. From lemma 4.2 we see that the distribution laws of $(1/cA_n)\{(\partial/\partial\theta)\log L_n\}$ converges to the normal law $\Phi(x)=(1/\sqrt{2\pi})\int_{-\infty}^{x} e^{-y^2/2}dy$ uniformly in anycompact subset of Θ.

Since from (4.23)

$$cA_n(T_n^*-\theta)=\frac{(1/cA_n)[(\partial/\partial\theta)\log L_n]_{\theta=\theta}}{(-1/c^2A_n^2)[(\partial^2/\partial\theta^2)\log L_n]_{\theta=\theta_n^*}},$$

it follows that the distribution laws of $cA_n(T_n^*-\theta)$ coverges to the normal $\Phi(x)$ uniformly in any compact subset of Θ.

In order to prove that $\{T_n^*: n=1, 2, \cdots\}$ is a $\{A_n\}$-consistent estimators, it is sufficient to show that for any $\varepsilon>0$ we can choose L satisfying $\int_{-CL}^{CL}(1/\sqrt{2\pi})e^{-x^2/2}dx>1-\varepsilon$ and that (2.1) holds.

Since

$$\begin{aligned}&P_\theta^{(n)}(\{A_n|T_n^*-\theta|\geq L)\\ &=P_\theta^{(n)}(\{cA_n|T_n^*-\theta|\geq cL\})\\ &=1-P_\theta^{(n)}(\{cA_n|T_n^*-\theta|<cL\})\end{aligned}$$

it follows that for every $\vartheta\in\Theta$ there exists $\delta>0$ such that

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\ \sup_{\theta:|\theta-\vartheta|<\delta} P_\theta^{(n)}(\{A_n|T_n^*-\theta|\geq L\})\\ &=1-\int_{-CL}^{CL}(1/\sqrt{2\pi})e^{-x^2/2}dx\\ &<\varepsilon.\end{aligned}$$

Hence it is shown that $\{T_n^*\}$ is $\{(n\log n)^{1/2}\}$-consistent.

3) $\alpha>2$. It follows from Assumption (C) that $E_\theta(Z_\theta)=0$ and $E_\theta(Z_{\theta\theta})+E_\theta(Z_\theta^2)=0$, where $Z_\theta=(\partial/\partial\theta)\log f(x-\theta)$ and $Z_{\theta\theta}=(\partial^2/\partial\theta^2)\log f(x-\theta)$. Further it is seen from lemma 4.6 that $E_\theta(Z_\theta^2)<\infty$. Hence the distribution law of $\sqrt{nI}(T_n^*-\theta)$ converges to the normal law $\Phi(x)$ uniformly in any compact subset of Θ, where $I=E_\theta(Z_\theta^2)$ (Cramér [2]). Therefore it is shown in the same way as the case $\alpha=2$ that $\{T_n^*: n=1, 2, \cdots\}$ is a $\{n^{1/2}\}$-consistent estimator. Thus we complete the proof.

5. Bounds for the Order of Convergence of Consistent Estimators

In this section we shall show that for each α, there does not exist a consistent estimator with the order greater than values as given in Table 1 of Theorem 4.1, that is, the order given by Table 1 is bound of the order of convergence of consistent estimators. Before proceeding to the next theorem, we shall prove the following lemmas.

LEMMA 5.1. Let f be a density function satisfying Assumption (A). Suppose that for $0<\Delta<b-a$, there exists a measurable function $g(\cdot)$ on $\mathfrak{X}$ such that $g(x)>0$ if $a-\Delta<x<b$, $g(x)=0$ otherwise and $\int_{\mathfrak{X}}g(x)dx=1$. Then

$$d_n(\theta-\Delta, \theta)\leq\left[\left\{\int_{a-\Delta}^{b}\frac{(f(x+\Delta)-g(x))^2}{g(x)}dx+1\right\}^n-1\right]^{1/2} \tag{5.1}$$

$$+\left[\left\{\int_{a-\Delta}^{b}\frac{(f(x)-g(x))^2}{g(x)}dx+1\right\}^n-1\right]^{1/2}.$$

PROOF. First we have

$$\begin{aligned}
&d_n(\theta-\Delta,\ \theta)\\
&=\int_{\mathfrak{X}^{(n)}}\left|\prod_{i=1}^{n}f(x_i-\theta+\Delta)-\prod_{i=1}^{n}f(x_i-\theta)\right|\prod_{i=1}^{n}dx_i\\
&\leq\int_{\mathfrak{X}^{(n)}}\left|\prod_{i=1}^{n}f(x_i-\theta+\Delta)-\prod_{i=1}^{n}g(x_i-\theta)\right|\prod_{i=1}^{n}dx_i+\int_{\mathfrak{X}^{(n)}}\left|\prod_{i=1}^{n}f(x_i-\theta)-\prod_{i=1}^{n}g(x_i-\theta)\right|\prod_{i=1}^{n}dx_i\\
&=\int_{a-\Delta}^{b}\cdots\int_{a-\Delta}^{b}\left|\prod_{i=1}^{n}f(x_i+\Delta)-\prod_{i=1}^{n}g(x_i)\right|\prod_{i=1}^{n}dx_i+\int_{a-\Delta}^{b}\cdots\int_{a-\Delta}^{b}\left|\prod_{i=1}^{n}f(x_i)-\prod_{i=1}^{n}g(x_i)\right|\prod_{i=1}^{n}dx_i\\
&=\int_{a-\Delta}^{b}\cdots\int_{a-\Delta}^{b}\left|\prod_{i=1}^{n}\frac{f(x_i+\Delta)}{g(x_i)}-1\right|\prod_{i=1}^{n}g(x_i)\prod_{i=1}^{n}dx_i+\int_{a-\Delta}^{b}\cdots\int_{a-\Delta}^{b}\left|\prod_{i=1}^{n}\frac{f(x_i)}{g(x_i)}-1\right|\prod_{i=1}^{n}g(x_i)\prod_{i=1}^{n}dx_i\\
&\leq\left[\int_{a-\Delta}^{b}\cdots\int_{a-\Delta}^{b}\left\{\prod_{i=1}^{n}\frac{f(x_i+\Delta)}{g(x_i)}-1\right\}^2\prod_{i=1}^{n}g(x_i)\prod_{i=1}^{n}dx_i\right]^{1/2}\\
&\quad+\left[\int_{a-\Delta}^{b}\cdots\int_{a-\Delta}^{b}\left\{\prod_{i=1}^{n}\frac{f(x_i)}{g(x_i)}-1\right\}^2\prod_{i=1}^{n}g(x_i)\prod_{i=1}^{n}dx_i\right]^{1/2}. \qquad (5.2)
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
&\int_{a-\Delta}^{b}\cdots\int_{a-\Delta}^{b}\left\{\prod_{i=1}^{n}\frac{f(x_i+\Delta)}{g(x_i)}-1\right\}^2\prod_{i=1}^{n}g(x_i)\prod_{i=1}^{n}dx_i\\
&=\int_{a-\Delta}^{b}\cdots\int_{a-\Delta}^{b}\prod_{i=1}^{n}\left\{\frac{f(x_i+\Delta)}{g(x_i)}\right\}^2\prod_{i=1}^{n}g(x_i)\prod_{i=1}^{n}dx_i-2\int_{a-\Delta}^{b}\cdots\int_{a-\Delta}^{b}\prod_{i=1}^{n}f(x_i+\Delta)\prod_{i=1}^{n}dx_i\\
&\quad+\int_{a-\Delta}^{b}\cdots\int_{a-\Delta}^{b}\prod_{i=1}^{n}g(x_i)\prod_{i=1}^{n}dx_i\\
&=\left[\int_{a-\Delta}^{b}\left\{\frac{f(x+\Delta)}{g(x)}\right\}^2g(x)dx\right]^n-1\\
&=\left[\int_{a-\Delta}^{b}\frac{\{(f(x+\Delta)-g(x))+g(x)\}^2}{g(x)}dx\right]^n-1\\
&=\left[\int_{a-\Delta}^{b}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx+2\int_{a-\Delta}^{b}\{f(x+\Delta)-g(x)\}dx+\int_{a-\Delta}^{b}g(x)dx\right]^n-1\\
&=\left[\int_{a-\Delta}^{b}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx+1\right]^n-1. \qquad (5.3)
\end{aligned}$$

Similarly we have

$$\int_{a-\Delta}^{b}\cdots\int_{a-\Delta}^{b}\left\{\prod_{i=1}^{n}\frac{f(x_i)}{g(x_i)}-1\right\}^2\prod_{i=1}^{n}g(x_i)\prod_{i=1}^{n}dx_i=\left[\int_{a-\Delta}^{b}\frac{\{f(x)-g(x)\}^2}{g(x)}dx+1\right]^n-1. \qquad (5.4)$$

It follows from (5.2), (5.3) and (5.4) that (5.1) holds. Thus we complete the proof.

If the assumptions of Lemma 5.1 hold, we can define an information I such that

$$I=\int_{a-\Delta}^{b}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx.$$

Henceforth for $0<\Delta<b-a$, we put $g(x)=\frac{1}{2}\{f(x+\Delta)+f(x)\}$. Then it is easily seen that $g(\cdot)$ satisfies the assumption of Lemma 5.1. Since

$$f(x+\Delta)-g(x)=\frac{1}{2}\{f(x+\Delta)-f(x)\}$$

and

$$f(x)-g(x)=\frac{1}{2}\{f(x)-f(x+\Delta)\},$$

it follows from (5.1) that

$$d_n(\theta-\Delta,\ \theta)\leq 2\left[\left\{\int_{a-\Delta}^{b}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx+1\right\}^n-1\right]^{1/2}=2\{(I+1)^n-1\}^{1/2}. \tag{5.5}$$

Henceforth we suppose that $f(x)$ satisfies Assumptions (A), (B) and (C). Then there exist positive numbers K_i, K_i' $(i=1, 2, 3)$ and ε such that

$$0<K_1\leq(x-a)^{1-\alpha}f(x)\leq K_2 \quad \text{for } a<x<a+\varepsilon, \tag{5.6}$$

$$0<K_1'\leq(b-x)^{1-\beta}f(x)\leq K_2' \quad \text{for } b-\varepsilon<x<b, \tag{5.7}$$

$$(x-a)^{2-\alpha}|f'(x)|\leq K_3 \quad \text{for } a<x<a+\varepsilon, \tag{5.8}$$

$$(b-x)^{2-\beta}|f'(x)|\leq K_3' \quad \text{for } b-\varepsilon<x<b, \tag{5.9}$$

$$0<\varepsilon<\min\left\{1,\ \frac{b-a}{2}\right\}.$$

Let $0<\Delta<\frac{\varepsilon}{2}$.

Now we devide I into six parts I_0, I_1, I_2, I_3, I_4 and I_5, that is,

$$I=\sum_{i=0}^{5}I_i,$$

where

$$I_0=\int_{a-\Delta}^{a}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx, \qquad I_1=\int_{a}^{a+\Delta}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx,$$

$$I_2=\int_{a+\Delta}^{a+\varepsilon}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx, \qquad I_3=\int_{a+\varepsilon}^{b-\varepsilon}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx,$$

$$I_4=\int_{b-\varepsilon}^{b-\Delta}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx, \qquad I_5=\int_{b-\Delta}^{b}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx.$$

LEMMA 5.2. For each $\alpha>0$, the orders of I_0, I_1, I_2, I_3, I_4, I_5 and I are given by Table 2.

Table 2

α	I_0	I_1	I_2	I_3	I_4	I_5	I
$0<\alpha<2$	$O(\Delta^\alpha)$	$O(\Delta^\alpha)$	$O(\Delta^\alpha)$	$O(\Delta^2)$	$O(\Delta^2)$ if $\beta\neq 2$; $O(\Delta^2\lvert\log\Delta\rvert)$ if $\beta=2$	$O(\Delta^\beta)$	$O(\Delta^\alpha)$
$\alpha=2$	$O(\Delta^2)$	$O(\Delta^2)$	$O(\Delta^2\lvert\log\Delta\rvert)$				$O(\Delta^2\lvert\log\Delta\rvert)$
$\alpha>2$	$O(\Delta^\alpha)$	$O(\Delta^\alpha)$	$O(\Delta^2)$				$O(\Delta^2)$

PROOF. i) I_0 and I_1. It follows from (5.6) that

$$I_0=\int_{a-\Delta}^{a}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx=\int_{a-\Delta}^{a}\frac{f(x+\Delta)}{2}dx=O(\Delta^\alpha). \tag{5.10}$$

Since

$$\begin{aligned}I_1&=\int_{a}^{a+\Delta}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx\\&=\int_{a}^{a+\Delta}\frac{\{(f(x+\Delta)-f(x))/2\}^2}{\{(f(x+\Delta)+f(x))/2\}^2}\{(f(x+\Delta)+f(x))/2\}dx\leq\frac{1}{2}\int_{a}^{a+\Delta}\{f(x+\Delta)+f(x)\}dx,\end{aligned}$$

it follows from (5.6) that

$$I_1=O(\Delta^{\alpha}). \tag{5.11}$$

ii) I_2. It follows by the mean value theorem that

$$\begin{aligned} I_2&=\int_{a+\Delta}^{a+\varepsilon}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx \\ &=\int_{a+\Delta}^{a+\varepsilon}\frac{\{(f(x+\Delta)-f(x))/2\}^2}{\{(f(x+\Delta)+f(x))/2\}^2}dx\leq\frac{1}{2}\int_{a+\Delta}^{a+\varepsilon}\frac{\{f(x+\Delta)-f(x)\}^2}{f(x)}dx \\ &=\frac{1}{2}\int_{a+\Delta}^{a+\varepsilon}\Delta^2\frac{\{f'(\xi(x,\ \Delta))\}^2}{f(x)}dx, \end{aligned} \tag{5.12}$$

where

$$a+\Delta<x<\xi(x,\ \Delta)<x+\Delta<a+\varepsilon+\Delta.$$

If $0<\alpha<2$, then it follows from (5.6), (5.8) and (5.12) that

$$\begin{aligned} I_2&\leq\int_{a+\Delta}^{a+\varepsilon}\Delta^2 C_1\frac{(\xi-a)^{2\alpha-4}}{(x-a)^{\alpha-1}}dx\leq C_1\Delta^2\int_{a+\Delta}^{a+\varepsilon}\frac{(x-a)^{2\alpha-4}}{(x-a)^{\alpha-1}}dx=C_1\Delta^2\int_{\Delta}^{\varepsilon}x^{\alpha-3}dx \\ &=\frac{C_1}{\alpha-2}\varepsilon^{\alpha-2}\Delta^2-\frac{C_1}{\alpha-2}\Delta^{\alpha}, \end{aligned} \tag{5.13}$$

where C_1 is some positive constant. If $\alpha=2$, then it follows from (5.8) that $f'(x)$ is bounded on $(a,\ a+\varepsilon)$. From (5.6) and (5.12) we have

$$I_2\leq C_2\Delta^2\int_{a+\Delta}^{a+\varepsilon}\{1/(x-a)\}dx=C_2\Delta^2(\log\varepsilon-\log\Delta), \tag{5.14}$$

where C_2 is some positive constant. If $\alpha>2$, then it follows from (5.6), (5.8) and (5.12) that

$$\begin{aligned} I_2&\leq C_3\int_{a+\Delta}^{a+\varepsilon}\Delta^2\frac{(\xi-a)^{2\alpha-4}}{(x-a)^{\alpha-1}}dx\leq C_3\Delta^2\int_{a+\Delta}^{a+\varepsilon}\frac{(x-a+\Delta)^{2\alpha-4}}{(x-a)^{\alpha-1}}dx \\ &=C_3\Delta^2\int_{\Delta}^{\varepsilon}x^{\alpha-3}\left(1+\frac{\Delta}{x}\right)^{2\alpha-4}dx\leq 2^{2\alpha-4}C_3\Delta^2\int_{\Delta}^{\varepsilon}x^{\alpha-3}dx\leq 2^{2\alpha-4}C_3\varepsilon^{\alpha-2}\Delta^2-\frac{2^{2\alpha-4}}{\alpha-2}C_3\Delta^{\alpha}, \end{aligned} \tag{5.15}$$

where C_3 is some positive constant. Hence it follows from (5.13), (5.14) and (5.15) that

$$I_2=\begin{cases} O(\Delta^{\alpha}) & \text{if } 0<\alpha<2, \\ O(\Delta^2|\log\Delta|) & \text{if } \alpha=2, \\ O(\Delta^2) & \text{if } \alpha>2. \end{cases} \tag{5.16}$$

iii) I_3. Since $f(x)$ and $f'(x)$ are continuous functions on $(a,\ b)$, it follows that

$$\begin{aligned} I_3&=\int_{a+\varepsilon}^{b-\varepsilon}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx\leq\frac{1}{2}\int_{a+\varepsilon}^{b-\varepsilon}\frac{\{f(x+\Delta)-f(x)\}^2}{g(x)}dx \\ &=\frac{1}{2}\int_{a+\varepsilon}^{b-\varepsilon}\frac{\Delta^2\{f'(\xi(x,\ \Delta))\}^2}{f(x)}dx\leq C_4'\Delta^2\int_{a+\varepsilon}^{b-\varepsilon}\{1/f(x)\}dx \\ &=C_4\Delta^2, \end{aligned} \tag{5.17}$$

where

$$a+\varepsilon<x<\xi(x,\ \Delta)<x+\Delta<b-(\varepsilon/2),$$

and C_4' and C_4 are certain positive constants. Hence we have

$$I_3=O(\Delta^2). \tag{5.18}$$

iv) I_4. It follows by the mean value theorem that

$$\begin{aligned} I_4&=\int_{b-\varepsilon}^{b-\Delta}\frac{\{f(x+\Delta)-g(x)\}^2}{g(x)}dx\leq\frac{1}{2}\int_{b-\varepsilon}^{b-\Delta}\frac{\{f(x+\Delta)-f(x)\}^2}{f(x)}dx \\ &=\frac{1}{2}\int_{b-\varepsilon}^{b-\Delta}\frac{\Delta^2\{f'(\xi(x,\ \Delta))\}^2}{f(x)}dx, \end{aligned} \tag{5.19}$$

where

$$b-\varepsilon<x<\xi(x,\ \Delta)<x+\Delta<b-(\varepsilon/2).$$

If $0<\beta<2$, then it follows from (5.7), (5.9) and (5.19) that

$$I_4 \leq C_5 \Delta^2 \int_{b-\varepsilon}^{b-\Delta} \frac{(b-\xi)^{2\beta-4}}{(b-x)^{\beta-1}} dx \leq C_5 \Delta^2 \left(\frac{\varepsilon}{2}\right)^{2\beta-4} \int_{\Delta}^{\varepsilon} x^{1-\beta} dx$$
$$= \frac{C_5}{2-\beta} \frac{\varepsilon^{\beta-2}}{2^{2\beta-4}} \Delta^2 - \frac{C_5}{2-\beta} \left(\frac{\varepsilon}{2}\right)^{2\beta-4} \Delta^{4-\beta}, \tag{5.20}$$

where C_5 is some positive constant. If $2 \leq \beta$, then it follows from (5.7), (5.9) and (5.19) that

$$I_4 \leq \int_{b-\varepsilon}^{b-\Delta} C_6 \Delta^2 \frac{(b-\xi)^{2\beta-4}}{(b-x)^{\beta-1}} dx \leq C_6 \Delta^2 \int_{b-\varepsilon}^{b-\Delta} \frac{(b-x)^{2\beta-4}}{(b-x)^{\beta-1}} dx$$
$$= C_6 \Delta^2 \int_{\Delta}^{\varepsilon} x^{\beta-3} dx$$
$$= \begin{cases} C_6 \Delta^2 (\log \varepsilon - \log \Delta) & \text{if } \beta = 2, \\ C_6 \Delta^2 \dfrac{1}{\beta-2} (\varepsilon^{\beta-2} - \Delta^{\beta-2}) & \text{if } \beta > 2, \end{cases} \tag{5.21}$$

where C_6 is some positive constant. Hence it follows from (5.20) and (5.21) that

$$I_4 = \begin{cases} O(\Delta^2) & \text{if } \beta \neq 2, \\ O(\Delta^2 |\log \Delta|) & \text{if } \beta = 2. \end{cases} \tag{5.22}$$

v) I_5. It follows from (5.8) that

$$I_5 = \int_{b-\Delta}^{b} \frac{\{f(x+\Delta) - g(x)\}^2}{g(x)} bx$$
$$= \int_{b-\Delta}^{b} \frac{f(x)}{2} dx$$
$$= O(\Delta^\beta). \tag{5.23}$$

Since $I = \sum_{i=0}^{5} I_i$, it follows from (5.10), (5.11), (5.16), (5.18), (5.22) and (5.23) that

$$I = \begin{cases} O(\Delta^\alpha) & \text{if } 0 < \alpha < 2, \\ O(\Delta^2 |\log \Delta|) & \text{if } \alpha = 2, \\ O(\Delta^2) & \text{if } \alpha > 2. \end{cases}$$

Thus we complete the proof.

REMARK. We also define an information I^* such that

$$I^* = \int_a^b \frac{\{f(x+\Delta) - f(x)\}^2}{f(x)} dx.$$

Since

$$\frac{\{f(x+\Delta) - g(x)\}^2}{g(x)} \leq \frac{1}{2} \frac{\{f(x+\Delta) - f(x)\}^2}{f(x)} \qquad \text{for } a < x < b,$$

it follows that $I_i \leq I_i^*$ $(i=1, 2, 3, 4, 5)$, where $I^* = \sum_{i=1}^{5} I_i^*$,

$$I_1^* = \int_a^{a+\Delta} \frac{\{f(x+\Delta) - f(x)\}^2}{f(x)} dx, \qquad I_2^* = \int_{a+\Delta}^{a+\varepsilon} \frac{\{f(x+\Delta) - f(x)\}^2}{f(x)} dx,$$
$$I_3^* = \int_{a+\varepsilon}^{b-\varepsilon} \frac{\{f(x+\Delta) - f(x)\}^2}{f(x)} dx, \qquad I_4^* = \int_{b-\varepsilon}^{b-\Delta} \frac{\{f(x+\Delta) - f(x)\}^2}{f(x)} dx,$$
$$I_5^* = \int_{b-\Delta}^{b} \frac{\{f(x+\Delta) - f(x)\}^2}{f(x)} dx.$$

It follows from the proof of Lemma 5.2 that for each $\alpha > 0$, the orders of I_i^* $(i=2, 3, 4, 5)$ given by Table 2 respectively. Furthermore if $0 < \alpha \leq 1$, then it follows from (5.6) that there exists a positive constant C_7 such that

$$0 < \frac{f(x+\Delta)}{f(x)} \leq \frac{K_2 (x+\Delta-a)^{\alpha-1}}{K_1 (x-a)^{\alpha-1}} = \frac{K_2}{K_1} \left(1 + \frac{\Delta}{x-a}\right)^{\alpha-1} \leq C_7 \qquad \text{for } a < x < a+\Delta \tag{5.24}$$

and the following hold:

$$\int_a^{a+\Delta} f(x)dx = O(\Delta^\alpha), \tag{5.25}$$

$$\int_a^{a+\Delta} f(x+\Delta)dx = O(\Delta^\alpha). \tag{5.26}$$

From (5.24) we have

$$\begin{aligned} I_1^* &= \int_a^{a+\Delta} \frac{\{f(x+\Delta)-f(x)\}^2}{f(x)} dx \\ &= \int_a^{a+\Delta} \frac{\{f(x+\Delta)\}^2}{f(x)} dx - 2\int_a^{a+\Delta} f(x+\Delta)dx + \int_a^{a+\Delta} f(x)dx \\ &\leq (C_7^2+1)\int_a^{a+\Delta} f(x)dx - 2\int_a^{a+\Delta} f(x+\Delta)dx \end{aligned} \tag{5.27}$$

It follows from (5.25), (5.26) and (5.27) that $I_1^* = O(\Delta^\alpha)$. Hence if $0<\alpha\leq 1$, the order of I_1^* is equal to the order of I_1.

From Lemmas 5.1 and 5.2 and (5.5) we get the following lemma.

LEMMA 5.3.

$$d_n(\theta-\Delta,\ \theta) = \begin{cases} 2[\{1+O(\Delta^\alpha)\}^n - 1]^{1/2} & \text{if } 0<\alpha<2, \\ 2[\{1+O(\Delta^2|\log \Delta|)\}^n - 1]^{1/2} & \text{if } \alpha=2, \\ 2[\{1+O(\Delta^2)\}^n - 1]^{1/2} & \text{if } \alpha>2. \end{cases}$$

THEOREM 5.1. Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent identically distributed random variables with a density function satisfying Assumptions (A), (B) and (C). For each α, the order given by Table 1 of Theorem 4.1 is the bound of the order of convergence of consistent estimators, that is, there does not exist a consistent estimator with the order greater than values as given by Table 1.

PROOF.

1) $0<\alpha<2$. From Lemma 5.3 we obtain for sufficiently large n and every $t>0$,

$$d_n(\theta-tc_n^{-1},\ \theta) \leq 2[\{1+O((tc_n^{-1})^\alpha)\}^n - 1]^{1/2}.$$

If order $\{c_n\}$ is greater than order $\{n^{1/\alpha}\}$, then $\lim_{n\to\infty} d_n(\theta-tc_n^{-1},,\ \theta)=0$ for all $t>0$ and all $\theta\in\Theta$. Hence it follows from Theorem 3.3 that there does not a consistent estimator with the order greater than order $\{n^{1/\alpha}\}$.

2) $\alpha=2$. From Lemma 5.3 we obtain for sufficiently large n and every $t>0$,

$$d_n(\theta-tc_n^{-1},\ \theta) \leq 2[\{1+O((tc_n^{-1})^2|\log tc_n^{-1}|)\} - 1]^{1/2}.$$

If order $\{c_n\}$ is greater than order $\{(n\log n)^{1/2}\}$, then $\lim_{n\to\infty} d(\theta-tc_n^{-1},\ \theta)=0$ for all $t>0$ and all $\theta\in\Theta$. Hence it follows from Theorem 3.3 that there does not exist a consistent estimator with the order greater than order $\{(n\log n)^{1/2}\}$.

3) $\alpha>2$. From Lemma 5.3 we have for sufficiently large n and every $t>0$,

$$d_n(\theta-tc_n^{-1},\ \theta) \leq 2[\{1-O((tc_n^{-1})^2)\}^n - 1]^{1/2}.$$

If order $\{c_n\}$ is greater than $\{n^{1/2}\}$, then $\lim_{n\to\infty} d_n(\theta-tc_n^{-1},\ \theta)=0$ for all $t>0$ and all $\theta\in\Theta$. Hence it follows from Theorem 3.3 that there does not exist a consistent estimator with the order greater than order $\{n^{1/2}\}$. Thus we complele the proof.

REMARK. Since $A(\theta)=(a+\theta,\ b+\theta)$, it follows from Assumptions (A) and (B) that for every $t>0$ and sufficiently large n

$$\begin{aligned} \{P_\theta(A(\theta-tc_n^{-1}))\}^n &= \left\{1-\int_{b+\theta-tc_n^{-1}}^{b+\theta} f(x-\theta)dx\right\}^n \\ &= \exp\left[n\log\left\{1-\int_{b-tc_n^{-1}}^{b} f(x)dx\right\}\right] \end{aligned}$$

$$=\exp\left[n\left\{-\frac{M}{\beta}t^{\beta}c_n{}^{-\beta}+O(c_n{}^{-2\beta})\right\}\right];$$
$$\{P_{\theta-tc_n{}^{-1}}(A(\theta))\}^n$$
$$=\exp\left[n\left\{-\frac{M}{\alpha}t^{\alpha}c_n{}^{-\alpha}+O(c_n{}^{-2\alpha})\right\}\right],$$

where M is some positive constant. From Lemma 4.3 we obtain the following results.

If $0<\alpha<2$ and $\alpha<\beta$, then every $\theta\in\Theta$ and every $t>0$,

$$\lim_{n\to\infty} L(\theta-tn^{-\frac{1}{\alpha}},\ \theta)=1-e^{-\frac{M}{\alpha}t^{\alpha}},$$

$$\lim_{n\to\infty} R(\theta-tn^{-\frac{1}{\alpha}},\ \theta)=0,$$

$$\lim_{n\to\infty} M(\theta-tn^{-\frac{1}{\alpha}},\ \theta)=\begin{cases}(e^{Kt^{\alpha}}-2e^{-\frac{M}{\alpha}t^{\alpha}}+1)^{1/2} & \text{for } 0<\alpha\leq 1,\\ \infty & \text{for } 1<\alpha<2,\end{cases}$$

$$\overline{\lim_{n\to\infty}}\, d_n(\theta-tn^{-\frac{1}{\alpha}},\ \theta)\leq 1-e^{-\frac{M}{\alpha}t^{\alpha}}+(e^{Kt^{\alpha}}-2e^{-\frac{M}{\alpha}t^{\alpha}}+1)^{1/2} \quad \text{for } 0<\alpha\leq 1,$$

$$\overline{\lim_{n\to\infty}}\, d_n(\theta-tn^{-\frac{1}{\alpha}},\ \theta)\leq 2(e^{ct^{\alpha}}-1)^{1/2} \quad \text{for } 1<\alpha<2,$$

where c is some positive constant and K is some constant. If $0<\alpha<2$ and $\alpha=\beta$, then for every $\theta\in\Theta$ and every $t>0$,

$$\lim_{n\to\infty} L(\theta-tn^{-\frac{1}{\alpha}},\ \theta)=\lim_{n\to\infty} R(\theta-tn^{-\frac{1}{\alpha}},\ \theta)=1-e^{-\frac{M}{\alpha}t^{\alpha}},$$

$$\lim_{n\to\infty} M(\theta-tn^{-\frac{1}{\alpha}},\ \theta)=\begin{cases}(e^{Kt^{\alpha}}-e^{-\frac{M}{\alpha}t^{\alpha}})^{1/2} & \text{for } 0<\alpha<1,\\ 0 & \text{for } \alpha=1,\\ \infty & \text{for } 1<\alpha<2,\end{cases}$$

$$\overline{\lim_{n\to\infty}}\, d_n(\theta-tn^{-\frac{1}{\alpha}},\ \theta)\leq 2(1-e^{-\frac{M}{\alpha}t^{\alpha}})+(e^{Kt^{\alpha}}-e^{-\frac{M}{\alpha}t^{\alpha}})^{1/2} \quad \text{for } 0<\alpha<1,$$

$$\overline{\lim_{n\to\infty}}\, d_n(\theta-tn^{-1},\ \theta)\leq 2(1-e^{-Mt}) \quad \text{for } \alpha=1,$$

$$\overline{\lim_{n\to\infty}}\, d_n(\theta-tn^{-\frac{1}{\alpha}},\ \theta)\leq 2(e^{ct^{\alpha}}-1)^{1/2} \quad \text{for } 1<\alpha<2.$$

If $\alpha=2$, then for every $\theta\in\Theta$ and every $t>0$,

$$\lim_{n\to\infty} L(\theta-t(n\log n)^{-\frac{1}{2}},\ \theta)=\lim_{n\to\infty} R(\theta-t(n\log n)^{-\frac{1}{2}},\ \theta)=0,$$

$$\lim_{n\to\infty} M(\theta-t(n\log n)^{-\frac{1}{2}},\ \theta)=\infty,$$

but

$$\overline{\lim_{n\to\infty}}\, d_n(\theta-t(n\log n)^{-\frac{1}{2}},\ \theta)\leq 2(e^{c}-1)^{\frac{1}{2}},$$

where c is some positive constant. If $\alpha>2$, then for every $\theta\in\Theta$ and every $t>0$,

$$\lim_{n\to\infty} L(\theta-tn^{-\frac{1}{2}},\ \theta)=\lim_{n\to\infty} R(\theta-tn^{-\frac{1}{2}},\ \theta)=0,$$

$$\lim_{n\to\infty} M(\theta-tn^{-\frac{1}{2}},\ \theta)=\infty,$$

but

$$\overline{\lim_{n\to\infty}}\, d_n(\theta-tn^{-\frac{1}{2}},\ \theta)\leq 2(e^{c'}-1)$$

where c' is some positive constant.

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Department of Mathematics
University of Electro-Communications
Chofugaoka, Chofu-shi, Tokyo 182, Japan