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A Remark on Asymptotic Sufficiency of Statistics in Non-Regular Cases*

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Abstract

Suppose that $X_1, X_2, \cdots, X_n, \cdots$ *is a sequence of independent identically distributed random variables with the density* $f(x:\theta)$ *with a compact support, where* θ *is a real valued parameter. We suppose that a strongly* $\{c_n\}$*-consistent estimator of* θ *exists. Then we show that a statistic* $(\min_{1\leq i\leq n} X_i, \max_{1\leq i\leq n} X_i)$ *is asymptotically sufficient in non-regular cases.*

1. Introduction

A consistent estimator with order $\{c_n\}$ (or a $\{c_n\}$-consistent estimator) is defined and discussed in Akahira [1], where the necessary conditions for the existence of such an estimator are established and the bounds of the orders of convergence of consistent estimators are obtained for non-regular cases. Further the asymptotic accuracies of $\{c_n\}$-consistent estimators are discussed in Akahira [2].

Asymptotic sufficiency has been discussed under regularity conditions by LeCam [4]. In this paper we extend a similar approach te non-regular cases.

Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent identically distributed random variables with the density $f(x:\theta)$ with a compact support, where θ is a real valued parameter. We suppose that a strongly $\{c_n\}$-consistent estimator of θ exists. Then we shall obtain that a statistic $(\min_{1\leq i\leq n} X_i, \max_{1\leq i\leq n} X_i)$ is asymptotically sufficient in non-regular cases.

2. Notations and definitions

Let $\mathcal{X}$ be an abstract sample space whose generic point is denoted by x, $\mathcal{B}$ a σ-field of subsets of $\mathcal{X}$ and $\{P_\theta : \theta\in\Theta\}$ a set of probability measures on $\mathcal{B}$, where Θ is called a parameter space. We suppose that Θ is an open set in a Euclidean 1-space R^1. Consider n-fold direct products $(\mathcal{X}^{(n)}, \mathcal{B}^{(n)})$ of $(\mathcal{X}, \mathcal{B})$ and the corresponding product measure $P_\theta^{(n)}$ of P_θ. For each $n=1, 2, \cdots$, the points of $\mathcal{X}^{(n)}$ will be denoted by $\tilde{x}_n=(x_1, \cdots, x_n)$ and the corresponding random variable by $\tilde{x}_n$. An estimator of θ is defined to be a sequence $\{\hat{\theta}_n\}$ of $\mathcal{B}^{(n)}$-measurable function $\hat{\theta}_n$ on $\mathcal{X}^{(n)}$ into Θ. For a sequence of positive numbers $\{c_n\}$ (c_n tending to infinity) an estimator $\{\hat{\theta}_n\}$ is called strongly consistent with order $\{c_n\}$ (or strongly $\{c_n\}$-consistent for short) if for every $\varepsilon>0$ and for every compact subset K of Θ, there exists a sufficiently large positive number L satisfying the following:

$$\varlimsup_{n\to\infty} \sup_{\theta\in K} P_\theta^{(n)}(\{c_n|\hat{\theta}_n-\theta|\geq L\})<\varepsilon.$$

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A weaker definition of a $\{c_n\}$-consistent estimator than that of the above form has been given in Akahira [1].

We suppose that every $P_\theta(\cdot)$ $(\theta\in\Theta)$ is absolutely continuous with respect to a σ-finite measure μ. Then we denote the density $dP_\theta/d\mu$ by $f(\cdot:\theta)$. If the distribution of $\tilde{x}_n$ is the product measure $P_\theta{}^{(n)}$, then the corresponding density with respect to the prodcuct measure $\mu^{(n)}$ will be denoted by $\prod_{i=1}^{n} f(x_i:\theta)$. A statistic $T_n(\tilde{X}_n)$ is called asymptotically sufficient if there exist a nonnegative function $p_n(\tilde{x}_n:\theta)$, each the product of a function of $\tilde{x}_n$ only by a function of T_n and θ only such that

$$\lim_{n\to\infty}\sup_{\theta\in K}\int_{\mathcal{X}^{(n)}}\left|\prod_{i=1}^{n} f(x_i:\theta)-p_n(\tilde{x}_n:\theta)\right|d\mu^{(n)}=0$$

for any compact subset K of Θ (LeCam [4]).

3. Asymptotically sufficient statistics

Before discussing the asymptotic sufficiency in detail we shall give a definition and a lemma.

Definition. (Generalized from Gnedenko and Kolmogorov [3]) For each $\theta\in\Theta$ the sums

$$Y_n(\theta)=X_1(\theta)+X_2(\theta)+\cdots+X_n(\theta)$$

of postive independent random variables $X_1(\theta), X_2(\theta), \cdots, X_n(\theta), \cdots$ are said to be uniformly relatively stable for constants $B_n(\theta)$ if there exist positive constants $B_n(\theta)$ such that for any $\varepsilon>0$

$$P_\theta{}^{(n)}\left(\left\{\left|\frac{Y_n(\theta)}{B_n(\theta)}-1\right|>\varepsilon\right\}\right)\to 0$$

as $n\to\infty$ uniformly in any compact subset of Θ.

In the subsequent lemma we use the notation that for each k and each $\theta\in\Theta$, $F_{\theta k}(x)$ is the distribution function of $X_k(\theta)$.

Lemma. (Gnedenko and Kolmogorov [3]).
For each $\theta\in\Theta$, let $X_1(\theta), X_2(\theta), \cdots, X_n(\theta), \cdots$ be a sequence of positive independent random variables. The sums

$$Y_n(\theta)=X_1(\theta)+X_2(\theta)+\cdots+X_n(\theta)$$

are uniformly relatively stable for constants $B_n(\theta)$, if there exists a sequence of positive constants $B_1(\theta), B_2(\theta), \cdots, B_n(\theta), \cdots$ such that for any $\varepsilon>0$

$$\sum_{k=1}^{n}\int_{\varepsilon B_n}^{\infty} dF_{\theta k}(x)\to 0$$

as $n\to\infty$ uniformly in any compact subset of Θ,

$$\frac{1}{B_n(\theta)}\sum_{k=1}^{n}\int_{0}^{\varepsilon B_n(\theta)} x\,dF_{\theta k}(x)\to 1$$

as $n\to\infty$ uniformly in any compact subset of Θ.

Let $\mathcal{X}=R^1$, Now we suppose that every $P_\theta(\cdot)$ $(\theta\in\Theta)$ is absolutely continuous with respect to a Lebesgue measure m. Then we denote the density dP_θ/dm by $f(\cdot:\theta)$ and by $A(\theta)\subset\mathcal{X}$ the set of points in the space of $\mathcal{X}$ for which $f(x:\theta)>0$ and suppose $f(x:\theta)=f(x-\theta)$. We make the following assumptions (A), (B) and (C).

Assumption (A).
$$f(x)>0 \quad \text{for} \quad a\leq x\leq b;$$
$$f(x)=0 \quad \text{for} \quad x<a,\ x>b,$$

and $f(a)$ and $f(b)$ are finite.

Assumption (B). $f(x)$ is twice continuously differentiable in the interval (a, b).
Define

$$\varphi(\theta)=\int_0^\infty w dF(w : \theta),$$

where $F(w : \theta)$ is the distribution function of

$$W(X : \theta)=\chi_{(a,b)\cap A(\theta)}(X)\left|\log\frac{f(x-\theta)}{f(x)}\right|$$

($\chi_{(a,b)\cap A(\theta)}(\cdot)$ denotes the indicator of $(a, b)\cap A(\theta)$).

Let $T_n=(Y, Z)$, where $Y=\min_{1\leq i\leq n} X_i$ and $Z=\max_{1\leq i\leq n} X_i$. We suppose that $\{\hat{\theta}_n(T_n)\}$ is a $\{c_n\}$-consistent estimator. The existence of the estimator is guaranteed (See Theorem 4. 1 of [1]). Then for any $\delta>0$ and any compact subset K of Θ there exists a sufficiently large positive number L satisfying the following:

$$\overline{\lim_{n\to\infty}}\sup_{\theta\in K} P_\theta^{(n)}(\{|\hat{\theta}_n(T_n)-\theta|>Lc_n^{-1}\})<\delta. \tag{3. 1}$$

Assumption (C). The following (3. 2)~(3. 4) hold:

$$\lim_{n\to\infty} n\varphi(Lc_n^{-1})=0 \tag{3. 2}$$

$$\lim_{n\to\infty}\sup_{\theta\in K} n\int_{\varepsilon n\varphi(Lc_n^{-1})}^\infty dF(w : \theta)=0 \tag{3. 3}$$

for any $\varepsilon>0$ and any compact subset K of Θ;

$$\lim_{n\to\infty}\sup_{\theta\in K}\frac{1}{\varphi(Lc_n^{-1})}\int_0^{\varepsilon n\varphi(Lc_n^{-1})} w dF(w : \theta)=1 \tag{3. 4}$$

for any $\varepsilon>0$ and any compact subset K of Θ.

Theorem. Under Assumptions (A), (B) and (C), the statistic T_n, i.e. $(\min_{1\leq i\leq n} X_i, \max_{1\leq i\leq n} X_i)$, is asymptotically sufficient.

Proof. Let ε be an arbitrary positive number. We define $h(T_n, \theta)$ and $g(\tilde{x}_n, \hat{\theta}_n(T_n))$ as follows:

$$h(T_n, \theta)=\chi_\theta(y, z)=\begin{cases}1, & \text{if } z-b<\theta<y-a;\\ 0, & \text{otherwise}\end{cases} \tag{3. 5}$$

$$g(\tilde{x}_n, \hat{\theta}_n(T_n))=\prod_{i=1}^n f(x_i-\hat{\theta}_n(T_n)). \tag{3. 6}$$

It follows from (3. 3), (3. 4) and Lemma that $\sum_{i=1}^n W(X_i : \hat{\theta}_n(T_n)-\theta)$ is uniformly relatively stable for $n\varphi(Lc_n^{-1})$. Hence we have for any compact subset K of Θ

$$\lim_{n\to\infty}\inf_{\theta\in K} P_\theta^{(n)}(A_n(\hat{\theta}_n(T_n)-\theta : \varepsilon))=1,$$

where $A_n(\hat{\theta}_n(T_n)-\theta : \varepsilon)=\left\{\tilde{x}_n : \left|\frac{1}{n\varphi(Lc_n^{-1})}\sum_{i=1}^n W(x_i : \hat{\theta}_n(T_n)-\theta)-1\right|<\varepsilon\right\}$.

It follows from (3. 1), (3. 2) and (3. 5)~(3. 7) that for any compact subset K of Θ

$$\overline{\lim_{n\to\infty}}\sup_{\theta\in K}\int_{\mathfrak{X}^{(n)}}\left|\prod_{i=1}^n f(x_i-\theta)-h(T_n, \theta)g(\tilde{x}_n, \hat{\theta}_n(T_n))\right|\prod_{i=1}^n dx_i$$

$$\leq\overline{\lim_{n\to\infty}}\sup_{\theta\in K}\left(\int_{\{|\hat{\theta}_n(T_n)-\theta|\leq Lc_n^{-1}\}\cap A_n(\hat{\theta}_n(T_n)-\theta : \varepsilon)}+\int_{\{|\hat{\theta}_n(T_n)-\theta|>Lc_n^{-1}\}}+\int_{A_n(\hat{\theta}_n(T_n)-\theta : \varepsilon)^c}\right)$$

$$\cdot\left|\prod_{i=1}^n f(x_i-\theta)-h(T_n, \theta)g(\tilde{x}_n, \hat{\theta}_n(T_n))\right|\prod_{i=1}^n dx_i$$

$$
\begin{aligned}
&= \varlimsup_{n\to\infty} \sup_{\theta\in K} \int_{\{|\hat{\theta}_n(T_n)-\theta|\le Lc_n^{-1}\}\cap A_n(\hat{\theta}_n(T_n)-\theta\,:\,\varepsilon)\cap\{z-b<\theta<y-a\}} \left| \prod_{i=1}^n f(x_i-\theta) - \prod_{i=1}^n f(x_i-\hat{\theta}_n(T_n)) \right| \prod_{i=1}^n dx_i \\
&\quad + \varlimsup_{n\to\infty} \sup_{\theta\in K} 2\int_{\{|\hat{\theta}_n(T_n)-\theta|>Lc_n^{-1}\}} \prod_{i=1}^n f(x_i-\theta) \prod_{i=1}^n dx_i \\
&\quad + \varlimsup_{n\to\infty} \sup_{\theta\in K} \{1-P_\theta^{(n)}(A_n(\hat{\theta}_n)-\theta\,:\,\varepsilon))\} \\
&\le \varlimsup_{n\to\infty} \sup_{\theta\in K} \int_{\{|\hat{\theta}_n(T_n)-\theta|\le Lc_n^{-1}\}\cap A_n(\hat{\theta}_n(T_n)-\theta\,:\,\varepsilon)\cap\{z-b<\theta<y-a\}} \left| \prod_{i=1}^n \frac{f(x_i-\hat{\theta}_n(T_n))}{f(x_i-\theta)} - 1 \right| \prod_{i=1}^n f(x_i-\theta) \prod_{i=1}^n dx_i + 2\delta \\
&\le \varlimsup_{n\to\infty} \int_{\mathscr{X}^{(n)}} \left| \left\{ \exp \sum_{i=1}^n W(x_i\,:\,Lc_n^{-1}) \right\} - 1 \right| \prod_{i=1}^n f(x_i) \prod_{i=1}^n dx_i + 2\delta \\
&\le \varlimsup_{n\to\infty} |\exp(1+\varepsilon)n\varphi(Lc_n^{-1}) - 1| + 2\delta \\
&= 2\delta
\end{aligned}
$$

Letting $\delta\to 0$, we complete the proof of the theorem.

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