ISOMETRIES OF THE SPECIAL ORTHOGONAL GROUP

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Abstract. In this paper we describe all isometries on the special orthogonal group. As an application we give a form of spectrally multiplicative map on the special orthogonal group.

1. Introduction

There is vast literature on these so called linear isometries on matrix spaces. Here we will be interested in isometries on groups of matrices, not linear spaces. The third author and L. Molnár studied surjective isometries (with respect to the metric induced by the operator norm) on unitary groups on Hilbert spaces in [6] (cf. [7]). By their results isometries (with respect to the metric induced by the operator norm) on $U(n)$ into itself are only automorphisms or anti-automorphisms up to unitary multiplications. In this paper we give a complete description of isometries on $SO(n)$. As a consequence of the result we will show that these are automorphisms, anti-automorphisms up to multiplications, and exceptional ones for $n = 4$.

In the following of the paper let n be a positive interger greater than 1 and $M_n(\mathbb{R})$ the real algebra of all $n \times n$ matrices of real entries with the identity matrix E_n. Denote by $SO(n)$ and $O(n)$ the groups of all special orthogonal matrices and all orthogonal matrices in $M_n(\mathbb{R})$ respectively. Let $\mathbb{R}^n_{+}\!\downarrow$ denote the set of all nonzero vectors $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ of the Euclidean n-space satisfying $x_1 \geq \cdots \geq x_n \geq 0$. For any $\mathfrak{c} = (c_1, \dots, c_n) \in \mathbb{R}^n_{+}\!\downarrow$ we define the so called c-spectral norm on $A \in M_n(\mathbb{R})$ by

$$\|A\|_{\mathfrak{c}} = \sum_{i=1}^{n} c_i \sigma_i(A),$$

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where $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$ are the singular values of A. In the following of the paper we assume that $c_1 = 1$ for $\mathfrak{c} = (c_1, \ldots, c_n) \in \mathbb{R}^n_+\!\downarrow$. This assumption does not lose generality in the paper. Note that the operator norm coincides with the c-spectral norm for $\mathfrak{c} = (1, 0, \ldots, 0)$.

In this paper we give a complete description of all isometries on $SO(n)$ with respect to the metric induced by the c-spectral norm. Isometries on $SO(n)$ are of the expected forms in one hand; automorphism or anti-automorphism followed by the multiplication, and of an exceptional form for $n = 4$ in another hand. Let $K_n(\mathbb{R})$ be the space of all skew-symmetric matrices. For $A \in K_4(\mathbb{R})$ the matrix $\widetilde{A}$ is obtained from A by interchanging its $(1, 4)$ and $(2, 3)$ entries, and interchanging the $(4, 1)$ and $(3, 2)$ entries respectively. Then as we will show later in this paper that $T(\exp(A)) = \exp(\widetilde{A})$ for $A \in K_4(\mathbb{R})$ defines a surjective isometries from $SO(4)$ onto itself, which is neither an automorphism nor an anti-automorphism. Nevertheless we note that *any* isometry T on $SO(n)$ preserves the structure of the so-called twisted subgroups, that is, $T(XY^{-1}X) = T(X)(T(Y))^{-1}T(X)$ for all $X, Y \in SO(n)$ for any isometry T from $SO(n)$ into itself.

2. The main result

The main result of the paper is the following.

Theorem 1. *Let T be a map from $SO(n)$ into itself and $\mathfrak{c} \in \mathbb{R}^n_+\!\downarrow$. Then the following* (i) *and* (ii) *are equivalent to each other.*

(i) *T is an isometry with respect to the metric induced by $\|\cdot\|_{\mathfrak{c}}$; $\|T(X) - T(Y)\|_{\mathfrak{c}} = \|X - Y\|_{\mathfrak{c}}$ for every pair $X, Y \in SO(n)$.*

(ii) *There exists $O \in O(n)$ such that T is of one of the following form:*

- (a): *$T(X) = T(E_n)OXO^{-1}$ for every $X \in SO(n)$,*
- (b): *$T(X) = T(E_n)OX^{-1}O^{-1}$ for every $X \in SO(n)$,*
- (c): *$n = 4$ and $T(X) = T(E_4)O(\exp(\widetilde{A}))O^{-1}$ for every $X \in SO(n)$, where $A \in K_4(\mathbb{R})$ with $\exp(A) = X$,*
- (d): *$n = 4$ and $T(X) = T(E_4)O(\exp(-\widetilde{A}))O^{-1}$ for every $X \in SO(n)$, where $A \in K_4(\mathbb{R})$ with $\exp(A) = X$.*

In this case T is surjective. Further if $T(E_n) = E_n$, then T is an automorphism on $SO(n)$ for (a)*; T is an anti-automorphism on $SO(n)$ for* (b)*; T is neither multiplicative nor anti-multiplicative on $SO(4)$ for* (c) *and* (d)*. On the other hand in any case T preserves the inverted Jordan product; $T(XY^{-1}X) = T(X)(T(Y))^{-1}T(X)$ for all $X, Y \in SO(n)$.*

Note that $\exp(A) = \exp(B)$ if and only if $\exp(\widetilde{A}) = \exp(\widetilde{B})$. We will show a proof of Theorem 1 in the later section.

3. Necessary conditions for isometries

Nobuya Watanabe [11] has notified us the following lemma which is appeared in [2, Excercise 2.4.1].

Lemma 2. *Suppose that T is an isometry from a compact metric space into itself. Then T is surjective.*

A proof is elementary and is omitted.

Lemma 3. *Suppose that $T : SO(n) \to SO(n)$ is an isometry with respect to the metric d induced by the norm $\|\cdot\|_{\mathfrak{c}}$. Then*

$$T(YX^{-1}Y) = T(Y)(T(X))^{-1}T(Y) \tag{1}$$

for every pair $X, Y \in SO(n)$. In particular, if T further satisfies that $T(E_n) = E_n$, then

$$T(YXY) = T(Y)T(X)T(Y) \tag{2}$$

for every pair $X, Y \in SO(n)$.

Proof. We first note that T is surjective by Lemma 2 as $SO(n)$ is compact. We can prove the equation (1) for $X, Y \in SO(n)$ with $d(X,Y) < \frac{1}{2}$ by a proof similar to that of Theorem 6 in [6] as follows. The conditions $\mathrm{C}_1(T(Y), T(YX^{-1}Y))$ and (B1) of B(X, Y) are apparently satisfied. (See [6, Definitions 1, 2 and 3] or [4] for the definitions of these conditions $\mathrm{C}_1(\cdot,\cdot)$, $\mathrm{B}(\cdot,\cdot)$, (B$\cdot$) and others.) It remains to check the condition (B2). Let $X, Y \in SO(n)$ such that $d(X,Y) < \frac{1}{2}$. We assert that with $K = 2 - 2d(X,Y) > 1$, the inequality

$$d(YW^{-1}Y, W) \geq K d(W, Y)$$

holds for every $W \in L_{X,Y}$, where

$$L_{X,Y} = \{W \in SO(n) : d(X, W) = d(YX^{-1}Y, W) = d(X, Y)\}.$$

To see this, let $W \in L_{X,Y}$. Then we see that

$$d(W, Y) \leq d(W, X) + d(X, Y) = 2d(X, Y)$$

and thus

$$2 - d(W, Y) \geq 2 - 2d(X, Y) = K.$$

We compute

$$d(W, Y) = \|W - Y\|_{\mathfrak{c}} = \|YW^{-1} - E_n\|_{\mathfrak{c}},$$

$$d(YW^{-1}Y, W) = \|YW^{-1}Y - W\|_{\mathfrak{c}} = \|YW^{-1}YW^{-1} - E_n\|_{\mathfrak{c}}$$
$$= \|(YW^{-1} + E_n)(YW^{-1} - E_n)\|_{\mathfrak{c}},$$

and

$$2\|YW^{-1} - E_n\|_{\mathfrak{c}} - \|(YW^{-1} + E_n)(YW^{-1} - E_n)\|_{\mathfrak{c}}$$
$$\leq \| \big(2E_n - (YW^{-1} + E_n)\big)(YW^{-1} - E_n)\|_{\mathfrak{c}} \leq \|YW^{-1} - E_n\|_{\mathfrak{c}}^2,$$

where the last inequality follows from the assumption that $c_1 = 1$. Thus

$$Kd(W, Y) \leq \big(2 - d(W, Y)\big)d(W, Y)$$
$$= 2\|YW^{-1} - E_n\|_{\mathfrak{c}} - \|YW^{-1} - E_n\|_{\mathfrak{c}}^2$$
$$\leq \|(YW^{-1} + E_n)(YW^{-1} - E_n)\|_{\mathfrak{c}} = d(YW^{-1}Y, W).$$

This gives us that the condition (B2) holds. Applying [6, Proposion 4] we have

$$T(YX^{-1}Y) = T(Y)(T(X))^{-1}T(Y)$$

for all $X, Y \in SO(n)$ with $d(X, Y) < \frac{1}{2}$.

Next we consider the general $X, Y \in SO(n)$. Since $X^{-1}Y \in SO(n)$, there exists a $Z \in K_n(\mathbb{R})$ such that $X^{-1}Y = \exp(Z)$. Let m be a positive integer such that $\exp(\frac{\|Z\|_{\mathfrak{c}}}{2^m}) - 1 < \frac{1}{2}$. As $c_1 = 1$ we have

$$\| \exp \frac{Z}{2^m} - E_n\|_{\mathfrak{c}} \leq \exp \frac{\|Z\|_{\mathfrak{c}}}{2^m} - 1 < \frac{1}{2}.$$

Let

$$A_k = X \exp \frac{kZ}{2^m}$$

for each $k = 0, 1, 2, \ldots, 2^{m+1}$. Then we have $A_0 = X$, $A_{2^m} = Y$, and $A_{2^{m+1}} = YX^{-1}Y$. It is easy to check that

$$A_{k+1}(A_k)^{-1}A_{k+1} = A_{k+2}$$

for every $k = 0, 1, 2, \ldots, 2^{m+1} - 2$. We also have

$$\|A_{k+1} - A_k\|_{\mathfrak{c}} = \| \exp \frac{Z}{2^m} - E_n\|_{\mathfrak{c}}$$

for every $k = 0, 1, 2, \ldots, 2^{m+1} - 1$. Then by the first part of the proof

$$T(A_{k+1}(A_k)^{-1}A_{k+1}) = T(A_{k+1})(T(A_k))^{-1}T(A_{k+1})$$

holds for every $k = 0, 1, 2, \ldots, 2^{m+1} - 2$. Applying [6, Lemma 7] we deduce that

$$T(YX^{-1}Y) = T(A_{2^m}(A_0)^{-1}A_{2^m})$$
$$= T(A_{2^m})(T(A_0))^{-1}T(A_{2^m}) = T(Y)(T(X))^{-1}T(Y);$$

we have (1).

In particular if $T(E_n) = E_n$, then letting $Y = E_n$ in (1), we observe $T(X^{-1}) = (T(X))^{-1}$ for every $X \in SO(n)$, whence the equation (2) holds. □

Lemma 4. *Let $A \in K_n(\mathbb{R})$. Suppose that $T_0 : SO(n) \to SO(n)$ is an isometry with respect to $\|\cdot\|_{\mathfrak{c}}$ such that $T_0(E_n) = E_n$. Let $S_A(t) = T_0(\exp(tA))$ for $t \in \mathbb{R}$. Then $S_A : \mathbb{R} \to SO(n)$ is a one-parameter group.*

Proof. As T_0 preserves the unit, for every $X \in SO(n)$ and for any integer l

$$T_0(X^l) = T_0(X)^l$$

is satisfied by the equation (2) and $T(X^{-1}) = (T(X))^{-1}$. We prove that $S_A(t+t') = S_A(t)S_A(t')$ holds for every pair t, t' of real numbers. First let $r = \frac{l}{m}$ and $r' = \frac{l'}{m'}$ be rational numbers with integers m, m', l, l'. We compute

$$\begin{aligned} S_A(r+r') &= T_0(\exp(\frac{lm'+ml'}{mm'}A)) \\ &= T_0(\exp(\frac{1}{mm'}A))^{lm'+ml'} = T_0(\exp(\frac{1}{mm'}A))^{lm'}T_0(\exp(\frac{1}{mm'}))^{ml'} \\ &= S_A(r)S_A(r'). \end{aligned}$$

As T_0 is continuous we observe that $S_A(t+t') = S_A(t)S_A(t')$ for every pair t, t' of real numbers. □

In the following we describes the necessary condition for the isometries between $SO(n)$, which is a part of Theorem 1. We remark that the main idea of the proofs of Lemmas 3 and 4, and Proposition 5 employing one parameter groups and a non-commutative generalization of the Mazur-Ulam theorem [4] have been motivated by recent paper of the third author and Molnár [7] where they describe the structure of surjective isometries between unitary groups of C^*-algebras. In particular the one-parameter-group argument had come from Sakai's paper [10] where he described the structure of the uniformly continuous group isomorphisms of unitary groups in AW^*-factors.

Proposition 5. *Suppose that $T : SO(n) \to SO(n)$ is an isometry with respect to $\|\cdot\|_{\mathfrak{c}}$. Then T is surjective and there exists $O \in O(n)$ such that one of the following holds.*

(a): $T(X) = T(E_n)OXO^{-1}$ *for every* $X \in SO(n)$.
(b): $T(X) = T(E_n)OX^{-1}O^{-1}$ *for every* $X \in SO(n)$.

(c): $n = 4$ *and* $T(\exp(A)) = T(E_4)O(\exp(\widetilde{A}))O^{-1}$ *for every* $A \in K_4(\mathbb{R})$.
(d): $n = 4$ *and* $T(\exp(A)) = T(E_4)O(\exp(-\widetilde{A}))O^{-1}$ *for every* $A \in K_4(\mathbb{R})$.

Proof. By Lemma 2 T is surjective. Put $T_0(X) = (T(E_n))^{-1}T(X)$ for $X \in SO(n)$. Then T_0 is a surjective isometry. By Lemma 4 $S_A : \mathbb{R} \to SO(n)$ is a one-parameter group for any $A \in K_n(\mathbb{R})$. It is well-known that there exists a unique element $f(A) \in K_n(\mathbb{R})$ such that $S_A(t) = \exp(tf(A))$ for every real number t; we constitute the map $f : K_n(\mathbb{R}) \to K_n(\mathbb{R})$.

We claim that f is surjective. As $(T_0)^{-1}$ is also a surjective isometry between $SO(n)$, in the same way as above there is a $g : K_n(\mathbb{R}) \to K_n(\mathbb{R})$ such that $T_0^{-1}(\exp(tA)) = \exp(tg(A))$ for every real number t and $A \in K_n(\mathbb{R})$. We have $\exp(tA) = T_0(\exp(tg(A))) = \exp(t(f(g(A))))$ for all t and $A \in K_n(\mathbb{R})$. Hence $f(g(A)) = A$ for every A; f is surjective.

We next show that f is a real-linear isometry. It is easy to check by the definition that $f(0) = 0$. As T_0 is an isometry,

$$\begin{aligned}
&\|A - B\|_{\mathfrak{c}} \\
&= \lim_{t\to 0} \left\| \frac{\exp(tA) - \exp(tB)}{t} \right\|_{\mathfrak{c}} = \lim_{t\to 0} \left\| \frac{T_0(\exp(tA)) - T_0(\exp(tB)}{t} \right\|_{\mathfrak{c}} \\
&\qquad = \lim_{t\to 0} \left\| \frac{\exp(tf(A)) - \exp(tf(B))}{t} \right\|_{\mathfrak{c}} = \|f(A) - f(B)\|_{\mathfrak{c}}
\end{aligned}$$

for every pair A and B in $K_n(\mathbb{R})$. We observe that f is a surjective isometry from $K_n(\mathbb{R})$ onto itself. Then by the celebrated Mazur-Ulam theorem f is a real-linear isometry.

Then by [8, Theorem 4.2] there exists an $O \in O(n)$ such that one of the following hold:
(aa) $f(A) = OAO^{-1}$ for every $A \in K_n(\mathbb{R})$;
(bb) $f(A) = -OAO^{-1}$ for every $A \in K_n(\mathbb{R})$;
(cc) $n = 4$ and $f(A) = O\widetilde{A}O^{-1}$ for every $A \in K_4(\mathbb{R})$;
(dd) $n = 4$ and $f(A) = -O\widetilde{A}O^{-1}$ for every $A \in K_4(\mathbb{R})$.
If f is of the form of (aa), then we have (a);

$$T_0(X) = \exp(f(A)) = \exp(OAO^{-1}) = O\exp(A)O^{-1} = OXO^{-1}$$

for every $X \in SO(n)$, where $X = \exp(A)$ for an $A \in K_n(\mathbb{R})$. In the same way we have (b), (c) and (d) from (bb), (cc), and (dd) respectively. □

4. The B-C-H formula of Fujii and Suzuki

If a map $T : SO(n) \to SO(n)$ is of the form (a) or (b) in Proposition 5, then T is apparently a surjective isometry, which is an automorphism followed by the multiplication for (a) and an anti-automorphism followed by the multiplication for (b). Fujii and Suzuki [3] describe a closed form of the Baker-Cambell-Hausdorff (B-C-H for short) formula in $SO(4)$ (Theorem 7). It will be applied to prove that T of the form of (c) or (d) is also a surjective isometry. We also show that it is not an automorphism nor an anti-automorphism by Theorem 7.

Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & -i & -1 & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & i \end{pmatrix},$$

which is called the magic matrix by Makhlin [3, p.900]. To prove Theorem 7 Fujii and Suzuki applied B-C-H formula in $SU(2)$, the special unitary group of the degree 2 [3, (11)]. A special emphasis is on the range of $\sin^{-1}\rho$, which is not stated clearly in [3], that $0 \leq \sin^{-1}\rho \leq \pi$ depending not only on ρ itself but also the value

$$\cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}(\boldsymbol{x}\cdot\boldsymbol{y}).$$

For the convenience of the reader we restate it here with a proof.

Let

$$H_0(2;\mathbb{C}) = \{X = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 : x_1, x_2, x_3 \in \mathbb{R}\}.$$

Then $iH_0(2;\mathbb{C})$ is the Lie algebra of the group $SU(2)$. For any element

$$X = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3,$$

of $H_0(2;\mathbb{C})$ we denote

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The following is the B-C-H formula in $SU(2)$ [3].

Theorem 6 (Fujii and Suzuki [3])**.** *Let $X, Y \in H_0(2;\mathbb{C})$. Then*

$$\exp(iX)\exp(iY) = \exp(iZ_0) \tag{3}$$

for $Z_0 = \alpha X + \beta Y + \frac{i}{2}\gamma(XY - YX)$*, where*

$$\alpha \equiv \alpha(\boldsymbol{x}, \boldsymbol{y}) = \frac{\sin^{-1}\rho}{\rho}\frac{\sin|\boldsymbol{x}|\cos|\boldsymbol{y}|}{|\boldsymbol{x}|},$$

$$\beta \equiv \beta(\boldsymbol{x}, \boldsymbol{y}) = \frac{\sin^{-1}\rho}{\rho}\frac{\cos|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{y}|}, \tag{4}$$

$$\gamma \equiv \gamma(\boldsymbol{x}, \boldsymbol{y}) = \frac{\sin^{-1}\rho}{\rho}\frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}$$

with

(5)

$$\rho \equiv \rho(\boldsymbol{x}, \boldsymbol{y}) = \Big\{\sin^2|\boldsymbol{x}|\cos^2|\boldsymbol{y}| + \sin^2|\boldsymbol{y}| - \frac{\sin^2|\boldsymbol{x}|\sin^2|\boldsymbol{y}|}{|\boldsymbol{x}|^2|\boldsymbol{y}|^2}(\boldsymbol{x}\cdot\boldsymbol{y})^2 + \frac{2\sin|\boldsymbol{x}|\cos|\boldsymbol{x}|\sin|\boldsymbol{y}|\cos|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}(\boldsymbol{x}\cdot\boldsymbol{y})\Big\}^{\frac{1}{2}}$$

and

$$0 \le \sin^{-1}\rho \le \pi,$$

$$\cos(\sin^{-1}\rho) = \cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}(\boldsymbol{x}\cdot\boldsymbol{y}). \tag{6}$$

Proof. As $SU(2) = \exp(iH_0(2;\mathbb{C}))$ there exists $Z \in H_0(2;\mathbb{C})$ such that $\exp(iX)\exp(iY) = \exp(iZ)$. By (10) in [3] we have

$$\exp(iX)\exp(iY) = \Big\{\cos|\boldsymbol{x}|E_2 + \frac{\sin|\boldsymbol{x}|}{|\boldsymbol{x}|}iX\Big\}\Big\{\cos|\boldsymbol{y}|E_2 + \frac{\sin|\boldsymbol{y}|}{|\boldsymbol{y}|}iY\Big\}$$

$$= \Big\{\cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}(\boldsymbol{x}\cdot\boldsymbol{y})\Big\}E_2$$

$$+ i\Big\{\frac{\sin|\boldsymbol{x}|\cos|\boldsymbol{y}|}{|\boldsymbol{x}|}X + \frac{\cos|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{y}|}Y + \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\frac{i}{2}(XY - YX)\Big\},$$

$$\exp(iZ) = \cos|\boldsymbol{z}|E_2 + \frac{\sin|\boldsymbol{z}|}{|\boldsymbol{z}|}iZ.$$

Comparing the coefficients we obtain

$$\cos|\boldsymbol{z}| = \cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}(\boldsymbol{x}\cdot\boldsymbol{y}),$$

$$\frac{\sin|\boldsymbol{z}|}{|\boldsymbol{z}|}Z = \frac{\sin|\boldsymbol{x}|\cos|\boldsymbol{y}|}{|\boldsymbol{x}|}X + \frac{\cos|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{y}|}Y + \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\frac{i}{2}(XY - YX).$$

Hence

$$\left|\cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}(\boldsymbol{x}\cdot\boldsymbol{y})\right| \le 1$$

and

$$\begin{aligned}
\sin^2|\boldsymbol{z}| &= 1-\cos^2|\boldsymbol{z}| \\
&= 1-\left\{\cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}(\boldsymbol{x}\cdot\boldsymbol{y})\right\}^2 \\
&= \sin^2|\boldsymbol{x}|\cos^2|\boldsymbol{y}| + \sin^2|\boldsymbol{y}| - \frac{\sin^2|\boldsymbol{x}|\sin^2|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}(\boldsymbol{x}\cdot\boldsymbol{y})^2 \\
&\qquad + \frac{2\sin|\boldsymbol{x}|\cos|\boldsymbol{x}|\sin|\boldsymbol{y}|\cos|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}(\boldsymbol{x}\cdot\boldsymbol{y}) \\
&= \rho^2.
\end{aligned}$$

From this, we can choose $r \in [0,\pi]$ such that

$$\begin{aligned}
\cos r &= \cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}(\boldsymbol{x}\cdot\boldsymbol{y}), \\
\sin r &= \rho.
\end{aligned}$$

Denote $r = \sin^{-1}\rho$ and put

$$\begin{aligned}
Z_0 &= \alpha X + \beta Y + \gamma\frac{i}{2}(XY-YX) \\
&= \frac{\sin^{-1}\rho}{\rho}\left\{\frac{\sin|\boldsymbol{x}|\cos|\boldsymbol{y}|}{|\boldsymbol{x}|}X + \frac{\cos|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{y}|}Y\right. \\
&\qquad\qquad \left. + \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}\frac{i}{2}(XY-YX)\right\} \\
&= \frac{\sin^{-1}\rho}{\rho}\frac{\sin|\boldsymbol{z}|}{|\boldsymbol{z}|}Z.
\end{aligned}$$

Then

$$|\boldsymbol{z}_0| = \frac{\sin^{-1}\rho}{\rho}\frac{|\sin|\boldsymbol{z}||}{|\boldsymbol{z}|}|\boldsymbol{z}| = \sin^{-1}\rho = r$$

because $\rho = |\sin|\boldsymbol{z}||$. Therefore

$$\begin{aligned}
\sin|\boldsymbol{z}_0| &= \sin r = \rho, \\
\cos|\boldsymbol{z}_0| &= \cos r = \cos|\boldsymbol{x}|\cos|\boldsymbol{y}| - \frac{\sin|\boldsymbol{x}|\sin|\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|}(\boldsymbol{x}\cdot\boldsymbol{y})
\end{aligned}$$

and hence

$$\frac{\sin|\boldsymbol{z}_0|}{|\boldsymbol{z}_0|}Z_0 = \frac{\sin|\boldsymbol{z}|}{|\boldsymbol{z}|}Z.$$

Consequently, $\exp(iZ_0) = \exp(iZ) = \exp(iX)\exp(iY)$ by (10) in [3]. □

Let $A = (a_{ij}) \in K_4(\mathbb{R})$. Note that $a_{ii} = 0$ and $a_{ij} = -a_{ji}$ for every $1 \le i \le 4$ and $1 \le j \le 4$. Define

$$\varphi_1(A) = \frac{a_{12} + a_{34}}{2},\ \varphi_2(A) = \frac{a_{13} - a_{24}}{2},\ \varphi_3(A) = \frac{a_{14} + a_{23}}{2},$$
$$\psi_1(A) = \frac{a_{12} - a_{34}}{2},\ \psi_2(A) = -\frac{a_{13} + a_{24}}{2},\ \psi_3(A) = \frac{a_{14} - a_{23}}{2}.$$

Define

$$\Phi(A) = \varphi_1(A)\sigma_1 + \varphi_2(A)\sigma_2 + \varphi_3(A)\sigma_3,$$
$$\Psi(A) = \psi_1(A)\sigma_1 + \psi_2(A)\sigma_2 + \psi_3(A)\sigma_3,$$

and

$$\overrightarrow{\Phi}(A) = \begin{pmatrix} \varphi_1(A) \\ \varphi_2(A) \\ \varphi_3(A) \end{pmatrix},\ \overrightarrow{\Psi}(A) = \begin{pmatrix} \psi_1(A) \\ \psi_2(A) \\ \psi_3(A) \end{pmatrix}.$$

For $A, B \in K_4(\mathbb{R})$ we also define

$$\alpha_1(A,B) = \alpha(\overrightarrow{\Phi}(A), \overrightarrow{\Phi}(B)),\ \alpha_2(A,B) = \alpha(\overrightarrow{\Psi}(A), \overrightarrow{\Psi}(B)),$$
$$\beta_1(A,B) = \beta(\overrightarrow{\Phi}(A), \overrightarrow{\Phi}(B)),\ \beta_2(A,B) = \beta(\overrightarrow{\Psi}(A), \overrightarrow{\Psi}(B)),$$
$$\gamma_1(A,B) = \gamma(\overrightarrow{\Phi}(A), \overrightarrow{\Phi}(B)),\ \gamma_2(A,B) = \gamma(\overrightarrow{\Psi}(A), \overrightarrow{\Psi}(B)),$$

where $\alpha(\cdot,\cdot)$, $\beta(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$ are defined as in Theorem 6. Put

$$(7)\quad fs(A,B) = \Big\{\alpha_1(A,B)\Phi(A) + \beta_1(A,B)\Phi(B) + \frac{i}{2}\gamma_1(A,B)\left(\Phi(A)\Phi(B) - \Phi(B)\Phi(A)\right)\Big\} \otimes E_2 + E_2 \otimes \Big\{\alpha_2(A,B)\Psi(A) + \beta_2(A,B)\Psi(B) + \frac{i}{2}\gamma_2(A,B)\left(\Psi(A)\Psi(B) - \Psi(B)\Psi(A)\right)\Big\}.$$

It is apparent by the definitions that

$$\alpha_i(A,B) = \alpha_i(\widetilde{A}, \widetilde{B}),\ \beta_i(A,B) = \beta_i(\widetilde{A}, \widetilde{B}),\ \gamma_i(A,B) = \gamma_i(\widetilde{A}, \widetilde{B}).$$

Define

$$(8)\qquad BCH(A,B) = iR^* fs(A,B)R.$$

The following is the B-C-H formula of Fujii and Suzuki which was proved by applying Theorem 6 [3].

Theorem 7 ([3]). *Let $A, B \in K_4(\mathbb{R})$. Then*

$$\exp(A)\exp(B) = \exp(BCH(A,B)).$$

5. Exceptional isometries on $SO(4)$

Lemma 8. *For every pair of $A, B \in K_4(\mathbb{R})$ the characteristic polynomials for $BCH(A,B)$ and $BCH(\widetilde{A}, \widetilde{B})$ coincides with each other.*

Proof. By the definition of $BCH(A,B)$ it is enough to show that the characteristic polynomials of $fs(A,B)$ and $fs(\widetilde{A}, \widetilde{B})$ coincide with each other. To simplify a proof put

$$\begin{aligned}
X_1(A,B) =& \alpha_1(A,B)\varphi_1(A) + \beta_1(A,B)\varphi_1(B) \\
&- \gamma_1(A,B)(\varphi_2(A)\varphi_3(B) - \varphi_3(A)\varphi_2(B)), \\
X_2(A,B) =& \alpha_1(A,B)\varphi_2(A) + \beta_1(A,B)\varphi_2(B) \\
&- \gamma_1(A,B)(\varphi_3(A)\varphi_1(B) - \varphi_1(A)\varphi_3(B)), \\
X_3(A,B) =& \alpha_1(A,B)\varphi_3(A) + \beta_1(A,B)\varphi_3(B) \\
&- \gamma_1(A,B)(\varphi_1(A)\varphi_2(B) - \varphi_2(A)\varphi_1(B)), \\
Y_1(A,B) =& \alpha_2(A,B)\psi_1(A) + \beta_2(A,B)\psi_1(B) \\
&- \gamma_2(A,B)(\psi_2(A)\psi_3(B) - \psi_3(A)\psi_2(B)), \\
Y_2(A,B) =& \alpha_2(A,B)\psi_2(A) + \beta_2(A,B)\psi_2(B) \\
&- \gamma_2(A,B)(\psi_3(A)\psi_1(B) - \psi_1(A)\psi_3(B)), \\
Y_3(A,B) =& \alpha_2(A,B)\psi_3(A) + \beta_2(A,B)\psi_3(B) \\
&- \gamma_2(A,B)(\psi_1(A)\psi_2(B) - \psi_2(A)\psi_1(B)).
\end{aligned}$$

Then by a computation we observe that

$$\begin{aligned}
\sum_{j=1}^{3} X_j(A,B)\sigma_j = \alpha_1(A,B)\Phi(A) + \beta_1(A,B)\Phi(B) \\
+ \frac{i}{2}\gamma_1(A,B)\left(\Phi(A)\Phi(B) - \Phi(B)\Phi(A)\right),
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^{3} Y_j(A,B)\sigma_j = \alpha_2(A,B)\Psi(A) + \beta_2(A,B)\Psi(B) \\
+ \frac{i}{2}\gamma_2(A,B)\left(\Psi(A)\Psi(B) - \Psi(B)\Psi(A)\right).
\end{aligned}$$

Thus we obtain

$$fs(A,B) = \begin{pmatrix} X_3(A,B)+Y_3(A,B) & Y_1(A,B)-iY_2(A,B) & X_1(A,B)-iX_2(A,B) & 0 \\ Y_1(A,B)+iY_2(A,B) & X_3(A,B)-Y_3(A,B) & 0 & X_1(A,B)-iX_2(A,B) \\ X_1(A,B)+iX_2(A,B) & 0 & -X_3(A,B)+Y_3(A,B) & Y_1(A,B)-iY_2(A,B) \\ 0 & X_1(A,B)+iX_2(A,B) & Y_1(A,B)+iY_2(A,B) & -X_3(A,B)-Y_3(A,B) \end{pmatrix}$$

and

$$P_{fs(A,B)}(t) = t^4 - 2\left\{\sum_{j=1}^{3} X_j(A,B)^2 + \sum_{j=1}^{3} Y_j(A,B)^2\right\} t^2 + \left\{\sum_{j=1}^{3} X_j(A,B)^2 - \sum_{j=1}^{3} Y_j(A,B)^2\right\}^2, \tag{9}$$

where $P_M(t)$ denotes the characteristic polynomial for the matrix M. We also have

$$P_{fs(\widetilde{A},\widetilde{B})}(t) = t^4 - 2\left\{\sum_{j=1}^{3} X_j(\widetilde{A},\widetilde{B})^2 + \sum_{j=1}^{3} Y_j(\widetilde{A},\widetilde{B})^2\right\} t^2 + \left\{\sum_{j=1}^{3} X_j(\widetilde{A},\widetilde{B})^2 - \sum_{j=1}^{3} Y_j(\widetilde{A},\widetilde{B})^2\right\}^2, \tag{10}$$

By an elementary calculation we see that

$$\begin{aligned}\sum_{j=1}^{3} X_j(A,B)^2 = {} & \alpha_1(A,B)^2 \sum_{j=1}^{3} \varphi_j(A)^2 + \beta_1(A,B)^2 \sum_{j=1}^{3} \varphi_j(B)^2 \\ & + \gamma_1(A,B)^2 \Big\{ \big(\varphi_2(A)\varphi_3(B) - \varphi_3(A)\varphi_2(B)\big)^2 \\ & + \big(\varphi_3(A)\varphi_1(B) - \varphi_1(A)\varphi_3(B)\big)^2 + \big(\varphi_1(A)\varphi_2(B) - \varphi_2(A)\varphi_1(B)\big)^2 \Big\} \\ & + 2\alpha_1(A,B)\beta_1(A,B)\big(\varphi_1(A)\varphi_1(B) + \varphi_2(A)\varphi_2(B) + \varphi_3(A)\varphi_3(B)\big),\end{aligned}$$

$$\begin{aligned}\sum_{j=1}^{3} Y_j(A,B)^2 = {} & \alpha_2(A,B)^2 \sum_{j=1}^{3} \psi_j(A)^2 + \beta_2(A,B)^2 \sum_{j=1}^{3} \psi_j(B)^2 \\ & + \gamma_2(A,B)^2 \Big\{ \big(\psi_2(A)\psi_3(B) - \psi_3(A)\psi_2(B)\big)^2 \\ & + \big(\psi_3(A)\psi_1(B) - \psi_1(A)\psi_3(B)\big)^2 + \big(\psi_1(A)\psi_2(B) - \psi_2(A)\psi_1(B)\big)^2 \Big\} \\ & + 2\alpha_2(A,B)\beta_2(A,B)\big(\psi_1(A)\psi_1(B) + \psi_2(A)\psi_2(B) + \psi_3(A)\psi_3(B)\big).\end{aligned}$$

Since

$$\varphi_j(\widetilde{C}) = \varphi_j(C),\ \psi_i(\widetilde{C}) = \psi_i(C),\ \psi_3(\widetilde{C}) = -\psi_3(C)$$

and

$$\alpha_i(\widetilde{A}, \widetilde{B}) = \alpha_i(A, B),\ \beta_i(\widetilde{A}, \widetilde{B}) = \beta_i(A, B),\ \gamma_i(\widetilde{A}, \widetilde{B}) = \gamma_i(A, B)$$

for $i = 1, 2,\ j = 1, 2, 3$, and $C = A, B$ we observe that

$$\sum_{j=1}^{3} X_j(A, B)^2 = \sum_{j=1}^{3} X_j(\widetilde{A}, \widetilde{B})^2,$$

$$\sum_{j=1}^{3} Y_j(A, B)^2 = \sum_{j=1}^{3} Y_j(\widetilde{A}, \widetilde{B})^2.$$

It follows that by the equations (9) and (10) that $P_{fs(A,B)}(t) = P_{fs(\widetilde{A},\widetilde{B})}$. As $BCH(A, B) = iR^*fs(A, B)R$ (resp. $BCH(\widetilde{A}, \widetilde{B}) = iR^*fs(\widetilde{A}, \widetilde{B})R$) we obtain the statement. □

Theorem 9. *For every pair of* $A, B \in K_4(\mathbb{R})$

$$\| \exp(A) - \exp(B)\|_{\mathfrak{c}} = \| \exp(\widetilde{A}) - \exp(\widetilde{B})\|_{\mathfrak{c}}.$$

Proof. By Lemma 8 the characteristic polynomials of $BCH(A, -B)$ and $BCH(\widetilde{A}, -\widetilde{B})$ coincide with each other. Applying the spectral mapping theorem we see at once that the eigenvalues of $\exp(BCH(A, -B)) - E_n$ and $\exp(BCH(\widetilde{A}, -\widetilde{B})) - E_n$ coincides with each other. As $\exp(BCH(A, -B)) - E_n$ and $\exp(BCH(\widetilde{A}, -\widetilde{B})) - E_n$ are normal matrices, the singular values of $\exp(BCH(A, -B)) - E_n$ and $\exp(BCH(\widetilde{A}, -\widetilde{B})) - E_n$ are the absolute value of the eigenvalues, whence they coincides with each other. It follows that

$$\| \exp(A) \exp(-B) - E_n\|_{\mathfrak{c}} = \| \exp(\widetilde{A}) \exp(-\widetilde{B}) - E_n\|_{\mathfrak{c}}.$$

As $\| \cdot \|_{\mathfrak{c}}$ is unitarily invariant we observe the desired equation. □

6. Proof of the main result

In this section we complete a proof of Theorem 1. In the following Lemmas 10 and 11 $\| \cdot \|$ denotes the operator norm.

Lemma 10. *Let* $A \in M_4(\mathbb{R})$ *with* $\|A\| < \frac{1}{2}$. *Suppose that* $\exp(A) = E_4$. *Then* $A = 0$.

Proof. Suppose that $\exp(A) = E_4$ for an $A \in M_4(\mathbb{R})$ with $\|A\| < \frac{1}{2}$. Then $\sum_{n=1}^{\infty} \frac{A^n}{n!} = 0$. Hence

$$\|A\| \leq \sum_{n=2}^{\infty} \frac{\|A\|^n}{n!} \leq \sum_{n=2}^{\infty} \|A\|^n = \frac{\|A\|^2}{1 - \|A\|}. \tag{11}$$

If $A \neq 0$, then by the hypothesis $\|A\| < \frac{1}{2}$ we obtain $\frac{\|A\|^2}{1-\|A\|} < \|A\|$, which contradicts to (11). Therefore we have $A = 0$. $\square$

Lemma 11. *There exists an $\varepsilon > 0$ such that the following holds: for any pair $A = (a_{ij}), B = (b_{ij}) \in M_4(\mathbb{R})$ with $\exp(A) = \exp(B)$ and $|a_{ij}| < \varepsilon$, $|b_{ij}| < \varepsilon$ for $1 \leq i, j \leq 4$, the equation $A = B$ holds.*

Proof. Let α be a positive real number such that $\exp \alpha < 2$ and $\alpha - \log(2 - \exp \alpha) < \frac{1}{2}$. There exists an $\varepsilon > 0$ such that $|a_{ij}| < \varepsilon$ $(1 \leq i, j \leq 4)$ implies that $\|A\| < \alpha$ for $A = (a_{ij}) \in M_4(\mathbb{R})$. We show this ε is the desired one. Suppose that $A = (a_{ij}), B = (b_{ij}) \in M_4(\mathbb{R})$ satisfy that $\exp(A) = \exp(B)$, $|a_{ij}| < \varepsilon$ and $|b_{ij}| < \varepsilon$ for every $1 \leq i, j \leq 4$. Then

$$\|E_4 - \exp(A)\| \leq \exp \|A\| - 1 < 1.$$

Then the series

$$\hat{A} = \sum_{n=1}^{\infty} \frac{-(E_4 - \exp(A))^n}{n}.$$

converges absolutely ($\sum_{n=1}^{\infty} \frac{\|E_4 - \exp(A)\|^n}{n} < \infty$), as in the scalar case, substituting this series into the series expansion for $\exp(\hat{A})$ yields $\exp(\hat{A}) = \exp(A)$. We see that

$$\|\hat{A}\| \leq \sum_{n=1}^{\infty} \frac{\|E_4 - \exp(A)\|^n}{n} \leq \sum_{n=1}^{\infty} \frac{(\exp \|A\| - 1)^n}{n} < -\log(2 - \exp \alpha).$$

As A and $E_4 - \exp(A)$ commute, we see that $\hat{A}$ and A commute. Thus

$$\exp(\hat{A} - A) = \exp(\hat{A}) \exp(-A) = E_4.$$

Since $\|\hat{A} - A\| \leq \alpha - \log(2 - \exp \alpha) < \frac{1}{2}$ we see that $\hat{A} - A = 0$ by Lemma 10. In the same way we see that $\hat{A} - B = 0$. Therefore we have $A = B$. $\square$

Proof of Theorem 1. Suppose that T is an isometry; i.e., T satisfies (i). Then by Proposition 5 T is surjective and one of the (a), (b), (c), or (d) holds. By Lemma 3 T preserves the inverted Jordan products.

Conversely if T is of the form of (a) or (b), then it is apparent that T is an isometry from $SO(n)$ onto itself.

Suppose that T satisfies (c). Note that T is well-defined in the sense that $\exp(\widetilde{A}) = \exp(\widetilde{A'})$ if $\exp(A) = \exp(A')$ for $A, A' \in K_4(\mathbb{R})$ by Theorem 9. As $\| \cdot \|_{\mathfrak{c}}$ is unitarily invariant

$$\|T(\exp(A)) - T(\exp(B))\|_{\mathfrak{c}} = \|\exp(\widetilde{A}) - \exp(\widetilde{B})\|_{\mathfrak{c}}.$$

Therefore by Theorem 9 we observe that

$$\|T(\exp(A)) - T(\exp(B))\|_{\mathfrak{c}} = \|\exp(A) - \exp(B)\|_{\mathfrak{c}},$$

that is, T satisfies (i).

In a way similar to the above we see that T satisfies (i) if T is of the form of (d).

Suppose further that $T(E_n) = E_n$. Then T is an automorphism for (a) and T is an anti-automorphism for (b).

Suppose that T is of the form of (c). We show T is not multiplicative. Suppose contrary that T is multiplicative. Let $A, B \in K_4(\mathbb{R})$ be such that the absolute value of each of entries of A and B are sufficiently small so that the absolute value of each entry of $BCH(A, B)$ and $BCH(\widetilde{A}, \widetilde{B})$ are less than ε which appears in Lemma 11. According to the BCH formula (8) and (7) this is possible. As we have assumed that T is multiplicative

$$\exp\left(\widetilde{BCH(A, B)}\right) = \exp(BCH(\widetilde{A}, \widetilde{B})).$$

It follows by Lemma 11 that

$$\widetilde{BCH(A, B)} = BCH(\widetilde{A}, \widetilde{B}).$$

By the definition of $\widetilde{\cdot}$ the $(1,2)$-entry of $\widetilde{BCH(A, B)}$ and that of $BCH(A, B)$ coincides. Thus we have that $(1,2)$-entries of both $BCH(A, B)$ and $BCH(\widetilde{A}, \widetilde{B})$ coincides with each other. On the other hand by a direct computation of $BCH(A, B) = iR^* fs(A, B)R$ we see that $(1,2)$-entry of $BCH(A, B)$ is

(12)

$$\alpha_1(A,B)\varphi_1(A)+\beta_1(A,B)\varphi_1(B)-\gamma_1(A,B)(\varphi_2(A)\varphi_3(B)-\varphi_3(A)\varphi_2(B))$$
$$+\alpha_2(A,B)\psi_1(A)+\beta_2(A,B)\psi_1(B)+\gamma_2(A,B)(\psi_2(A)\psi_3(B)-\psi_3(A)\psi_2(B)).$$

Hence by this formula it is easy to see that for appropriate matrices A and B such that the absolute value of each element is sufficiently small, the $(1,2)$-entry of $BCH(A, B)$ does not coincides with that of $BCH(\widetilde{A}, \widetilde{B})$, which is a contradiction proving that T is not multiplicative. Applying the equation (12) we also see that T is not anti-multiplicative.

Suppose that T is of the form of (d). In the same way as above we see that T is neither multiplicative nor anti-multiplicative. □

7. Multiplicatively spectral preserving maps on $SO(n)$

In this section we present an application of Theorem 1. A map T between subsets of unital complex algebras is called spectrally multiplicative (or multiplicatively spectrum-preserving) if it satisfies

$$\sigma(T(a)T(b)) = \sigma(ab)$$

for all a, b, where $\sigma(\cdot)$ denotes the spectrum. Study on spectrally multiplicative maps on certain unital Banach algebras was initiated by Molnár [9]. It has been interested by many authors partly because such maps are closely related to isomorphisms without linearity and multiplicativity being prerequested. One of the latest interesting paper on this topics is [1]. See also a recent survey [5]. We characterize the spectrally multiplicative maps on $SO(n)$.

Theorem 12. *Let $T : SO(n) \to SO(n)$. Then T is a spectrally multiplicative map if and only if the following holds. There exists $O \in O(n)$ such that T is of one of the following form:*

(1) $T(X) = OXO^{-1}$ *for every* $X \in SO(n)$,

(2) $T(X) = -OXO^{-1}$ *for every* $X \in SO(n)$,

(3) $T(X) = OX^{-1}O^{-1}$ *for every* $X \in SO(n)$,

(4) $T(X) = -OX^{-1}O^{-1}$ *for every* $X \in SO(n)$,

(5) $n = 4$ *and* $T(X) = O(\exp(\widetilde{A}))O^{-1}$ *for every* $X \in SO(n)$, *where* $A \in K_4(\mathbb{R})$ *with* $\exp(A) = X$,

(6) $n = 4$ *and* $T(X) = -O(\exp(\widetilde{A}))O^{-1}$ *for every* $X \in SO(n)$, *where* $A \in K_4(\mathbb{R})$ *with* $\exp(A) = X$,

(7) $n = 4$ *and* $T(X) = O(\exp(-\widetilde{A}))O^{-1}$ *for every* $X \in SO(n)$, *where* $A \in K_4(\mathbb{R})$ *with* $\exp(A) = X$.

(8) $n = 4$ *and* $T(X) = -O(\exp(-\widetilde{A}))O^{-1}$ *for every* $X \in SO(n)$, *where* $A \in K_4(\mathbb{R})$ *with* $\exp(A) = X$.

In this case T is a surjective isometry with respect $\|\cdot\|_{\mathfrak{c}}$ for any $\mathfrak{c} \in \mathbb{R}^n_+\!\downarrow$.

Proof. Suppose that T satisfies (1) or (2). Then the map T is apparently spectrally multiplicative. Next, we recall that the important equality $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$ holds for elements a, b in a unital complex algebra, whence

$$\sigma(XY) = \sigma(YX) = \sigma((YX)^t) = \sigma(X^{-1}Y^{-1}) \tag{13}$$

for $X, Y \in SO(n)$. We see at once by (13) that T is spectrally multiplicative if T is of the form (3) or (4). By Theorem 7 and Lemma 8, and applying the spectral mapping theorem we see that $\sigma(\exp(A)\exp(B)) = \sigma(\exp(\widetilde{A})\exp(\widetilde{B}))$ for $A, B \in K_4(\mathbb{R})$. It follows by this equality and (13) that T is multiplicatively spectrum preserving if T is of one of the form of (5) through (8).

Conversely, suppose that T is multiplicatively spectrum preserving. As

$$\sigma(T(X)T(X^{-1})) = \sigma(E_n) = \{1\}$$

for $X \in SO(n)$, $T(X^{-1}) = (T(X))^{-1}$ for every $X \in SO(n)$. We infer that $\sigma(XY^{-1}) = \sigma(T(X)(T(Y))^{-1})$, hence

$$\sigma(XY^{-1} - E_n) = \sigma(T(X)(T(Y))^{-1} - E_n)$$

for all $X, Y \in SO(n)$. As $UV^{-1} - E_n$ is a normal matrix, we observe that

$$\|XY^{-1} - E_n\| = \|T(X)(T(Y))^{-1} - E_n\|,$$

hence

$$\|X - Y\| = \|T(X) - T(Y)\|$$

for every pair X and Y in $SO(n)$, where $\| \cdot \|$ denote the operator norm. By Theorem 1 there exists $O \in O(n)$ that T is of the one of the following form:

(a) $T(X) = T(E_n)OXO^{-1}$ for every $X \in SO(n)$,

(b) $T(X) = T(E_n)OX^{-1}O^{-1}$ for every $X \in SO(n)$,

(c) $n = 4$ and $T(X) = T(E_4)O(\exp(\widetilde{A}))O^{-1}$ for every $X \in SO(n)$, where $A \in K_4(\mathbb{R})$ with $\exp(A) = X$,

(d) $n = 4$ and $T(X) = T(E_4)O(\exp(-\widetilde{A}))O^{-1}$ for every $X \in SO(n)$, where $A \in K_4(\mathbb{R})$ with $\exp(A) = X$.

The rest is to prove that $T(E_n) = E_n$ or $-E_n$. We give a proof for the case of (a). Proofs for the rest of the cases are similar and are omtted. Since

$$\sigma((T(E_n))^2) = \sigma(E_n) = \{1\}$$

we infer $(T(E_n))^2 = E_n$ since $T(E_n)^2$ is a special orthogonal matrix. Then we see that the standard form of $T(E_n)$ is a diagonal matrix whose diagonal entries are 1 or -1, whence there exists $U \in SO(n)$ such that $UT(E_n)U^{-1}$ is a diagonal matrix whose entries are 1 or -1. Furthermore these entries are all 1 or all -1. We prove this. Suppose that there are both 1 and -1 within the diagonal entries of $UT(E_n)U^{-1}$. We will show a contradiction. Without loss of generality we may assume that the $(1,1)$-entry of $UT(E_n)U^{-1}$ is 1 and $(2,2)$-entry of $UT(E_n)U^{-1}$ is -1. Choose $X \in SO(n)$ as follows: the $(1,2)$-entries of $UOXO^{-1}U^{-1}$ is 1, the $(2,1)$-entries of $UOXO^{-1}U^{-1}$ is -1, (k,k) entries for $k \geq 3$ (if $n \geq 3$) are all 1, and all other entries are 0. Then we easily see that

$$\left(UT(E_n)U^{-1}(UOXO^{-1}U^{-1})\right)^2 = E_n$$

and the $(1,1)$ and $(2,2)$ entries are both -1, other diagonal entries (if $n \geq 3$) are all 1, and other entries are all 0 for $(UOXO^{-1}U^{-1})^2$. Hence

$$\sigma\left(\left(UT(E_n)U^{-1}(UOXO^{-1}U^{-1})\right)^2\right) \neq \sigma\left(\left(UOXO^{-1}U^{-1}\right)^2\right) = \sigma(X^2).$$

As

$$\sigma\left(\left(UT(E_n)U^{-1}(UOXO^{-1}U^{-1})\right)^2\right) = \sigma\left(\left(T(E_n)OXO^{-1}\right)^2\right) = \sigma(T(X)^2)$$

we arrive at

$$\sigma(T(X)^2) \neq \sigma(X^2),$$

which is a contradiction proving that the entries of the diagonal of $UT(E_n)U^{-1}$ are all 1 or all -1; $T(E_n) = E_n$ or $T(E_n) = -E_n$. $\square$

Note that the maps of the form of (c) or (d) preserve the inverted Jordan triple products in the sense that $T(XY^{-1}X) = T(X)(T(Y))^{-1}T(X)$ for all $X, Y \in SO(n)$ by Lemma 3; preserve the structure of $SO(n)$ as the twisted subgroups.

We complete this paper with a remark. If a map $T : SO(n) \to SO(n)$ is of the form (a), (b), (c) or (d) of (ii) of Theorem 1, then T is an isometry on $SO(n)$ with respect the metric induced by *any* c-spectral norm, hence T is an isometry with respect to the metric induced by any *unitarily invariant* norm. The authors do not know whether the form of (a), (b), (c) or (d) are only the form of isometries with respect to the metric induced by a given unitarily invariant norm.

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