SEQUENTIAL INTERVAL ESTIMATION OF A LOCATION PARAMETER WITH THE FIXED WIDTH IN THE UNIFORM DISTRIBUTION WITH AN UNKNOWN SCALE PARAMETER

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ABSTRACT

We consider a sequential interval estimation with fixed width of a location parameter θ of a sequence of uniform random variables with unknown scale ξ. A stopping rule is proposed and its asymptotic properties are investigated. Numerical evaluations are also done. Further, the exact distribution of the size of sample is given.

1. INTRODUCTION

Suppose that we are to estimate a location parameter θ of a sequence of random observations $X_1, X_2, \ldots, X_n, \ldots$ with unknown scale ξ. We would like to obtain sequentially a confidence interval of fixed width d with confidence coefficient $1 - \alpha$. Obviously we can not obtain a fixed sample size procedure if ξ is unknown.

In regular cases, if the maximum likelihood estimator (MLE) $\hat{\theta}$ of θ is at least asymptotically independent of the MLE $\hat{\xi}$ of ξ, and asymptotically normally distributed with variance $(nI_\xi)^{-1}$, we can obtain an asymptotically valid procedure when d is small by continuing sampling as long as

$$\sqrt{nI_{\hat{\xi}_n}} u_{\alpha/2} > d,$$

where $u_{\alpha/2}$ is the upper $100(\alpha/2)$ percentile, and stop sampling when $\sqrt{nI_{\hat{\xi}_n}} u_{\alpha/2} \leq d$, and construct the interval $\left[\hat{\theta}_n - d, \hat{\theta}_n + d\right]$, where I_ξ is the Fisher information on ξ. Here $\hat{\theta}_n$ and $\hat{\xi}_n$ are the MLEs based on $X_1, \ldots, X_n$. It may be necessary to prove the above statement rigorously with a set of regularity conditions, but anyway it is possible to prove the validity of the statement if we assume the appropriate set of regularity conditions, and we would not go further into the details of the proof. There are many works on the fixed-width interval estimation of normal mean (see, e.g. Ghosh et al. (1997)).

But in the non-regular case, it may happen that even asymptotically $\hat{\xi}_n$ and $\hat{\theta}_n$ are independent, then the above procedure would fail to be valid. In this paper, we consider the case of the uniform distribution on the interval $[\theta - (\xi/2), \theta + (\xi/2)]$, where θ and ξ are unknown, and obtain sequentially a confidence interval of θ with fixed width d and confidence coefficient $1 - \alpha$. In the case of the uniform distribution on the interval $(0, \theta)$, a similar problem

was studied by Graybill and Connell (1964), Cooke (1971), Akahira (1993), Bose (2001), Mukhopadhyay and Cicconetti (2002) and others. A sequential point estimation of θ of the uniform distribution $U(\theta - (1/2), \theta + (1/2))$ was also discussed by Wald (1950). A uniform distribution may be used as a model in some practical problems (see, e.g. Perruzzi and Hilliard (1984) and Zimmels (1983)).

2. CONFIDENCE INTERVAL OF θ

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent and identically distributed (i.i.d.) random variables according to the uniform distribution on the interval $[\theta - (\xi/2), \theta + (\xi/2)]$, where $\theta(\in \mathbb{R}^1)$ and $\xi \ (> 0)$ are unknown.

Let $X_{(1)} := \min_{1 \le i \le n} X_i$, $X_{(n)} := \max_{1 \le i \le n} X_i$. Then the midrange and the range are

$$M_n := \frac{1}{2}\left(X_{(1)} + X_{(n)}\right), \ \ R_n := X_{(n)} - X_{(1)},$$

respectively. When a confidence coefficient $1 - \alpha$ and a width $d \ (> 0)$ are given, we want to determine n such that

$$P_{\theta,\xi}\left\{|M_n - \theta| \le d\right\} \ge 1 - \alpha. \tag{2.1}$$

Now we consider the conditional probability given R_n

$$P_{\theta,\xi}\left\{\frac{M_n - \theta}{R_n} \le t \ \bigg| \ R_n\right\} = P_{\theta,\xi}\{M_n - \theta \le tR_n \mid R_n\}$$

$$= P_{\theta,\xi}\left\{\frac{M_n - \theta}{\xi} \le t\frac{R_n}{\xi} \ \bigg| \ \frac{R_n}{\xi}\right\} \tag{2.2}$$

for $t \in \mathbb{R}^1$. Letting $Y_i := (X_i - \theta)/\xi \ (i = 1, 2, \ldots, n, \ldots)$, we see that $Y_1, Y_2, \ldots, Y_n, \ldots$ is a sequence of i.i.d. random variables according to the

uniform distribution on $[-1/2, 1/2]$. Then $Y_{(1)} := \min_{1 \le i \le n} Y_i = (X_{(1)} - \theta)/\xi$ and $Y_{(n)} := \max_{1 \le i \le n} Y_i = (X_{(n)} - \theta)/\xi$. We also have

$$\frac{M_n - \theta}{\xi} = \frac{1}{2}(Y_{(1)} + Y_{(n)}) =: M_n', \quad \frac{R_n}{\xi} = Y_{(n)} - Y_{(1)} =: R_n'.$$

From (2.2) we obtain

$$P_{\theta,\xi}\{M_n - \theta \le tR_n \mid R_n\} = P_{\theta,\xi}\{M_n' \le tR_n' \mid R_n'\}. \tag{2.3}$$

Lemma 2.1. The conditional probability of M_n' given $R_n' = r$ is given by

$$P_{\theta,\xi}\{M_n' \le tr \mid R_n' = r\} = \begin{cases} 0 & \text{for} \quad t \le -\frac{1}{2}\left(\frac{1}{r} - 1\right), \\ \frac{1}{2} + \frac{tr}{1-r} & \text{for} \quad -\frac{1}{2}\left(\frac{1}{r} - 1\right) < t \le \frac{1}{2}\left(\frac{1}{r} - 1\right), \\ 1 & \text{for} \quad t \ge \frac{1}{2}\left(\frac{1}{r} - 1\right). \end{cases} \tag{2.4}$$

Proof. Since the joint density of M_n' and R_n' is given by

$$f_{M_n', R_n'}(m, r) = \begin{cases} n(n-1)r^{n-2} & \text{for} \quad 0 < r < 1, \ |r| \le 1 - 2|m|, \\ 0 & \text{otherwise,} \end{cases} \tag{2.5}$$

it follows that the marginal density of R_n' is

$$f_{R_n'}(r) = \begin{cases} n(n-1)r^{n-2}(1-r) & \text{for} \quad 0 < r < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Hence the conditional density of M_n' given R_n' is

$$f_{M_n'|R_n'}(m \mid r) = \begin{cases} \frac{1}{1-r} & \text{for} \quad |m| \le \frac{1}{2} - \frac{r}{2}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

where $0 < r < 1$. Note that this is the uniform density on the interval $[-(1/2) + (R_n'/2), (1/2) - (R_n'/2)]$. From (2.7) we have for $|t|r \le (1-r)/2$

$$P_{\theta,\xi}\{M_n' \le tr \mid R_n' = r\} = \int_{-\frac{1}{2} + \frac{r}{2}}^{tr} \frac{1}{1-r} dm = \frac{1}{2} + \frac{tr}{1-r},$$

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and (2.4) for other cases. This completes the proof. □

From (2.3) and Lemma 2.1 we have the following.

Theorem 2.1. For $t > 0$, the conditional probability given R_n is given by

$$P_{\theta,\xi}\{|M_n - \theta| \leq tR_n \mid R_n\} = \begin{cases} \frac{t}{\frac{1}{2}\left(\frac{\xi}{R_n}-1\right)} & \text{for} \quad 0 < t \leq \frac{1}{2}\left(\frac{\xi}{R_n}-1\right), \\ 1 & \text{for} \quad t > \frac{1}{2}\left(\frac{\xi}{R_n}-1\right). \end{cases}$$

Proof. From (2.2) and Lemma 2.1 we have for $t > 0$

$$P_{\theta,\xi}\{|M_n - \theta| \leq tR_n \mid R_n\} = P_{\theta,\xi}\{|M'_n| \leq tR'_n \mid R'_n\}$$

$$= \begin{cases} \frac{t}{\frac{1}{2}\left(\frac{1}{R'_n}-1\right)} & \text{for} \quad 0 < t \leq \frac{1}{2}\left(\frac{1}{R'_n}-1\right), \\ 1 & \text{for} \quad t > \frac{1}{2}\left(\frac{1}{R'_n}-1\right) \end{cases}$$

$$= \begin{cases} \frac{t}{\frac{1}{2}\left(\frac{\xi}{R_n}-1\right)} & \text{for} \quad 0 < t \leq \frac{1}{2}\left(\frac{\xi}{R_n}-1\right), \\ 1 & \text{for} \quad t > \frac{1}{2}\left(\frac{\xi}{R_n}-1\right). \end{cases}$$

This completes the proof. □

Letting $d = tR_n$ and

$$\frac{d/R_n}{\frac{1}{2}\left(\frac{\xi}{R_n}-1\right)} \geq 1 - \alpha, \tag{2.8}$$

we have from (2.2) and Theorem 2.1

$$P_{\theta,\xi}\{|M_n - \theta| \leq d \mid R_n\} \geq 1 - \alpha$$

hence

$$E_{\theta,\xi}\left[P_{\theta,\xi}\{|M_n - \theta| \leq d \mid R_n\}\right] \geq 1 - \alpha. \tag{2.9}$$

From (2.1), (2.8) and (2.9) we stop when

$$\xi - R_n \leq \frac{2d}{1-\alpha}, \tag{2.10}$$

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but it is not useful because ξ is unknown. Since

$$E_{\theta,\xi}[\xi - R_n] = \frac{2\xi}{n+1},$$

it follows that

$$E_{\theta,\xi}[\xi - R_n] = E\left[\frac{2}{n-1} R_n\right].$$

So, instead of $\xi - R_n$ in (2.10), we use $2R_n/(n-1)$. Hence, we stop when

$$R_n \le \frac{(n-1)d}{1-\alpha}, \tag{2.11}$$

and construct the confidence interval of θ

$$[M_n - d, M_n + d],$$

that is, we consider the sequential estimation procedure $(\tau_0, [M_{\tau_0} - d, M_{\tau_0} + d])$, where τ is given by

$$\tau_0 = \tau\left(\frac{d}{1-\alpha}\right) := \inf\left\{n > 1 \;\middle|\; \frac{R_n}{n-1} \le \frac{d}{1-\alpha}\right\}.$$

Note that this stopping rule is bounded with probability 1 since $0 \le R_n \le \xi$ with probability 1 and (2.11), that is, $P_{\theta,\xi}\left(\tau_0 \le \left\lceil\frac{\xi(1-\alpha)}{d}\right\rceil + 1\right) = 1$ where $\lceil x \rceil$ means the least integer greater than or equal to x.

However, this sequential interval estimation procedure is *not* asymptotically consistent. So, instead of τ_0, we consider another stopping rule:

$$\tau_1 = \tau\left(-\frac{2d}{\log\alpha}\right) := \inf\left\{n \ge 2 \;\middle|\; \frac{R_n}{n-1} \le -\frac{2d}{\log\alpha}\right\}. \tag{2.12}$$

Consider the interval estimation of θ when ξ is known. Since the density function of M'_n is

$$f_{M'_n}(m) = \begin{cases} n(1-2|m|) & \text{for} \quad |m| \le 1/2, \\ 0 & \text{otherwise} \end{cases}$$

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from (2.5). Then the coverage probability is

$$
\begin{aligned}
P\left\{|M_n - \theta| \le d\right\} &= P\{|M_n - \theta|/\xi \le d/\xi\} \\
&= P\{|M'_n| \le d/\xi\} \\
&= \int_{-d/\xi}^{d/\xi} n(1 - 2|m|)^{n-1} dm \\
&= 1 - \left(1 - \frac{2d}{\xi}\right)^n
\end{aligned}
\tag{2.13}
$$

for $0 \le d \le \xi/2$. In order to satisfy $(2.13) \ge 1 - \alpha$, we have to take

$$
n \ge \frac{\log \alpha}{\log\left(1 - (2d/\xi)\right)} =: n^* \quad \text{(say)}.
$$

Here, we substitute the modified stopping rule (2.12) for the optimal fixed sample size n^* since $n^* \simeq -\xi \log \alpha/(2d)$ for small d. Then we have the following theorem.

Theorem 2.2. For the sequential estimation procedure $(\tau_1, [M_{\tau_1} - d, M_{\tau_1} + d])$, we have the following.
(i) $\lim_{d \to 0+} P\{|M_{\tau_1} - \theta| \le d) = 1 - \alpha$ (i.e. asymptotic consistency).
(ii) $\tau_1/n^* \overset{\text{a.s.}}{\to} 1 \ (d \to 0+)$.
(iii) $E(\tau_1)/n^* \to 1 \ (d \to 0+)$.

Proof. (i) From Lemma 1 of Chow and Robbins (1965), the stopping rule τ_1 given by (2.12) satisfies

$$
\lim_{d \to 0+} -\frac{2d\tau_1}{\xi \log \alpha} = 1 \quad \text{a.s.}
\tag{2.14}
$$

Since $n(M_n - \theta)$ converges in distribution to a two-sided exponential distribution with density $\exp(-2|x|)$ as $n \to \infty$, it follows from Theorem 1

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of Anscombe (1952) that $\tau_1(M_{\tau_1} - \theta)$ converges in distribution to the same distribution as $d \to 0+$. Hence, since

$$\frac{d\tau_1}{\xi} \overset{\text{a.s.}}{\to} -\frac{\log\alpha}{2} \quad \text{as} \ \ d \to 0+$$

from (2.14), it follows that

$$\lim_{d\to 0+} P\{|M_{\tau_1} - \theta| \leq d\} = \lim_{d\to 0+} P\{\tau_1 |M_{\tau_1} - \theta|/\xi \leq d\tau_1/\xi\}$$

$$= \int_{(\log\alpha)/2}^{-(\log\alpha)/2} \exp(-2|x|)dx$$

$$= 1 - \alpha.$$

(ii) From (2.14) and L'Hopital's theorem, we have

$$\frac{\tau_1}{n^*} = \frac{\tau_1 \log\{1 - (2d/\xi)\}}{\log\alpha} \overset{\text{a.s.}}{\to} 1 \quad \text{as} \ \ d \to 0+.$$

(iii) Since, from Lemma 2 of Chow and Robbins (1965),

$$\lim_{d\to 0+} \frac{2dE(\tau_1 - 1)}{-\xi\log\alpha} = \lim_{d\to 0+} \frac{2dE(\tau_1)}{-\xi\log\alpha} = 1$$

and

$$dn^* = \frac{d\log\alpha}{\log\{1 - (2d/\xi)\}} \to \frac{\log\alpha}{-2/\xi} \quad \text{as} \ \ d \to 0+,$$

we have

$$\lim_{d\to 0+} \frac{E(\tau_1)}{n^*} = \lim_{d\to 0+} \frac{dE(\tau_1)}{dn^*} = \frac{-(\xi\log\alpha)/2}{(\log\alpha)/(-2/\xi)} = 1.$$

Thus we complete the proof. $\square$

3. NUMERICAL EVALUATION

In this section we examine the coverage probability of the procedure $[M_{\tau_1} - d, M_{\tau_1} + d]$ by simulation based on 100000 repetitions. Since M_{τ_1} is location

equivariant, we may assume $\theta = 0$ without loss of generality. When $\alpha = 0.10$, $d = 0.01(0.01)0.05$, $\xi = 1(1)5$, Tables 1, 2 and 3 show the values of coverage probabilities, the average sample numbers (ASNs) and the standard deviations (SDs) of the sequential estimation procedure $(\tau_1, [M_{\tau_1} - d, M_{\tau_1} + d])$ for $\alpha = 0.10$, respectively. The result shows that we might need to improve the stopping rule, but provides us with a tentative information on θ. Consider another stopping rule with the initial sample size as 10 instead of (2.12), that is,

$$\tau_1' = \inf \left\{ n \geq 10 \ \middle| \ \frac{R_n}{n-1} \leq -\frac{2d}{\log \alpha} \right\}.$$

Tables 4, 5 and 6 show the values of coverage probabilities, the ASNs and the SDs of the sequential estimation procedure $(\tau_1', [M_{\tau_1'} - d, M_{\tau_1'} + d])$ for $\alpha = 0.10$, respectively. The result suggests that the estimation procedure is consistent for this case.

Table 1. Coverage probabilities of
$[M_{\tau_1} - d, M_{\tau_1} + d]$ for $\alpha = 0.10$

$\xi \setminus d$	0.01	0.02	0.03	0.04	0.05
1	0.7607	0.77045	0.77837	0.78468	0.79554
2	0.75503	0.76211	0.76583	0.76831	0.77439
3	0.75466	0.75862	0.76084	0.76462	0.76704
4	0.75086	0.75763	0.76043	0.76246	0.76471
5	0.75343	0.75475	0.75519	0.75825	0.75884

Table 2. ASNs of τ_1 for $\alpha = 0.10$

$\xi \setminus d$	0.01	0.02	0.03	0.04	0.05
1	75.3458	37.9159	26.3105	19.5954	15.5133
2	150.742	75.5505	51.2336	37.9784	30.5142
3	225.222	112.968	75.4855	56.9635	45.3646
4	300.76	150.42	101.419	75.4829	60.4096
5	375.119	187.819	126.251	94.736	75.7516

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Table 3. SDs of τ_1' for $\alpha = 0.10$

$\xi \setminus d$	0.01	0.02	0.03	0.04	0.05
1	43.4924	21.8577	14.7446	11.0086	9.17156
2	86.1962	43.4728	28.9139	21.8323	17.7156
3	129.429	64.72	43.4497	32.6691	26.3042
4	171.983	86.2782	57.6551	43.4353	34.8847
5	214.641	107.455	71.7378	53.9417	43.4604

Table 4. Coverage probabilities of
$[M_{\tau_1'} - d, M_{\tau_1'} + d]$ for $\alpha = 0.10$

$\xi \setminus d$	0.01	0.02	0.03	0.04	0.05
1	0.9093	0.9128	0.9217	0.9237	0.9265
2	0.9051	0.9142	0.9108	0.9186	0.9235
3	0.904	0.9022	0.907	0.9124	0.9066
4	0.9006	0.9011	0.9133	0.9042	0.9119
5	0.9059	0.9025	0.9002	0.9083	0.9084

Table 5. ASNs of τ_1' for $\alpha = 0.10$

$\xi \setminus d$	0.01	0.02	0.03	0.04	0.05
1	116.997	58.995	39.990	29.975	24.940
2	231.997	116.997	77.997	58.995	47.993
3	346.998	173.998	116.998	87.997	70.997
4	461.997	231.997	154.998	116.997	93.997
5	576.997	288.997	192.998	144.999	116.997

Table 6. SDs of τ_1' for $\alpha = 0.10$

$\xi \setminus d$	0.01	0.02	0.03	0.04	0.05
1	0.566183	0.518538	0.559546	0.689228	0.943005
2	0.859796	0.586055	0.41362	0.482571	0.567398
3	1.21506	0.655995	0.545588	0.461446	0.49555
4	1.0107	0.702022	0.480908	0.513148	0.496942
5	1.70099	1.18369	0.708751	0.539996	0.534994

4. SOME PROPERTIES OF THE ESTIMATION PROCEDURE

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of i.i.d. random variables according to the uniform distribution $[\theta - (\xi/2), \theta + (\xi/2)]$ ($\theta \in \mathbb{R}^1, \xi > 0$). Let $X_{(1|n)} \leq X_{(2|n)} \leq \cdots \leq X_{(n|n)}$ be the order statistics of $X_1, X_2, \ldots, X_n$ ($n \geq 1$). Consider a stopping rule such that $\tau(A) := \min\{n \geq 2 \mid R_n/(n-1) \leq A\}$ for a positive constant A, and a sequential interval estimation procedure $[M_{\tau(A)} - d, M_{\tau(A)} + d]$ of θ for a constant d (> 0). In this section we consider some properties of $(\tau(A), M_{\tau(A)})$. Hereafter we represent $P_{\theta,\xi}\{\cdot\}$ as $P\{\cdot\}$ for simplicity.

Theorem 4.1. The probability $P\{\tau(A) = n\}$ is given by

$$P\{\tau(A) = n\} = p_{n-1} - p_n \quad (n \geq 2)$$

recursively, where

$$p_1 := P\{\tau(A) > 1\} = 1,$$

$$p_2 := P\{\tau(A) > 2\} = \begin{cases} \left(\frac{\xi - A}{\xi}\right)^2 & \text{for} \quad 0 \leq A \leq \xi, \\ 0 & \text{otherwise}, \end{cases}$$

$$p_n := P\{\tau(A) > n\}$$

$$= \begin{cases} \frac{(n-1)(n-2)}{\xi^{n-1}} \left[\left\{1 - \frac{2}{\xi}(n-1)A\right\}(I_{n-1,n-3} - I_{n,n-3}) \right. \\ \left. + \frac{1}{\xi}(I_{n-1,n-2} - I_{n,n-2}) + I_{n,n-3} \right] \\ \quad \cdot \left\{1 - \left(\frac{(n-2)A}{\xi}\right)^{n-2}\left(n - 1 - \frac{(n-2)^2 A}{\xi}\right)\right\}^{-1} p_{n-1} \\ \quad \text{for} \quad 0 \leq (n-1)A \leq \xi, \\ 0 \quad \text{otherwise}, \end{cases}$$

for $n \geq 3$, with

$$I_{l,m} := \frac{1}{(m+1)(m+2)}\{\xi^{m+2} + (m+1)((l-1)A)^{m+2} - \xi(m+2)((l-1)A)^{m+1}\}$$

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for $l \geq 1$ and $m \geq 0$.

Proof. First we have

$$
\begin{aligned}
P\{\tau(A) = n\} &= P\{R_n \leq (n-1)A, \ R_i > (i-1)A \ (i = 2, \ldots, n-1)\} \\
&= P\{R_i > (i-1)A \ (i = 2, \ldots, n-1)\} \\
&\quad - P\{R_i > (i-1)A \ (i = 2, \ldots, n)\} \\
&= p_{n-1} - p_n,
\end{aligned}
$$

where $p_n = P\{R_i > (i-1)A \ (i = 2, \ldots, n)\} \ (n \geq 1)$. Since $p_2 = P\{R_2 > A\} = P\{R_2' > A/\xi\}$ and (2.6), it follows that $p_2 = \{(\xi - A)/\xi\}^2$ for $0 < A < \xi$ and $p_2 = 0$ for $A \geq \xi$. We also have

$$
\begin{aligned}
p_n &= P\{R_n > (n-1)A \mid R_i > (i-1)A \ (i = 2, \ldots, n-1)\}p_{n-1} \\
&= P\{R_n > (n-1)A \mid R_{n-1} > (n-2)A\}p_{n-1} \\
&= \frac{P\{R_n > (n-1)A, R_{n-1} > (n-2)A\}}{P\{R_{n-1} > (n-2)A\}}p_{n-1}.
\end{aligned} \tag{4.1}
$$

The denominator of (4.1) is

$$
P\{R_{n-1} > (n-2)A\} = 1 - \left(\frac{n-2}{\xi}A\right)^{n-2}\left(n - 1 - \frac{(n-2)^2}{\xi}A\right)
$$

for $0 < (n-2)A/\xi < 1$, since $P\{R_{n-1} > (n-2)A\} = P\{R_n' > (n-2)A/\xi\}$ and (2.6), and $P\{R_{n-1} > (n-2)A\} = 0$ for $(n-2)A/\xi \geq 1$. Denoting by $f_{X_{(1|n-1)},X_{(n-1|n-1)}}(u,v)$ the joint density of $X_{(1|n-1)}$ and $X_{(n-1|n-1)}$, we have

$$
\begin{aligned}
&P\{R_n > (n-1)A, \ R_{n-1} > (n-2)A\} \\
&= \iint_{B_{n-1}} P\{R_n > (n-1)A \mid X_{(1|n-1)} = u, X_{(n-1|n-1)} = v\} \\
&\qquad \cdot f_{X_{(1,|n-1)},X_{(n-1|n-1)}}(u,v)dudv \\
&= \left(\iint_{B_{n-1}-B_n} + \iint_{B_n}\right) P\{R_n > (n-1)A \mid X_{(1|n-1)} = u, X_{(n-1|n-1)} = v\} \\
&\qquad \cdot f_{X_{(1|n-1)},X_{(n-1|n-1)}}(u,v)dudv,
\end{aligned} \tag{4.2}
$$

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where $B_i := \{(u, v) \mid v - u > (i-1)A, \; \theta - (\xi/2) \le u \le v \le \theta + (\xi/2)\}$ $(i \ge 1)$. If $\left(X_{(1|n-1)}, X_{(n-1|n-1)}\right) = (u, v)$ is given for $\theta - (\xi/2) \le u \le v \le \theta + (\xi/2)$, then

$$\left(X_{(1|n)}, X_{(n|n)}\right) = \begin{cases} (X_n, v) & \text{with probability } F(u), \\ (u, v) & \text{with probability } F(v) - F(u), \\ (u, X_n) & \text{with probability } 1 - F(v), \end{cases}$$

where $F(\cdot)$ is the distribution function of X_i given by

$$F(x) = \begin{cases} 0 & \text{if } x < \theta - (\xi/2), \\ \{x - \theta + (\xi/2)\}/\xi & \text{if } \theta - (\xi/2) \le x \le \theta + (\xi/2), \\ 1 & \text{if } x > \theta + (\xi/2). \end{cases} \qquad (4.3)$$

So, $X_{(n|n)} > X_{(1|n)} + (n-1)A$ holds with probability 1 given (u, v) satisfying $v > u + (n-1)A$, and with probability $\{1 + F(v - (n-1)A) - F(u + (n-1)A)\}$ given (u, v) satisfying $u + (n-2)A < v < u + (n-1)A$. Then the right-hand side of (4.2) is equal to

$$\iint_{B_{n-1} - B_n} \{1 + F(v - (n-1)A) - F(u + (n-1)A)\}$$
$$\cdot f_{X_{(1|n-1)}, X_{(n-1,|n-1)}}(u, v)\,du\,dv$$
$$+ \iint_{B_n} f_{X_{(1|n-1)}, X_{(n-1|n-1)}}(u, v)\,du\,dv. \qquad (4.4)$$

Since

$$f_{X_{(1|n-1)}, X_{(n-1|n-1)}}(u, v) = \begin{cases} (n-1)(n-2)\{F(v) - F(u)\}^{n-3}/\xi^2 \\ \qquad \text{for} \quad \theta - (\xi/2) \le u \le v \le \theta + (\xi/2), \\ 0 \qquad \text{otherwise}, \end{cases}$$

it follows from (4.3) that (4.4) equals, by letting $I_{l,m} := \iint_{B_l}(v - u)^m\,du\,dv$ for $l \ge 1$ and $m \ge 0$,

$$\frac{(n-1)(n-2)}{\xi^{n-1}}\left[\left\{1 - \frac{2}{\xi}(n-1)A\right\}(I_{n-1,n-3} - I_{n,n-3})\right.$$
$$\left. + \frac{1}{\xi}(I_{n-1,n-2} - I_{n,n-2}) + I_{n,n-3}\right] \qquad (n \ge 3)$$

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with $I_{l,m} = \frac{1}{(m+1)(m+2)}\{\xi^{m+2} + (m+1)((l-1)A)^{m+2} - \xi(m+2)((l-1)A)^{m+1}\}$, which is the desired equality. $\qquad\square$

Remark. The values of ASN $E(\tau(A))$ can be calculated by using $\{p_n\}$. Indeed, since $p_n = P\{\tau(A) > n\}$ $(n \geq 2)$,

$$
\begin{aligned}
\sum_{i=2}^{\infty} p_i &= \sum_{i=2}^{\infty} P\{\tau(A) > i\} = \sum_{i=2}^{\infty} \sum_{j=i+1}^{\infty} P\{\tau(A) = j\} = \sum_{j=3}^{\infty} \sum_{i=2}^{j-1} P\{\tau(A) = j\} \\
&= \sum_{j=3}^{\infty} j P\{\tau(A) = j\} - 2 \sum_{j=3}^{\infty} P\{\tau(A) = j\} \\
&= E(\tau(A)) - 2P\{\tau(A) = 2\} - 2\{1 - P\{\tau(A) = 2\}\} \\
&= E(\tau(A)) - 2,
\end{aligned}
$$

we have

$$
E(\tau(A)) = \sum_{i=2}^{\infty} p_i + 2.
$$

ACKNOWLEDGEMENTS

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