微積分（9コマ目）

未定定数

$t=0$

連立1次方程式

$x' = y$

$y' = \cancel{bx^2} \; -b^2 x \qquad (b>0)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -b^2 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\varphi_A = |tE - A|$$

$$= \left| \begin{pmatrix} t & -1 \\ b^2 & t \end{pmatrix} \right|$$

$$= t^2 + b^2 = (t+ib)(t-ib)$$

$$t = \pm ib$$

jordanの標準形 $\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$

$$P^{-1} \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} P = \begin{pmatrix} 0 & 1 \\ -b^2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} \longleftrightarrow bi$$

$$\cancel{e^{t\begin{pmatrix} 0 & -b \\ -b & \end{pmatrix}}}$$

$$e^{t\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}} \qquad e^{tbi} = \cos bt + i \sin bt$$

$$\cos bt + i \sin bt$$

$$\Updownarrow$$

$$\begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$x = c_1 \cos bt - c_2 \sin bt$$

$$y = c_1 \sin bt + c_2 \cos bt$$

$$t\begin{pmatrix} 0 & 1 \\ -b^2 & 0 \end{pmatrix} = P^{-1} t\begin{pmatrix} & \\ & \end{pmatrix} P$$

未定定数 $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$e^{tA} = P^{-1} e^{t\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}} P$$

$$\begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}$$

$$\begin{cases} x = a_{11} \cos bt + a_{12} \sin bt \\ y = a_{21} \cos bt + a_{22} \sin bt \end{cases}$$

$$x' = -b a_{11} \sin bt + b a_{12} \cos bt$$
$$= a_{21} \cos bt + a_{22} \sin bt$$

$$y' = -b a_{21} \sin bt + b a_{22} \cos bt$$
$$= -b^2 \{ a_{11} \cos bt + a_{12} \sin bt \}$$

$$(\overset{b a_{12}}{\cancel{b}} - a_{21}) \cos bt + (-b a_{11} - a_{22}) \sin bt = 0$$

$(^\forall t \in \mathbb{R})$ $\alpha \cos bt + \beta \sin bt \Rightarrow \alpha = \beta = 0$

$$a_{21} = b a_{12} \quad a_{22} = -b a_{11}$$

$$(b a_{22} + b^2 a_{11}) \cos bt + (-b a_{21} + b^2 a_{12}) \sin bt = 0$$

$$a_{22} = -b a_{11} \qquad a_{21} = b a_{12}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ b a_{12} & -b a_{11} \end{pmatrix}$$

一般解 $\begin{cases} x = a_{11} \cos bt + a_{12} \sin bt \\ y = b a_{12} \cos bt - b a_{11} \sin bt \end{cases}$

初期条件 $t = 0$

$t = 1$

$$x^2 = a_{11}^2\cos^2 bt + a_{12}^2\sin^2 bt + 2a_{11}a_{12}\cos bt \sin bt$$
$$= a_{11}^2\cos^2 bt + a_{12}^2\sin^2 bt + a_{11}a_{12}\sin 2bt$$

$$y^2 = b^2 a_{12}^2\cos b^2 t + b^2 a_{11}^2\sin^2 bt - 2b^2 a_{11}a_{12}\sin bt\cos bt$$
$$= b^2(a_{12}^2\cos^2 t + a_{11}^2\sin^2 bt - a_{11}a_{12}\sin 2bt$$

$$b^2x^2 + y^2$$
$$= b^2(a_{11}^2 + a_{12}^2) \quad : 一定$$

⇓

楕円

$x' = y \qquad x'' = -b^2 x \qquad$ 周期運動

$y' = -bx^2$

report　来週の月曜日締めきり D705のレポートBOXへ

次の微分方程式を解け

I $\begin{cases} x' = -4y & x(0) = 0 \\ y' = x & y(0) = -7 \end{cases}$

II $\begin{cases} x' = x & x(0) = a \\ y' = x + y & y(0) = b \end{cases}$

III $\begin{cases} x' = 2x - y \\ y' = 2y \end{cases}$

IV $\begin{cases} x' = 2x - y \\ y' = x + 2y \end{cases}$

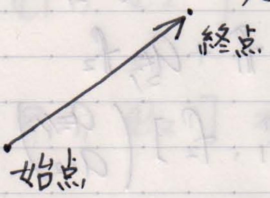
線型代数

基底

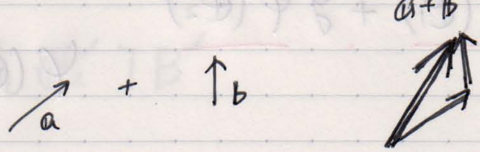
2次元のベクトル

平行移動で重なる矢印は同じもの

矢印



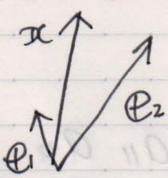
足し算



数

同一直線上にない2つのベクトル e_1, e_2 をとる。

基底



$$x = x_1 e_1 + x_2 e_2$$

一通りに書ける

基底を決めると…

$$x \longleftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

厳密には、

$$x = [e_1, e_2] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

平面ベクトルの全体から平面ベクトルの全体への写像 φ
(mapping)

$$\begin{cases} \varphi(x+y) = \varphi(x) + \varphi(y) \\ \varphi(\alpha x) = \alpha \varphi(x) \end{cases} \quad \alpha : \text{スカラー}$$

このような写像を 線形写像 という。

平面のベクトル $\xrightarrow{\varphi}$ 線型写像 平面のベクトル

基底 $\mathbf{e}_1, \mathbf{e}_2$　　　基底 $\mathbf{f}_1, \mathbf{f}_2$

$$\varphi(x\mathbf{e}_1 + y\mathbf{e}_2)$$
$$= x\varphi(\mathbf{e}_1) + y\varphi(\mathbf{e}_2)$$

$$\varphi(\mathbf{e}_1) = [\mathbf{f}_1, \mathbf{f}_2]\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = a_{11}\mathbf{f}_1 + a_{21}\mathbf{f}_2$$

$$\varphi(\mathbf{e}_2) = [\mathbf{f}_1, \mathbf{f}_2]\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = a_{12}\mathbf{f}_1 + a_{22}\mathbf{f}_2$$

$$= x[\mathbf{f}_1, \mathbf{f}_2]\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + y[\mathbf{f}_1, \mathbf{f}_2]\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

$$= [\mathbf{f}_1, \mathbf{f}_2]\left\{ x\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + y\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right\}$$

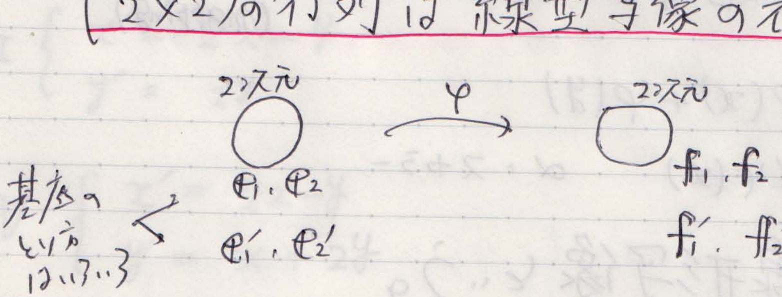
$$= [\mathbf{f}_1, \mathbf{f}_2]A\begin{pmatrix} x \\ y \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

∴ 線型写像 φ はそれぞれに基底

$\{\mathbf{e}_1, \mathbf{e}_2\}$　$\{\mathbf{f}_1, \mathbf{f}_2\}$ をとると

行列 A で表わせる

(2×2)の行列は 線型写像の表現



$$e_1' = [e_1, e_2]\begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$$

$$e_2' = [e_1, e_2]\begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{\cancel{22}\,21} & b_{22} \end{pmatrix}$$

$$[e_1', e_2'] = [e_1, e_2]\, B$$

$$[e_1, e_2] = [e_1', e_2']\, B'$$

$$[e_1', e_2'] = [e_1, e_2]\, B = [e_1', e_2']\, \overset{E}{B'B}$$

$$BB' = E$$

未定定数

$t=0$

<連立1次方程式

$x' = y \qquad (b>0)$

$y' = -b^2 x$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -b^2 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$= ($

$$\varphi_A = |tE - A|$$

$$= \begin{vmatrix} t & -1 \\ b^2 & t \end{vmatrix}$$

$$= t^2 + b^2$$

$$t = \pm ib$$

Jordanの標準形

$$\underline{P^{-1}\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}} P = \begin{pmatrix} 0 & 1 \\ -b^2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} \Longleftrightarrow bi$$

$$e^{t\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}}$$

$$e^{tbi} = \cos bt + i \sin bt$$

$\cos bt + i\sin bt$

$\Updownarrow$

$$\begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$x = c_1\cos bt - c_2\sin bt$

$y = c_1\sin bt + c_2\cos bt$

$$t\begin{pmatrix} 0 & 1 \\ -b^2 & 0 \end{pmatrix} = p^{-1}t\begin{pmatrix} & \\ & \end{pmatrix}p \;-$$

$$e^{tA} = \underline{p^{-1}}\ \underline{e^{t\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}}}\ \underline{p}$$

$$\begin{cases} x = a_{11}\cos bt + a_{12}\sin bt \\ y = a_{21}\cos bt + a_{22}\sin bt \end{cases}$$

未定定数

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$x' = -b\,a_{11}\sin bt + b\,a_{12}\cos bt$

$= a_{21}\cos bt + a_{22}\sin bt$

$y' = -b\,a_{21}\sin bt + b\,a_{22}\cos bt$

$= -b^2\{a_{11}\cos bt + a_{12}\sin bt\}$

$\alpha \cos bt + \beta \sin bt \Rightarrow \alpha = \beta = 0$
$(\forall t \in \mathbb{R})$

$(b a_{12} - a_{21}) \cos bt + (-b a_{11} - a_{22}) \sin bt = 0$

$a_{21} = b a_{12} \qquad a_{22} = -b a_{11}$

$(b a_{22} + b^2 a_{11}) \cos bt + (-b a_{21} + b^2 a_{12}) \sin bt = 0$

$a_{22} = -b a_{11}$

$a_{21} = b a_{12}$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ b a_{12} & -b a_{11} \end{pmatrix}$

一般解

$\begin{cases} x = a_{11} \cos bt + a_{12} \sin bt \\ y = b a_{12} \cos bt - b a_{11} \sin bt \end{cases}$

初期条件 $t = 0$
$t = 1$

$x' = y$
$y' = -b^2 x$

$x'' = -b^2 x$

$x^2 = a_{11}^2 \cos^2 bt + a_{12}^2 \sin^2 bt + a_{11} a_{12} \sin 2bt$

周期

$y^2 = b^2 a_{12}^2 \cos^2 bt + b^2 a_{11}^2 \sin^2 bt - b^2 a_{11} a_{12} \sin 2bt$

$b^2 x^2 + y^2 = 一定$ 　楕円

$= b^2 a_{11}^2 \cos^2 bt + b^2 a_{12}^2 \sin^2 bt + b^2 a_{11} a_{12} \sin 2bt$
$\quad + b^2 a_{12}^2 \cos^2 bt + b^2 a_{11}^2 \sin^2 bt - b^2 a_{11} a_{12} \sin 2bt$

$= b^2 a_{11}^2 + b^2 a_{12}^2$

$= b^2 (a_{11}^2 + a_{12}^2)$

report

次の微分方程式を解け

I $\begin{cases} x' = -4y & x(0) = 0 \\ y' = x & y(0) = -7 \end{cases}$

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III $\begin{cases} x' = 2x - y \\ y' = 2y \end{cases}$

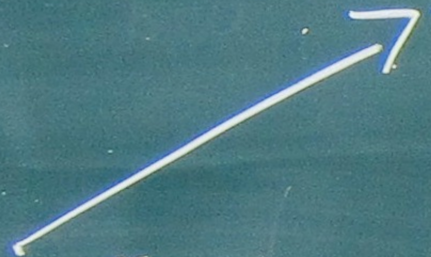
IV $\begin{cases} x' = 2x - y \\ y' = x + 2y \end{cases}$

線型代数

基底

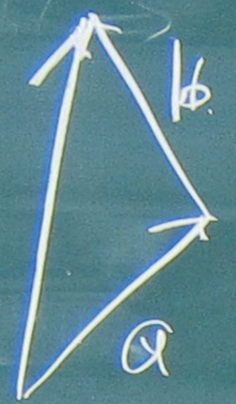
…元のベクトル

平行移動

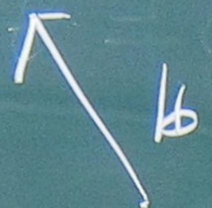


矢印

終点



数

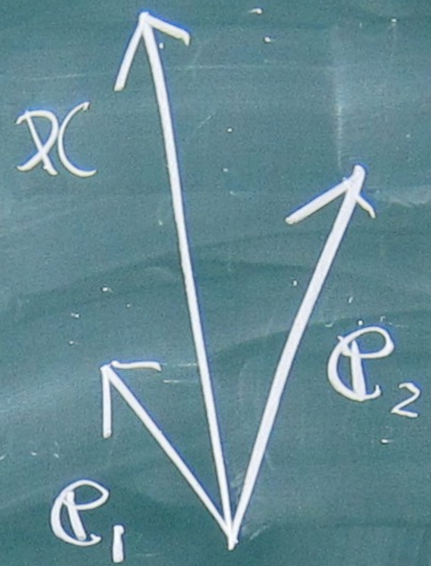


同一直線上にない

2つのベクトル

$\mathbf{e}_1, \mathbf{e}_2$ をとる

基底を決めると



$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$

$\mathbf{x} \leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\mathbf{x} = [\mathbf{e}_1, \mathbf{e}_2]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

平面ベクトルの全体から

平面ベクトルの全体への写像 φ (mapping)

線型写像

$$\begin{cases} \varphi(\mathbf{x}+\mathbf{y}) = \varphi(\mathbf{x}) + \varphi(\mathbf{y}) \\ \varphi(\alpha\mathbf{x}) = \alpha\varphi(\mathbf{x}), \quad \alpha: \text{スカラー} \end{cases}$$

平面のベクトル $\xrightarrow[\text{線型写像}]{\varphi}$ 平面のベクトル

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

基底 $\mathbf{e}_1, \mathbf{e}_2$　　　基底 $\mathbf{f}_1, \mathbf{f}_2$

$\varphi(x\mathbf{e}_1 + y\mathbf{e}_2)$

$= x\varphi(\mathbf{e}_1) + y\varphi(\mathbf{e}_2)$

$= x[\mathbf{f}_1, \mathbf{f}_2]\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + y[\mathbf{f}_1, \mathbf{f}_2]\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$

$\varphi(\mathbf{e}_1) = [\mathbf{f}_1, \mathbf{f}_2]\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$

$= a_{11}\mathbf{f}_1 + a_{21}\mathbf{f}_2$

$\varphi(\mathbf{e}_2) = [\mathbf{f}_1, \mathbf{f}_2]\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$

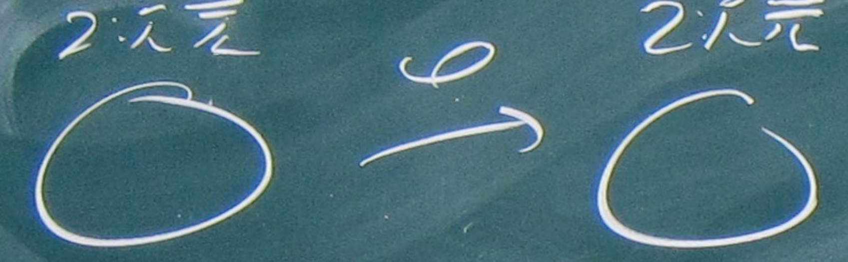
$= a_{12}\mathbf{f}_1 + a_{22}\mathbf{f}_2$

$= [\mathbf{f}_1, \mathbf{f}_2]\left\{x\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + y\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}\right\}$

$= [\mathbf{f}_1, \mathbf{f}_2]\,\underline{A}\begin{pmatrix} x \\ y \end{pmatrix}$

線型写像 φ はそれぞれに基底 $\{\mathbf{e}_1, \mathbf{e}_2\}$ $\{\mathbf{f}_1, \mathbf{f}_2\}$ をとると行列 A で表わせる

(2×2の)行列は線型写像の表現



$\mathbf{e}_1, \mathbf{e}_2$ $\quad$ $\mathbf{f}_1, \mathbf{f}_2$

$\mathbf{e}_1', \mathbf{e}_2'$ $\quad$ $\mathbf{f}_1', \mathbf{f}_2'$

$$\mathbf{e}_1' = [\mathbf{e}_1, \mathbf{e}_2]\begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\mathbf{e}_2' = [\mathbf{e}_1, \mathbf{e}_2]\begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix}$$

$$E = B'B$$

$$[\mathbf{e}_1', \mathbf{e}_2'] = [\mathbf{e}_1, \mathbf{e}_2]B = [\mathbf{e}_1', \mathbf{e}_2']B'B$$

$$[\mathbf{e}_1, \mathbf{e}_2] = [\mathbf{e}_1', \mathbf{e}_2']B'$$

$$BB' = E$$