The first and second order large-deviation efficiency for an exponential family and certain curved exponential models

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Abstract

The first and second order large-deviation efficiency is discussed for an exponential family of distributions. The lower bound for the tail probability of asymptotically median unbiased estimators is directly derived up to the second order by use of the saddlepoint approximation. The maximum likelihood estimator (MLE) is also shown to be second order large-deviation efficient in the sense that the MLE attains the lower bound. Further, in certain curved exponential model, the first and second order lower bounds are obtained, and the MLE is shown not to be first order large-deviation efficient.

Keywords: Large-deviation; Asymptotically median unbiased estimator; Saddlepoint approximation; Lower bound; Tail probability; Maximum likelihood estimator; Curved exponential model

AMS Subject Classification: 62F12; 62F10; 60F10

1 Introduction

Under suitable regularity conditions, the asymptotic efficiency of estimators including higher order has been investigated from the viewpoint of the concentration probability around the true parameter (see, e.g. Akahira and Takeuchi, 1981, 2003, Pfanzagl and Wefelmeyer, 1985, Ghosh, 1994). In such a case the Edgeworth expansion of the distribution of estimator plays an important part. For example, it is shown that the modified maximum likelihood estimator is third order asymptotically efficient in some class of estimators under regularity conditions.

On the other hand, from the viewpoint of large-deviation, the asymptotic efficiency can be also considered. For example, the Bahadur efficiency is well known. Indeed, for any consistent estimator $\hat{\theta}_n$ of an unknown real-valued parameter θ and any $\varepsilon > 0$, the tail probability

$$\alpha\left(\hat{\theta}_n, \theta, \varepsilon\right) := P_{\theta,n}\left\{|\hat{\theta}_n - \theta| > \varepsilon\right\}$$

tends to zero as $n \to \infty$. Under suitable conditions it is shown that the rate of convergence is exponential and has an asymptotic expansion of the form

$$\alpha\left(\hat{\theta}_n, \theta, \varepsilon\right) = e^{-n\beta(\hat{\theta}_n,\theta,\varepsilon)}\left(c_0 + \frac{c_1}{n} + \cdots\right),$$

where $\beta(\hat{\theta}_n, \theta, \varepsilon)$ is positive and c_i's are constants. Here the constant $\beta(\hat{\theta}_n, \theta, \varepsilon)$ is called an exponential rate. Bahadur (1971) shows that the upper bound for the exponential rate of consistent estimators is given by use of the amount of the Kullback-Leibler (K-L) information and discusses its attainment under suitable regularity conditions, which is called the Bahadur efficiency (see also Fu, 1973). Using the asymptotic expansion of the amount of the K-L information, the Bahadur type second order efficiency is also considered by Fu (1982) (see also Akahira, 1995).

Recently, from a different viewpoint from the Bahadur efficiency, the concept of first and second order large-deviation efficiency has been discussed by Akahira (2006). In the concept it is essential to consider the asymptotic relative ratio of the tail probability of any asymptotically median unbiased estimator to the first order lower bound up to the second order. Indeed, the first and second order lower bound for the tail probability of asymptotically median unbiased estimators are directly derived by use of the saddlepoint approximation. In this paper the derivation is introduced according to Akahira (2006), and for an exponential family of distributions, the lower bound is obtained up to the second order and the maximum likelihood estimator (MLE) is shown to be second order large-deviation efficient in the sense that the MLE attains the lower bound. Further, in certain curved exponential model, the lower bound is given up to the second order and the MLE is shown not to be first order large-deviation efficient.

2 Definitions

Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of independent and identically distributed (i.i.d.) random variables with a probability density function (p.d.f.) $f(x,\theta)$ with respect to a σ-finite measure μ, where $\theta \in \Theta$ and Θ is an open interval in $\boldsymbol{R}^1$. Put $\boldsymbol{X} := (X_1, \cdots, X_n)$. If an estimator $\hat{\theta}_n := \hat{\theta}_n(\boldsymbol{X})$ of θ satisfies

$$P_{\theta,n}\{\hat{\theta}_n \le \theta\} = \frac{1}{2} + o(1), \quad P_{\theta,n}\{\hat{\theta}_n \ge \theta\} = \frac{1}{2} + o(1),$$

as $n \to \infty$, then $\hat{\theta}_n$ is called asymptotically median unbiased (AMU for short) for θ. Let $\mathcal{A}$ be a class of all the AMU estimators of θ.

Definition(Akahira, 2006). If there exists an AMU estimator $\hat{\theta}_n^* = \hat{\theta}_n^*(\boldsymbol{X})$ such that for any $\hat{\theta}_n \in \mathcal{A}$, any $\theta \in \Theta$ and any $a > 0$

$$\begin{aligned} P_{\theta,n}\{|\hat{\theta}_n - \theta| \ge a\} &\ge P_{\theta,n}\{|\hat{\theta}_n^* - \theta| \ge a\}\{1 + o(1)\} \\ &=: B_n(a,\theta)\{1 + o(1)\} \end{aligned} \tag{2.1}$$

as $n \to \infty$, then $\hat{\theta}_n^*$ is called first order large-deviation efficient (LDE). If there exists an AMU estimator $\hat{\theta}_n^{**} = \hat{\theta}_n^{**}(\boldsymbol{X})$ such that for any $\hat{\theta}_n \in \mathcal{A}$, any $\theta \in \Theta$ and any $a > 0$

$$\frac{P_{\theta,n}\{|\hat{\theta}_n - \theta| \ge a\}}{B_n(a,\theta)} \ge 1 + \frac{b_1(a,\theta)}{n} + o\left(\frac{1}{n}\right) \tag{2.2}$$

as $n \to \infty$ and $\hat{\theta}_n^{**}$ attains the lower bound in (2.2) up to the order $o(1/n)$, then $\hat{\theta}_n^{**}$ is called second order LDE, where $B_n(a,\theta)$ is given by (2.1) and $b_1(a,\theta)$ is certain constant.

In order to discuss the higher order LDE, it is necessary to get the lower bound for the two-sided tail probability $P_{\theta,n}\{|\hat{\theta}_n - \theta| > a\}$ of $\hat{\theta}_n (\in \mathcal{A})$ up to the higher order.

3 First and second order lower bounds for the tail probability

Let θ_0 be any fixed value in Θ. Then we consider a problem of testing the hypothesis $H : \theta = \theta_0 + a$, against the alternative $K : \theta = \theta_0$, where $a > 0$. Let $\phi^*(\boldsymbol{X})$ be the most powerful (MP) test of level $1/2 + o(1)$. Letting $\hat{\theta}_n \in \mathcal{A}$ and putting

$$A_{\hat{\theta}_n} := \{\boldsymbol{x} | \hat{\theta}_n(\boldsymbol{x}) \leq \theta_0 + a\},$$

we see that the indicator $\chi_{A_{\hat{\theta}_n}}(\boldsymbol{x})$ is a test of level $1/2 + o(1)$, where $\boldsymbol{x} := (x_1, \cdots, x_n)$. Since

$$E_{\theta_0}(\phi^*) \geq E_{\theta_0}[\chi_{A_{\hat{\theta}_n}}] = P_{\theta_0,n}\{\hat{\theta}_n \leq \theta_0 + a\}$$

for large n, it follows that

$$P_{\theta_0,n}\{\hat{\theta}_n - \theta_0 > a\} \geq 1 - E_{\theta_0}(\phi^*). \tag{3.1}$$

In order to obtain the lower bound, *i.e.*, the right-hand side of (3.1), it is seen from the fundamental lemma of Neyman-Pearson that a test with the rejection region of type

$$\bar{Z}(\theta_0) := \frac{1}{n}\sum_{j=1}^{n} Z_j(\theta_0) > c$$

is MP, where

$$Z_j(\theta_0) := \log(f(X_j, \theta_0)/f(X_j, \theta_0 + a)) \quad (j = 1, \cdots, n)$$

and c is a constant chosen such that the asymptotic level of the test is $1/2 + o(1)$. Then $c = E_{\theta_0+a}[Z_1(\theta_0)] + o(1) =: \mu + o(1)$ (say). Note that $Z_1, \cdots, Z_n$ are i.i.d. and

$$\mu = -\int_{-\infty}^{\infty} \left\{\log \frac{f(x, \theta_0 + a)}{f(x, \theta_0)}\right\} f(x, \theta_0 + a) d\mu(x) =: -I(\theta_0 + a, \theta_0) < 0,$$

where $I(\theta_0 + a, \theta_0)$ is the amount of Kullback-Leibler information. Since

$$E_{\theta_0}(\phi^*) = P_{\theta_0,n}\{\bar{Z}(\theta_0) > c\},$$

it follows from (3.1) that for large n

$$P_{\theta_0,n}\{\hat{\theta}_n - \theta_0 > a\} \geq 1 - P_{\theta_0,n}\{\bar{Z}(\theta_0) > c\} = P_{\theta_0,n}\{\bar{Z}(\theta_0) \leq c\}. \tag{3.2}$$

for $a > 0$. In order to obtain the asymptotic expansion of the tail probability of $\bar{Z}(\theta_0)$ in (3.2), we use the saddlepoint approximation (Jensen, 1995). Let

$$M_{Z_1(\theta_0)}(t; \theta_0) := E_{\theta_0}[\exp\{tZ_1(\theta_0)\}], \quad K_{Z_1(\theta_0)}(t; \theta_0) := \log M_{Z_1(\theta_0)}(t; \theta_0)$$

for all t in some open interval involving the origin, that is, they are the moment generating function (m.g.f.) and the cumulant generating function (c.g.f.) of $Z_1(\theta_0)$, respectively. Let $\hat{t}(a)$ be a solution of t of the equation $(\partial/\partial t)K_{Z_1(\theta_0)}(t; \theta_0) = \mu$. In a similar way to the case $a > 0$, we have a lower bound in the case $a < 0$. Since θ_0 is arbitrary in Θ, henceforth we write θ instead of θ_0.

From (3.2) and the saddlepoint approximation we have the following.

Theorem 3.1(Akahira, 2006). For any $\hat{\theta}_n \in \mathcal{A}$, any $\theta \in \Theta$ and any $a > 0$, it holds that for large n

$$\begin{aligned} & P_{\theta,n}\{\hat{\theta}_n - \theta > a\} \\ & \geq \frac{1}{\lambda} M_{Z_1}^n(\hat{t}) e^{-n\mu\hat{t}} \left[B_0(\lambda) + \frac{\mathrm{sgn}(\hat{t})}{\sqrt{n}} \frac{\zeta_3(\hat{t})}{6} B_3(\lambda) + \frac{1}{n} \left\{ \frac{\zeta_4(\hat{t})}{24} B_4(\lambda) + \frac{\zeta_3^2(\hat{t})}{72} B_6(\lambda) \right\} \right. \\ & \left. \qquad + O\left(\frac{1}{n^2}\right) \right], \end{aligned} \tag{3.3}$$

where $\lambda = \sqrt{n}\left|\hat{t}\right| \sqrt{K''_{Z_1}(\hat{t})}$, $\zeta_3(t) := \kappa_{3,\theta+a}(Z_1)/\{K''_{Z_1}(t)\}^{3/2}$, $\zeta_4(t) := \kappa_{4,\theta+a}(Z_1)/\{K''_{Z_1}(t)\}^2$ with third and fourth cummulants $\kappa_{3,\theta+a}(Z_1)$ and $\kappa_{4,\theta+a}(Z_1)$ of $Z_1 = Z_1(\theta)$, and

$$\begin{aligned} B_0(\lambda) &:= \lambda e^{\lambda^2/2}\{1 - \Phi(\lambda)\}, \\ B_3(\lambda) &:= -\left\{ \lambda^3 B_0(\lambda) - \frac{1}{\sqrt{2\pi}}(\lambda^3 - \lambda) \right\}, \\ B_4(\lambda) &:= \lambda^4 B_0(\lambda) - \frac{1}{\sqrt{2\pi}}(\lambda^4 - \lambda^2), \\ B_6(\lambda) &:= \lambda^6 B_0(\lambda) - \frac{1}{\sqrt{2\pi}}(\lambda^6 - \lambda^4 + 3\lambda^2). \end{aligned}$$

For the case $a < 0$, we also obtain a similar lower bound to (3.3) for the tail probability $P_{\theta,n}\{\hat{\theta}_n - \theta < a\}$. If there exists an AMU estimator attaining the lower bound for the asymptotic relative ratio of the two-sided tail probability $P_{\theta,n}\{|\hat{\theta}_n - \theta| > a\}$ to the first order lower bound up to the order $o(1/n)$, then it is second order LDE.

4 The second order LDE for an exponential family of distributions

Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of i.i.d. random variables with a p.d.f. $f(x, \theta)$ (w.r.t. the Lebesgue measure) which belongs to an exponential family of distributions, i.e.

$$f(x, \theta) = \exp\{\theta x + C(\theta) + S(x)\} \tag{4.1}$$

for $x \in \mathcal{X} \subset \boldsymbol{R}^1$, where $\theta \in \Theta$ and Θ is an open interval of $\boldsymbol{R}^1$, $C(\cdot)$ is a four times differentiable real-valued function of Θ, and $S(\cdot)$ is a real valued function on $\mathcal{X}$.

In a similar way to Section 3, we obtain the lower bound for the tail probability. Since, for $a > 0$

$$Z_j(\theta) = \log(f(X_j, \theta)/f(X_j, \theta + a)) = C(\theta) - C(\theta + a) - aX_j$$

for $j = 1, \cdots, n$, the m.g.f. of $Z_1(\theta)$ is given by

$$M_{Z_1(\theta)}(t; \theta) = E_\theta[\exp\{tZ_1(\theta)\}]$$

$$= \exp[\{t\{C(\theta) - C(\theta + a)\} + C(\theta) - C(\theta - at)].$$

Since the c.g.f. of $Z_1(\theta)$ is

$$K_{Z_1(\theta)}(t;\theta) := \log M_{Z_1(\theta)}(t;\theta) = t\{C(\theta) - C(\theta+a)\} + C(\theta) - C(\theta - at),$$

it follows that

$$\frac{\partial}{\partial t} K_{Z_1(\theta)}(t;\theta) = C(\theta) - C(\theta + a) + aC^{(1)}(\theta - at). \tag{4.2}$$

On the other hand we have

$$\mu := E_{\theta+a}[Z_1(\theta)] = E_{\theta+a}[C(\theta) - C(\theta+a) - aX_1] = C(\theta) - C(\theta+a) + aC^{(1)}(\theta+a). \tag{4.3}$$

From (4.2) and (4.3) we have $\hat{t} = -1$ as a solution of the equation $(\partial/\partial t)K_{Z_1(\theta)}(t;\theta) = \mu$. Since

$$\begin{aligned} K_{Z_1(\theta)}(t;\theta+a) &= \log M_{Z_1(\theta)}(t;\theta+a) = \log E_{\theta+a}[\exp\{tZ_1(\theta)\}] \\ &= t\{C(\theta) - C(\theta+a)\} + C(\theta+a) - C(\theta + a - at), \end{aligned}$$

it follows that the second, third and fourth cumulants are

$$\begin{aligned} \kappa_{2,\theta+a}(Z_1(\theta)) &= V_{\theta+a}(Z_1(\theta)) = \frac{\partial^2}{\partial t^2} K_{Z_1(\theta)}(0;\theta+a) = -a^2 C^{(2)}(\theta+a) > 0, \\ \kappa_{3,\theta+a}(Z_1(\theta)) &= \frac{\partial^3}{\partial t^3} K_{Z_1(\theta)}(0;\theta+a) = a^3 C^{(3)}(\theta+a), \\ \kappa_{4,\theta+a}(Z_1(\theta)) &= \frac{\partial^4}{\partial t^4} K_{Z_1(\theta)}(0;\theta+a) = -a^4 C^{(4)}(\theta+a), \end{aligned}$$

respectively, where, for each $j = 2,3,4$, $C^{(j)}(\theta)$ are the j-th derivative of $C(\theta)$. From Theorem 3.1 we have the following.

Theorem 4.1 For any $\hat{\theta}_n \in \mathcal{A}$, any $\theta \in \Theta$ and any $a > 0$

$$\frac{P_{\theta,n}\{\hat{\theta}_n - \theta > a\}}{B_n(a,\theta)} \geq 1 + \frac{\exp\{(na^2/2)C^{(2)}(\theta+a)\}}{a\sqrt{-nC^{(2)}(\theta+a)}\left\{1 - \Phi\left(a\sqrt{-nC^{(2)}(\theta+a)}\right)\right\}} \left\{\frac{\triangle}{n} + O\left(\frac{1}{n^2}\right)\right\} \tag{4.4}$$

as $n \to \infty$, where

$$\begin{aligned} B_n(a,\theta) &:= \left\{1 - \Phi\left(a\sqrt{-nC^{(2)}(\theta+a)}\right)\right\} \\ &\quad \cdot \exp\left[n\left\{C(\theta) - C(\theta+a) + aC^{(1)}(\theta+a) - \frac{a^2}{2}C^{(2)}(\theta+a)\right\}\right], \\ \triangle &:= \frac{1}{24\sqrt{2\pi}\{C^{(2)}(\theta+a)\}^2}\left[\frac{12}{a}C^{(3)}(\theta+a) - 3C^{(4)}(\theta+a) + \frac{5\{C^{(3)}(\theta+a)\}^2}{C^{(2)}(\theta+a)}\right]. \end{aligned}$$

The proof is straightforward from Theorem 3.1, since $B_0(\lambda) \approx 1/\sqrt{2\pi}$, $B_3(\lambda) \approx -3/(\sqrt{2\pi}\lambda)$, $B_4(\lambda) \approx 3/\sqrt{2\pi}$ and $B_6(\lambda) \approx -15/\sqrt{2\pi}$, as $\lambda \to \infty$. In a similar way to Theorem 4.1, the lower bound for the probability $P_{\theta,n}\{\hat{\theta}_n - \theta < a\}$ for $a < 0$ is also obtained.

Theorem 4.2 The maximum likelihood estimator (MLE) $\hat{\theta}_{ML}$ is second order LDE.

Proof Without loss of generality, we assume that $\theta = 0$. First, since, for $a > 0$

$$P_{0,n}\{\hat{\theta}_{ML} > a\} = P_{0,n}\left\{\frac{1}{n}\sum_{j=1}^{n}\frac{\partial}{\partial a}\log f(X_j, a) > 0\right\},$$

it follows from (4.1) that

$$P_{0,n}\{\hat{\theta}_{ML} > a\} = P_{0,n}\{\bar{X} > -C^{(1)}(a)\}, \tag{4.5}$$

where $\bar{X} := (1/n)\sum_{i=1}^{n} X_i$. On the other hand we have from (4.1)

$$\bar{Z}(0) := \frac{1}{n}\sum_{j=1}^{n} Z_j(0) = \frac{1}{n}\sum_{j=1}^{n}\log\frac{f(X_j, 0)}{f(X_j, a)} = C(0) - C(a) - a\bar{X}. \tag{4.6}$$

Letting

$$c := E_a[Z_1(0)] = C(0) - C(a) - aE_a(X_1) = C(0) - C(a) + aC^{(1)}(a), \tag{4.7}$$

we obtain from (4.5) to (4.7)

$$\begin{aligned} P_{0,n}\{\hat{\theta}_{ML} > a\} &= P_{0,n}\left\{\bar{X} > \frac{1}{a}(-c + C(0) - C(a))\right\} \\ &= P_{0,n}\{c > C(0) - C(a) - a\bar{X}\} \\ &= 1 - P_{0,n}\{\bar{Z}(0) \geq c\}, \end{aligned}$$

which implies that the equality in (3.2) holds, hence the MLE $\hat{\theta}_{ML}$ satisfies the equality in (3.3) and (4.4). In the case when $a < 0$, we also have a similar result to the case $a > 0$. Therefore the MLE is second order LDE. □

5 The lower bound for the tail probability in certain curved exponential model

Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of independent and identically distributed random variables according to a normal distribution $N(2\theta, \theta^2)$, where $\theta > 0$, which belongs to a curved exponential family of distributions.

First, in order to obtain the bound for the tail probability of asymptotically median unbiased estimators of θ, we consider a problem of testing the hypothesis $H : \theta = \theta_0 + a$, against the alternative $K : \theta = \theta_0$, where θ_0 is arbitrarily fixed and $a > 0$. Since

$$\begin{aligned} Z_1(\theta_0) &= \log f(X, \theta_0) - \log f(X, \theta_0 + a) \\ &= -\log\theta_0 - \frac{1}{2\theta_0^2}(X - 2\theta_0)^2 + \log(\theta_0 + a) + \frac{1}{2(\theta_0 + a)^2}\{X - 2(\theta_0 + a)\}^2, \end{aligned}$$

we have

$$\begin{aligned}
\mu :&= E_{\theta_0+a}[Z_1(\theta_0)] \\
&= \log\frac{\theta_0+a}{\theta_0} - \frac{1}{2\theta_0^2}\{E_{\theta_0+a}[\{X-2(\theta_0+a)\}^2]+4a^2\}+\frac{1}{2} \\
&= \log\frac{\theta_0+a}{\theta_0} - \frac{1}{2\theta_0^2}\{(\theta_0+a)^2+4a^2\}+\frac{1}{2} \\
&= \log\frac{\theta_0+a}{\theta_0} - \frac{5}{2}\left(\frac{a}{\theta_0}\right)^2 - \frac{a}{\theta_0}.
\end{aligned} \tag{5.1}$$

Under the hypothesis $H:\theta=\theta_0+a$, the m.g.f. of $Z_1(\theta_0)$ is given by

$$\begin{aligned}
&M_{Z_1(\theta_0)}(t,\theta_0+a) := E_{\theta_0+a}[e^{tZ_1(\theta_0)}] \\
&= E_{\theta_0+a}\left[\exp\left\{t\left(\log\frac{\theta_0+a}{\theta_0} - \frac{1}{2\theta_0^2}(X-2\theta_0)^2 + \frac{1}{2(\theta_0+a)^2}(X-2(\theta_0+a))^2\right)\right\}\right] \\
&= \left(\frac{\theta_0+a}{\theta_0}\right)^t \frac{1}{\sqrt{2\pi}(\theta_0+a)}\int_{-\infty}^{\infty}\exp\left\{-\frac{t}{2\theta_0^2}(x-2\theta_0)^2 + \frac{t-1}{2(\theta_0+a)^2}(x-2(\theta_0+a))^2\right\}dx \\
&= \left(\frac{\theta_0+a}{\theta_0}\right)^t \frac{1}{\sqrt{2\pi}(\theta_0+a)}\int_{-\infty}^{\infty}\exp\left\{-\frac{t}{2\theta_0^2}(u+2a)^2 + \frac{t-1}{2(\theta_0+a)^2}u^2\right\}du \\
&= \left(\frac{\theta_0+a}{\theta_0}\right)^t \frac{\sqrt{A}}{\theta_0+a}\exp\left\{\frac{a^2t}{\theta_0^2}\left(\frac{2At}{\theta_0^2}-2\right)\right\}
\end{aligned} \tag{5.2}$$

for all $t > -\theta_0^2/\{a(a+2\theta_0)\}$, where

$$A := \frac{\theta_0^2(\theta_0+a)^2}{a(a+2\theta_0)t+\theta_0^2} \ .$$

Since, by (5.2),

$$\begin{aligned}
&K_{Z_1(\theta_0)}(t,\theta_0+a) = \log M_{Z_1(\theta_0)}(t,\theta_0+a) \\
&= t\log\frac{\theta_0+a}{\theta_0} - \frac{1}{2}\log\frac{a(a+2\theta_0)t+\theta_0^2}{\theta_0^2(\theta_0+a)^2} - \log(\theta_0+a) + \frac{2a^2t}{\theta_0^2}\left\{\frac{(\theta_0+a)^2t}{a(a+2\theta_0)t+\theta_0^2}-1\right\},
\end{aligned}$$

it follows that

$$\begin{aligned}
&K^{(1)}_{Z_1(\theta_0)}(t,\theta_0+a) := \frac{\partial}{\partial t}K_{Z_1(\theta_0)}(t,\theta_0+a) \\
&= \log\frac{\theta_0+a}{\theta_0} - \frac{a(a+2\theta_0)}{2(a(a+2\theta_0)t+\theta_0^2)} + \frac{2a^2}{\theta_0^2}\left\{\frac{(a+\theta_0)^2t}{a(a+2\theta_0)t+\theta_0^2}-1\right\} \\
&\qquad + \frac{2a^2(a+\theta_0)^2t}{(a(a+2\theta_0)t+\theta_0^2)^2},
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
&K^{(2)}_{Z_1(\theta_0)}(t,\theta_0+a) := \frac{\partial^2}{\partial t^2}K_{Z_1(\theta_0)}(t,\theta_0+a) \\
&= \frac{a^2(a+2\theta_0)^2}{2(a(a+2\theta_0)t+\theta_0^2)^2} + \frac{2a^2(a+\theta_0)^2}{(a(a+2\theta_0)t+\theta_0^2)^2} + \frac{2a^2(a+\theta_0)^2(\theta_0^2-a(a+2\theta_0)t)}{(a(a+2\theta_0)t+\theta_0^2)^3}, \\
&K^{(3)}_{Z_1(\theta_0)}(t,\theta_0+a) := \frac{\partial^3}{\partial t^3}K_{Z_1(\theta_0)}(t,\theta_0+a) \\
&= -\frac{a^3(a+2\theta_0)\{(a+2\theta_0)^2+6(a+\theta_0)^2\}}{\{a(a+2\theta_0)t+\theta_0^2\}^3} - \frac{6a^3(a+\theta_0)^2(a+2\theta_0)\{\theta_0^2-a(a+2\theta_0)t\}}{\{a(a+2\theta_0)t+\theta_0^2\}^4},
\end{aligned}$$

$$
\begin{aligned}
K^{(4)}_{Z_1(\theta_0)}(t,\theta_0+a) &:= \frac{\partial^4}{\partial t^4} K_{Z_1(\theta_0)}(t,\theta_0+a) \\
&= \frac{3a^4(a+2\theta_0)^2\{(a+2\theta_0)^2+6(a+\theta_0)^2\}}{\{a(a+2\theta_0)t+\theta_0^2\}^4} \\
&\quad + 6a^4(a+\theta_0)^2(a+2\theta_0)^2\left\{\frac{1}{(a(a+2\theta_0)t+\theta_0^2)^4}+\frac{4(\theta_0^2-a(a+2\theta_0)t)}{(a(a+2\theta_0)t+\theta_0^2)^5}\right\}.
\end{aligned}
$$

Hence the cumulants of $Z_1(\theta_0)$ under $H:\theta=\theta_0+a$ are given as follows.

$$
\begin{aligned}
\mu =& E_{\theta_0+a}[Z_1(\theta_0)] = K^{(1)}_{Z_1(\theta_0)}(0,\theta_0+a) = \log\frac{\theta_0+a}{\theta_0} - \frac{5a^2}{2\theta_0^2} - \frac{a}{\theta_0}, \\
& V_{\theta_0+a}(Z_1(\theta_0)) = K^{(2)}_{Z_1(\theta_0)}(0,\theta_0+a) = \frac{9}{2}\left(\frac{a}{\theta_0}\right)^4 + 10\left(\frac{a}{\theta_0}\right)^3 + 6\left(\frac{a}{\theta_0}\right)^2, \\
& \kappa_{3,\theta_0+a}(Z_1(\theta_0)) = K^{(3)}_{Z_1(\theta_0)}(0,\theta_0+a) = -\left(\frac{a}{\theta_0}\right)^3\left(\frac{a}{\theta_0}+2\right)\left(\frac{13a^2}{\theta_0^2}+\frac{28a}{\theta_0}+16\right), \\
& \kappa_{4,\theta_0+a}(Z_1(\theta_0)) = K^{(4)}_{Z_1(\theta_0)}(0,\theta_0+a) = 3\left(\frac{a}{\theta_0}\right)^4\left(\frac{a}{\theta_0}+2\right)^2\left\{17\left(\frac{a}{\theta_0}\right)^2+\frac{36a}{\theta_0}+20\right\}.
\end{aligned}
$$

Under the alternative $K:\theta=\theta_0$, the m.g.f. of $Z_1(\theta_0)$ is given by

$$
\begin{aligned}
M_{Z_1(\theta_0)}(t,\theta_0) &= E_{\theta_0}[e^{tZ_1(\theta_0)}] \\
&= E_{\theta_0}\left[\exp\left\{t\left(\log\frac{\theta_0+a}{\theta_0} - \frac{1}{2\theta_0^2}(X-2\theta_0)^2 + \frac{1}{2(\theta_0+a)^2}(X-2(\theta_0+a))^2\right)\right\}\right] \\
&= \left(\frac{\theta_0+a}{\theta_0}\right)^t \frac{1}{\sqrt{2\pi}\theta_0}\int_{-\infty}^{\infty}\exp\left\{-\frac{t+1}{2\theta_0^2}(x-2\theta_0)^2 + \frac{t}{2(\theta_0+a)^2}(x-2(\theta_0+a))^2\right\}dx \\
&= \left(\frac{\theta_0+a}{\theta_0}\right)^t \frac{1}{\theta_0}\left\{\exp\frac{2a^2t(\tilde{B}t+1)}{(\theta_0+a)^2}\right\}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2\tilde{B}(\theta_0+a)^2}(u+2a\tilde{B}t)^2\right\}du \\
&= \left(\frac{\theta_0+a}{\theta_0}\right)^{t+1} \frac{\theta_0}{\sqrt{a(a+2\theta_0)t+(\theta_0+a)^2}}\exp\left\{\frac{2a^2t(\tilde{B}t+1)}{(\theta_0+a)^2}\right\} \qquad (5.4)
\end{aligned}
$$

for all $t > -(\theta_0+a)^2/\{a(a+2\theta_0)\}$, where

$$
\tilde{B} = \frac{\theta_0^2}{a(a+2\theta_0)t+(\theta_0+a)^2}.
$$

Since

$$
\begin{aligned}
& K_{Z_1(\theta_0)}(t,\theta_0) := \log M_{Z_1(\theta_0)}(t,\theta_0) \\
& = (t+1)\{\log(\theta_0+a)-\log\theta_0\} + \log\theta_0 - \frac{1}{2}\log(a(a+2\theta_0)t+(\theta_0+a)^2) + \frac{2a^2t(\tilde{B}t+1)}{(\theta_0+a)^2},
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\frac{\partial K_{Z_1(\theta_0)}(t,\theta_0)}{\partial t} &= \log\frac{\theta_0+a}{\theta_0} - \frac{a(a+2\theta_0)}{2(a(a+2\theta_0)t+(\theta_0+a)^2)} \\
&\quad + \frac{2a^2}{(\theta_0+a)^2}(\tilde{B}'t^2+2\tilde{B}t+1), \qquad (5.5)
\end{aligned}
$$

where $\tilde{B}' = \partial\tilde{B}/\partial t$. Since

$$\tilde{B}' = -\frac{a\theta_0^2(a+2\theta_0)}{\{a(a+2\theta_0)t+(\theta_0+a)^2\}^2}\ ,$$

it follows from (5.5) that

$$\begin{aligned}\frac{\partial K_{Z_1(\theta_0)}(t,\theta_0)}{\partial t} &= \log\frac{\theta_0+a}{\theta_0} - \frac{a(a+2\theta_0)}{2\{a(a+2\theta_0)t+(\theta_0+a)^2\}} \\ &\quad + \frac{2a^3\theta_0^2(a+2\theta_0)t^2}{(\theta_0+a)^2\{a(a+2\theta_0)t+(\theta_0+a)^2\}^2} + \frac{4a^2\theta_0^2 t}{\{a(a+2\theta_0)t+(\theta_0+a)^2\}^2} \\ &\quad + \frac{2a^2}{(\theta_0+a)^2}\ . \end{aligned} \tag{5.6}$$

Since, by (5.1),

$$\frac{\partial K_{Z_1(\theta_0)}(-1,\theta_0)}{\partial t} = \log\frac{\theta_0+a}{\theta_0} - \frac{5a^2}{2\theta_0^2} - \frac{a}{\theta_0} = \mu,$$

it is seen that $t = \hat{t} = -1$ is a solution of the quation

$$\frac{\partial K_{Z_1(\theta_0)}(t,\theta_0)}{\partial t} = \mu.$$

From (5.6) we have

$$\begin{aligned}&\frac{\partial^2 K_{Z_1(\theta_0)}(t,\theta_0)}{\partial t^2} \\ &= \frac{a^2(a+2\theta_0)^2}{2\{a(a+2\theta_0)t+(\theta_0+a)^2\}^2} \\ &+ \frac{2a^3\theta_0^2(a+2\theta_0)[2t\{a(a+2\theta_0)t+(\theta_0+a)^2\}^2 - 2a(a+2\theta_0)t^2\{a(a+2\theta_0)t+(\theta_0+a)^2\}]}{(\theta_0+a)^2\{a(a+2\theta_0)t+(\theta_0+a)^2\}^4} \\ &+ \frac{4a^2\theta_0^2[\{a(a+2\theta_0)t+(\theta_0+a)^2\}^2 - 2a(a+2\theta_0)t\{a(a+2\theta_0)t+(\theta_0+a)^2\}]}{\{a(a+2\theta_0)t+(\theta_0+a)^2\}^4},\end{aligned}$$

which yields

$$\frac{\partial^2 K_{Z_1(\theta_0)}(-1,\theta_0)}{\partial t^2} = \frac{9a^4}{2\theta_0^4} + \frac{10a^3}{\theta_0^3} + \frac{6a^2}{\theta_0^2}. \tag{5.7}$$

Hence, from (3.3), for any $\hat{\theta}_n \in \mathcal{A}$ and any $a > 0$, it holds that for large n

$$\begin{aligned}P_{\theta,n}\{\hat{\theta}_n - \theta > a\} \geq \frac{B_0(\lambda)}{\lambda} M^n_{Z_1(\theta)}(\hat{t},\theta)e^{-n\mu\hat{t}}\Bigg[1 + \frac{sgn(\hat{t})}{\sqrt{n}}\frac{\zeta_3(\hat{t})}{6}\frac{B_3(\lambda)}{B_0(\lambda)} \\ + \frac{1}{n}\left\{\frac{\zeta_4(\hat{t})}{24}\frac{B_4(\lambda)}{B_0(\lambda)} + \frac{\zeta_3^2(\hat{t})}{72}\frac{B_6(\lambda)}{B_0(\lambda)}\right\} + O\left(\frac{1}{n^2}\right)\Bigg],\end{aligned} \tag{5.8}$$

where $\lambda := \sqrt{n}|\hat{t}|\sqrt{(\partial^2/\partial t^2)K_{Z_1(\theta)}(\hat{t},\theta)}$, $\zeta_3(t) := \kappa_{3,\theta+a}(Z_1(\theta))/\{(\partial^2/\partial t^2)K_{Z_1(\theta)}(t,\theta)\}^{3/2}$, $\zeta_4(t) := \kappa_{4,\theta+a}(Z_1(\theta))/\{(\partial^2/\partial t^2)K_{Z_1(\theta)}(t,\theta)\}^2$. From (5.7) we have

$$\lambda = \beta\sqrt{n\left(\frac{9}{2}\beta^2 + 10\beta + 6\right)},$$

where $\beta = a/\theta$. Since $\tilde{B} = 1$ for $t = -1$, it follows from (5.4) that

$$M_{Z_1(\theta)}(-1, \theta) = E_\theta[e^{-Z_1(\theta)}] = 1$$

Since

$$\begin{aligned}
&\frac{B_0(\lambda)}{\lambda} M^n_{Z_1(\theta)}(\hat{t}, \theta) e^{-n\mu\hat{t}} = \frac{1}{\lambda} e^{n\mu} \lambda e^{\lambda^2/2} \{1 - \Phi(\lambda)\} \\
&= \{1 - \Phi(\lambda)\} \exp\left[n \left\{ \log(1+\beta) - \frac{5}{2}\beta^2 - \beta \right\} + \frac{n}{2}\beta^2 \left(\frac{9}{2}\beta^2 + 10\beta + 6 \right) \right], \qquad (5.9)
\end{aligned}$$

it follows from (5.8) that for any $\hat{\theta}_n \in \mathcal{A}$

$$\begin{aligned}
P_{\theta,n}\{\hat{\theta}_n - \theta > a\} \geq &\left\{ 1 - \Phi\left(\frac{a}{\theta} \sqrt{n \left(\frac{9a^2}{2\theta^2} + \frac{10a}{\theta} + 6 \right)} \right) \right\} \\
&\cdot \left[\exp\left\{ n \left(\log\left(1 + \frac{a}{\theta}\right) - \frac{5}{2}\left(\frac{a}{\theta}\right)^2 - \frac{a}{\theta} \right) \right.\right. \\
&\left.\left. + \frac{n}{2}\left(\frac{a}{\theta}\right)^2 \left(\frac{9a^2}{2\theta^2} + \frac{10a}{\theta} + 6 \right) \right\} \right] \left\{ 1 + O\left(\frac{1}{n}\right) \right\}
\end{aligned}$$

as $n \to \infty$. From (5.9) and Mills' ratio we have for a fixed small $\beta = a/\theta$,

$$\begin{aligned}
\frac{B_0(\lambda)}{\lambda} M^n_{Z_1(\theta)}(\hat{t}, \theta) e^{-n\mu\hat{t}} &\approx \frac{1}{\lambda} \phi(\lambda) \exp\left(n\mu + \frac{\lambda^2}{2} \right) \\
&= \frac{\exp[n\{\log(1+\beta) - \frac{5}{2}\beta^2 - \beta\}]}{\sqrt{2\pi}\beta\sqrt{n(\frac{9}{2}\beta^2 + 10\beta + 6)}} \\
&\approx \frac{1}{\sqrt{2\pi}\sqrt{6n}\beta} e^{-3n\beta + O(n\beta^3)} \\
&= \phi(\sqrt{6n}\beta) \frac{1}{\sqrt{6n}\beta} (1 + O(n\beta^3)) \\
&\approx 1 - \Phi(\sqrt{6n}\beta), \qquad (5.10)
\end{aligned}$$

as $n \to \infty$. Hence we obtain for a fixed small $a > 0$,

$$P_{\theta,n}\{\hat{\theta}_n - \theta > a\} \geq \left\{ 1 - \Phi\left(\sqrt{6n}\frac{a}{\theta} \right) \right\} \left\{ 1 + O\left(\frac{1}{n}\right) \right\}. \qquad (5.11)$$

In a similar way to the case $a > 0$, we have for a fixed small $|a|$ $(a < 0)$

$$P_{\theta,n}\{\hat{\theta}_n - \theta < a\} \geq \Phi\left(\sqrt{6n}\frac{a}{\theta} \right) \left\{ 1 + O\left(\frac{1}{n}\right) \right\}$$

as $n \to \infty$. Hence we obtain for a fixed small $a > 0$

$$P_{\theta,n}\{|\hat{\theta}_n - \theta| > a\} \geq 2\left\{ 1 - \Phi\left(\sqrt{6n}\frac{a}{\theta} \right) \right\} \{1 + o(1)\}$$

as $n \to \infty$.

Next, we obtain the second order lower bound for the tail probability of AMU estimators $\hat{\theta}_n$. Since $\hat{t} = -1$, it follows from Theorem 3.1 that

$$\zeta_3(-1) = \kappa_{3,\theta+a}(Z_1(\theta)) \Big/ \left\{ \frac{\partial^2}{\partial t^2} K_{Z_1(\theta)}(-1; \theta) \right\}^{3/2}$$

$$
\begin{aligned}
&= \left\{ \frac{\partial^3}{\partial t^3} K_{Z_1(\theta)}(0; \theta + a) \right\} \Big/ \left\{ \frac{\partial^2}{\partial t^2} K_{Z_1(\theta)}(-1; \theta) \right\}^{3/2} \\
&= -\frac{\beta^3(\beta+2)(13\beta^2+28\beta+16)}{\left(\frac{9}{2}\beta^4+10\beta^3+6\beta^2\right)^{3/2}} = -\frac{(\beta+2)(13\beta^2+28\beta+16)}{\left(\frac{9}{2}\beta^2+10\beta+6\right)^{3/2}} \\
&= \frac{1}{\sqrt{6}}\left(-\frac{16}{3}+\frac{4}{3}\beta+\frac{193}{9}\beta^2\right)+O(\beta^3), \qquad (5.12)
\end{aligned}
$$

$$
\begin{aligned}
\zeta_4(-1) &= \kappa_{4,\theta+a}(Z_1(\theta)) \Big/ \left\{ \frac{\partial^2}{\partial t^2} K_{Z_1(\theta)}(-1; \theta) \right\}^2 \\
&= \left\{ \frac{\partial^4}{\partial t^4} K_{Z_1(\theta)}(0; \theta + a) \right\} \Big/ \left\{ \frac{\partial^2}{\partial t^2} K_{Z_1(\theta)}(-1; \theta) \right\}^2 \\
&= \frac{3\beta^4(\beta+2)^2(17\beta^2+36\beta+20)}{\left(\frac{9}{2}\beta^4+10\beta^3+6\beta^2\right)^2} = \frac{3(\beta+2)^2(17\beta^2+36\beta+20)}{36\left(1+\frac{5}{3}\beta+\frac{3}{4}\beta^2\right)^2} \\
&= \frac{20}{3}-\frac{32}{9}\beta+\frac{8}{3}\beta^2+O(\beta^3) \qquad (5.13)
\end{aligned}
$$

for small β. From (5.8), (5.10), (5.12) and (5.13) we obtain

$$
P_{\theta,n}\{\hat{\theta}_n - \theta > a\} \geq \left\{1-\Phi(\sqrt{6n}\beta)\right\}\left[1+\frac{1}{2\lambda\sqrt{n}}\zeta_3(-1)+\frac{1}{n}\left\{\frac{1}{8}\zeta_4(-1)-\frac{5}{24}\zeta_3^2(-1)\right\}+O\left(\frac{1}{n^2}\right)\right], \qquad (5.14)
$$

where

$$
\lambda = \beta\sqrt{n\left(\frac{9}{2}\beta^2+10\beta+6\right)}
$$

with $\beta = a/\theta$, and $\zeta_i(-1)$ $(i = 3, 4)$ are given by (5.12) and (5.13). From (5.12) to (5.14) we have for a fixed small $a > 0$

$$
\frac{P_{\theta,n}\left\{\hat{\theta}_n - \theta > a\right\}}{1-\Phi\left(\sqrt{6n}a/\theta\right)} \geq 1-\frac{1}{9n}\left(\frac{4\theta}{a}-\frac{53}{18}\right)+O\left(\frac{1}{n^2}\right) \qquad (5.15)
$$

as $n \to \infty$.

6 MLE in the curved exponential model

In this section, we obtain the tail probability of the maximum likelihood estimator (MLE) $\hat{\theta}_{ML}$ of θ and investigate whether the MLE attains the bound or not. First let

$$
\psi(x,\theta) = \frac{\partial}{\partial\theta}\log f(x,\theta).
$$

Then the MLE is given as the solution of the equation

$$
\sum_{j=1}^{n}\psi(X_j,\theta) = 0.
$$

Since $f(x,\theta)$ is the density of the normal distribution $N(2\theta, \theta^2)$, the MLE is uniquely determined by

$$\hat{\theta}_{ML} = -\bar{X} + \sqrt{\bar{X}^2 + \frac{1}{n}\sum_{i=1}^{n} X_i^2},$$

where $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$. Since

$$\begin{aligned} W = \psi(X,a) &= \frac{\partial}{\partial a}\log f(X,a) = \frac{1}{a^3}\{(X-2a)^2 + 2a(X-2a) - a^2\} \\ &= \frac{1}{a^3}\{(X-a)^2 - 2a^2\}, \end{aligned}$$

it follows that the m.g.f. of W is given by

$$\begin{aligned} M(t,a) &:= E_\theta(e^{tW}) \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\theta} \exp\left\{\left(\frac{t}{a^3} - \frac{1}{2\theta^2}\right)x^2 + \left(\frac{2}{\theta} - \frac{2t}{a^2}\right)x - \frac{t}{a} - 2\right\} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\theta} \exp\left\{-\frac{a^3 - 2\theta^2 t}{2a^3\theta^2}\left(x + \frac{2a\theta(a^2-\theta t)}{2\theta^2 t - a^3}\right)^2 - \frac{2(a^2-\theta t)^2}{a(2\theta^2 t - a^3)} - \frac{t}{a} - 2\right\} dx \quad (6.1) \end{aligned}$$

for all $t < a^3/(2\theta^2)$. Putting $\beta := a/\theta$ and $y := t/\theta$, we have

$$M(t,a) = \frac{\beta^{3/2}}{\sqrt{\beta^3 - 2y}} \exp\left\{\frac{2(\beta^2-y)^2}{\beta(\beta^3-2y)} - \frac{y}{\beta} - 2\right\} =: M_0(y,\beta),$$

which yields

$$K_0(y,\beta) = \log M_0(y,\beta) = \frac{3}{2}\log\beta - \frac{1}{2}\log(\beta^3 - 2y) + \frac{2(\beta^2-y)^2}{\beta(\beta^3-2y)} - \frac{y}{\beta} - 2\ . \quad (6.2)$$

Note that

$$\begin{aligned} \frac{\partial}{\partial t}K(t,a) &= \frac{\partial}{\partial t}K_0\left(\frac{t}{\theta}, \frac{a}{\theta}\right) = \frac{1}{\theta}K_0'\left(\frac{t}{\theta}, \frac{a}{\theta}\right) = \frac{1}{\theta}K_0'(y,\beta), \\ \frac{\partial^2}{\partial t^2}K(t,a) &= \frac{\partial^2}{\partial t^2}K_0\left(\frac{t}{\theta}, \frac{a}{\theta}\right) = \frac{1}{\theta^2}K_0''\left(\frac{t}{\theta}, \frac{a}{\theta}\right) = \frac{1}{\theta^2}K_0''(y,\beta), \end{aligned}$$

where $K_0'(y,\beta) = (\partial/\partial y)K_0(y,\beta)$ and $K_0''(y,\beta) = (\partial^2/\partial y^2)K_0(y,\beta)$.
From (6.2) we have

$$\begin{aligned} K_0'(y,\beta) &= \frac{1}{\beta^3 - 2y} + \frac{4(\beta^2-y)(y-\beta^3+\beta^2)}{\beta(\beta^3-2y)^2} - \frac{1}{\beta} \\ &= -\frac{1}{\beta(\beta^3-2y)^2}\{8y^2 - 2\beta(4\beta^2-1)y + \beta^4(\beta+5)(\beta-1)\}. \end{aligned}$$

Putting $z := \beta^3 - 2y$, we solve the equation $(\partial/\partial y)K_0(y,\beta) = 0$, i.e.

$$\begin{aligned} 0 &= 8y^2 - 2\beta(4\beta^2-1)y + \beta^4(\beta+5)(\beta-1) \\ &= 2(\beta^3-2y)^2 - \beta(\beta^3-2y) - \beta^4(\beta-2)^2 \\ &= 2z^2 - \beta z - \beta^4(\beta-2)^2. \quad (6.3) \end{aligned}$$

Since $z>0$, the solution of the equation on z is given by

$$z=\frac{\beta}{4}(1+\sqrt{1+8\beta^2(\beta-2)^2})=:z_0,$$

which yields

$$y=\frac{\beta}{8}(4\beta^2-1-\sqrt{1+8\beta^2(\beta-2)^2})=:y_0$$

corresponding to $z=z_0$. Then it follows from (6.2) that

$$K_0(y_0,\beta)=\frac{3}{2}\log\beta-\frac{1}{2}\log(\beta^3-2y_0)+\frac{2(\beta^2-y_0^2)}{\beta z_0}-\frac{y_0}{\beta}-2\ .$$

Now, the tail probability of the MLE is approximated by

$$\begin{aligned}P_{\theta,n}\{\hat{\theta}_{ML}>a\}=&1-\Phi\left(\sqrt{-2nK_0(y_0,\beta)}\right)\\&+\frac{1}{\sqrt{2\pi}}e^{nK_0(y_0,\beta)}\left\{\frac{1}{y_0\sqrt{nK_0''(y_0,\beta)}}-\frac{1}{\sqrt{-2nK_0(y_0,\beta)}}+o\left(\frac{1}{\sqrt{n}}\right)\right\}\end{aligned}$$

(see Lugannani and Rice, 1980, Jensen, 1995 and Barndorff-Nielsen and Cox, 1989). Putting $a+\theta$ instead of a in (6.2), we have

$$\begin{aligned}&P_{\theta,n}\{\hat{\theta}_{ML}-\theta>a\}\\&=1-\Phi\left(\sqrt{-2nK_0(y_0,1+\beta)}\right)\\&\quad+\frac{1}{\sqrt{2\pi}}e^{nK_0(y_0,1+\beta)}\left\{\frac{1}{y_0\sqrt{nK_0''(y_0,1+\beta)}}-\frac{1}{\sqrt{-2nK_0(y_0,1+\beta)}}+o\left(\frac{1}{\sqrt{n}}\right)\right\}\\&=\frac{1}{\sqrt{2\pi}}e^{nK_0(y_0,1+\beta)}\left\{\frac{1}{y_0\sqrt{nK_0''(y_0,1+\beta)}}+o\left(\frac{1}{\sqrt{n}}\right)\right\}\end{aligned}\tag{6.4}$$

as $n\to\infty$. Here,

$$K_0''(y_0,1+\beta)=\frac{2}{z_0^2}\left\{1+\frac{2}{z_0}(\beta+1)^3(\beta-1)^2\right\}\tag{6.5}$$

with

$$\frac{1}{z_0}=-\frac{1-\sqrt{1+8(\beta^2-1)^2}}{2(\beta+1)^3(\beta-1)^2}.\tag{6.6}$$

Since, for small β

$$\sqrt{1+8(\beta^2-1)^2}=3-\frac{8}{3}\beta^2+O(\beta^4),$$

it follows from (6.2) that

$$K_0(y_0,1+\beta)=-3\beta^2+O(\beta^3).\tag{6.7}$$

From (6.5) and (6.6) we have for small β

$$K_0''(y_0,1+\beta)=6\left\{1-2\beta+\frac{31}{9}\beta^2+O(\beta^3)\right\}.\tag{6.8}$$

Then it follows from (6.4), (6.7) and (6.8) that for a fixed small a

$$\begin{aligned} P_{\theta,n}\left\{\hat{\theta}_{ML}-\theta>a\right\} &= 1-\Phi\left(\sqrt{6n}\beta\right)+\frac{1}{\sqrt{6n}}\phi\left(\sqrt{6n}\beta\right)\left\{1+\frac{25}{18}\beta+o(1)\right\} \\ &= \left\{1-\Phi\left(\sqrt{6n}\beta\right)\right\}\left[1+\frac{1}{\sqrt{6n}}\frac{\phi\left(\sqrt{6n}\beta\right)}{1-\Phi\left(\sqrt{6n}\beta\right)}\left\{1+\frac{25}{18}\beta+o(1)\right\}\right] \end{aligned} \tag{6.9}$$

as $n \to \infty$. Since, by Mills' ratio,

$$\frac{\phi\left(\sqrt{6n}\beta\right)}{1-\Phi\left(\sqrt{6n}\beta\right)} = \sqrt{6n}\beta + O\left(\frac{1}{\sqrt{n}}\right),$$

it follows from (6.9) that for a fixed small a

$$\frac{P_{\theta,n}\left\{\hat{\theta}_{ML}-\theta>a\right\}}{1-\Phi\left(\sqrt{6n}a/\theta\right)} = 1+\frac{a}{\theta}\left(1+\frac{25a}{18\theta}\right)+o(1) \tag{6.10}$$

as $n \to \infty$. From (5.11) or (5.15), and (6.10), we see that the MLE $\hat{\theta}_{ML}$ does not attain the lower bound in the first order, hence the MLE is not first order large-deviation efficient.

7 Remarks

In Sections 5 and 6, we treat the normal distribution $N(2\theta, \theta^2)$ with $\theta > 0$ as a curved exponential model, and similar results to the above hold for $N(k\theta, \theta^2)$ with $k \neq 0$, and $\theta \neq 0$. Further, they may be extended to a more general curved exponential model. But, the problem whether the lower bound (3.3) is sharp or not in this case is still open. This seems to be interesting.

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