微積分(7コマ目)
べき級数(無限次の多項式)

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

xは実数
xを複素数 $z = x + iy$ におきかえる

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

足し算と掛け算が定ぎできなければならない

複素数の世界は足し算、掛け算~~が~~ができるので拡張できる

数C　行列
2×2の行列

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

足し算も掛け算もできる
↓
zをAにかえてみる

行列

$$e^{A} = E + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots$$

単位行列

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

■ e^Aの性質

- $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
- $e^O = E$
- $e^{A+B} \neq e^A e^B$ (指数法則)

行列の掛け算では $AB \neq BA$

- $\boxed{AB = BA \text{ ならば } e^{A+B} = e^A e^B}$

$B = P^{-1}AP$

ただし Pは正則行列

逆行列をもつ行列

$$e^B = e^{P^{-1}AP}$$

$$= E + P^{-1}AP + \frac{1}{2!}(P^{-1}AP)^2 + \frac{1}{3!}(P^{-1}AP)^3 + \cdots$$

$$E = P^{-1}EP$$

$$(P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}A^2P$$

$$(P^{-1}AP)^n = P^{-1}A^nP \cancel{= P^{-1}A}$$

$$= P e^A P$$

2×2の行列の分類

$A \sim B \overset{def}{\iff}$ 正則行列Pがあって $B = P^{-1}AP$

お友達

↓

相似

1 $A \sim A$ ($P = E$をとればよい)

2 $A \sim B \Rightarrow B \sim A$

$$B = \underset{\wedge \atop P}{}P^{-1}AP\underset{\wedge \atop P^{-1}}{} \Rightarrow A = PBP^{-1}$$

3 $A \sim B$ & $B \sim C \Rightarrow A \sim C$

$B = P^{-1}AP \quad C = Q^{-1}BQ$

$C = Q^{-1}P^{-1}APQ \quad \left((PQ)^{-1} = Q^{-1}P^{-1}\right)$

行列の分類論
(jordanの標準形)

$$A \quad \begin{pmatrix} \cancel{\alpha}^{a} & 0 \\ 0 & \cancel{\beta}b \end{pmatrix} \quad \begin{pmatrix} \cancel{\alpha}^{a} & 1 \\ 0 & \cancel{\alpha}_{b} \end{pmatrix} \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

~~α、β~~ は実数
a、b

- $$e^{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \frac{1}{2}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^2 + \frac{1}{3!}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^3 + \cdots$$

$$\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 + a + \frac{a^2}{2} + \frac{1}{3!}a^3 + \cdots & 0 \\ 0 & 1 + b + \frac{1}{2}b^2 + \frac{1}{3!}b^3 + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$$

$$A = P^{-1}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} P$$

$$e^A = P^{-1} e^{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} P$$

- $$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} = \underbrace{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}}_{aE} + \begin{pmatrix} \cancel{a}0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$e^{\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}} = e^{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}} e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

$$e^{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}} = e^a E = \begin{pmatrix} e^a & 0 \\ 0 & e^a \end{pmatrix}$$

$$e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2}_{\downarrow\ 0} + \cdots$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$$\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix}\begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} \overset{a}{a_1a_2 - b_1b_2} & \overset{-b}{-a_1b_2 - a_2b_1} \\ \underset{b}{a_2b_1 + a_1b_2} & \underset{a}{-b_1b_2 + a_1a_2} \end{pmatrix}$$

$$\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & -(b_1 + b_2) \\ b_1 + b_2 & a_1 + a_2 \end{pmatrix}$$

足し算

$$A = \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \overset{\text{対応}}{\longleftrightarrow} a_1 + b_1 i$$

$$B = \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} \longleftrightarrow a_2 + i b_2$$

$$A + B \leftrightarrow (a_1 + a_2) + i(b_1 + b_2)$$

掛け算

$$AB \leftrightarrow (a_1 + b_1 i)(a_2 + b_2 i)$$
$$= (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$$

$$e^z = e^{a+bi} = e^a e^{bi}$$
$$= e^a(\cos b + i \sin b)$$
$$= e^a \cos b + i e^a \sin b \leftrightarrow e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}$$

固有値

A （2×2の行列） tが変数

$\varphi_A(t) = |tE - A|$ 　tの2次式

Aの固有多項式

A

$$\varphi_{P^{-1}AP}(t) = |tE - P^{-1}AP|$$

$$= |P^{-1}(tE-A)P|$$

$$= |P^{-1}|\underbrace{|tE-A|}_{\varphi_A(t)}|P|$$

$$|P^{-1}||P| = |P^{-1}P| = |E| = 1$$

$$\varphi_{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}}(t)$$

$$= \left|tE - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right|$$

$$= \begin{vmatrix} t-a & 0 \\ 0 & t-b \end{vmatrix} = (t-a)(t-b)$$

$$\left(\underbrace{P^{-1}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}P}_{aE} = aE \right)$$

$$\varphi_{\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}}(t)$$

$$= \left| tE - \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \right|$$

$$= \begin{vmatrix} t-a & -1 \\ 0 & t-a \end{vmatrix} = (t-a)^2$$ 重根

$$\varphi_{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}(t)$$

$$= \left| tE - \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right|$$

$$= \begin{vmatrix} t-a & b \\ -b & t-a \end{vmatrix} = (t-a)^2 + b^2$$

$$t = a \pm bi$$

巾級数（無限次の多項式）

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$e^0 = 1$

xは実数

足し算　掛け算

$e^{a+b} = e^a e^b$

xを複素数 $z = x + iy$ に
おきかえる

$ab = ba$

$$e^{z} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

拡張　複素数の世界

数 C

行列

2×2の行列

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

足し算

掛け算

$AB \neq BA$

行列

$$e^{A} = E + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots$$

単位行列

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

e^Aの性質

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$B = A^2$

$e^O = E$

$\boxed{AB = BA \text{ ならば } e^{A+B} = e^A e^B \text{（指数法則）}}$

$$\boxed{B = P^{-1} A P}$$

ただしPは正則行列

(逆行列をもつ行列)

$$(P^{-1}AP)^n = P^{-1}A^nP$$

$$= P^{-1} e^{A} P$$

definition

$$e^{P^{-1}AP}$$

$$\ldots^{-1}AP + \frac{1}{2!}(P^{-1}AP)^2 + \frac{1}{3!}(P^{-1}AP)^3 + \cdots$$

$$(P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}A^2P$$

2x2の行列の分類

$$A \sim B \overset{def}{\Longleftrightarrow}$$ 正則行列Pがあって $B = P^{-1}A\ldots$

お友達

相似人

1) $A \sim A$ $\quad P=E$ $\quad (PQ)^{-1} = Q^{-1}P^{-1}$

2) $A \sim B \Rightarrow B \sim A$ $\quad A$

$B = P^{-1}AP$ $\quad A = PBP^{-1}$

P $\quad P^{-1}$

3) $A \sim B$ & $B \sim C \Rightarrow A \sim C$

$B = \underline{P^{-1}AP}$ $\quad C = Q^{-1}BQ = \underline{Q^{-1}P^{-1}}\,AP\underline{Q}$

行列の分類論
(Jordanの標準形)

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

a, bは実数

$$e^{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}a&0\\0&b\end{pmatrix} + \frac{1}{2}\begin{pmatrix}a&0\\0&b\end{pmatrix}^2 + \frac{1}{3!}\begin{pmatrix}a&0\\0&b\end{pmatrix}^3 + \cdots$$

$$\left\{ \begin{pmatrix}a&0\\0&b\end{pmatrix}^n = \begin{pmatrix}a^n&0\\0&b^n\end{pmatrix} \right\}$$

$$= \begin{pmatrix} 1+a+\frac{a^2}{2}+\frac{1}{3!}a^3+\cdots & 0 \\ 0 & 1+b+\frac{1}{2}b^2+\frac{1}{3!}b^3+\cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^a & 0 \\ 0 & \end{pmatrix}$$

$$A = P^{-1}\begin{pmatrix}a&0\\0&b\end{pmatrix}P$$

$$e^A = P^{-1}\, e^{\begin{pmatrix}a&0\\0&b\end{pmatrix}}\, P$$

$$\begin{pmatrix}a&0\\0&a\end{pmatrix}$$

$$\begin{pmatrix}a&1\\0&a\end{pmatrix} = \underline{\begin{pmatrix}a&0\\0&a\end{pmatrix}} + \begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$e^{\begin{pmatrix}a&1\\0&a\end{pmatrix}}$$

$$= \qquad \begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$e^{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}} = e^a E = \begin{pmatrix} e^a & 0 \\ 0 & e^a \end{pmatrix}$$

$$e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = \underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} + \frac{1}{2}\underline{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2} + \cdots$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix}\begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{array}{c} \begin{array}{cc} \quad a \quad & \quad -b \quad \end{array} \\ \begin{pmatrix} a_1a_2 - b_1b_2 & -a_1b_2 - a_2b_1 \\ a_2b_1 + a_1b_2 & -b_1b_2 + a_1a_2 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{array}{c} \begin{array}{cc} \quad b \quad & \quad a \quad \end{array} \\ \begin{pmatrix} a_1 + a_2 & -(b_1 + b_2) \\ b_1 + b_2 & a_1 + a_2 \end{pmatrix} \end{array}$$

运算 对应

$$A=\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \longleftrightarrow a_1+b_1 i$$

$$B=\begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} \longleftrightarrow a_2+ i b_2$$

$$A+B \longleftrightarrow (a_1+a_2)+i(b_1+b_2)$$

$$AB \longleftrightarrow (a_1+b_1 i)(a_2+b_2 i)$$

$$=(a_1 a_2-b_1 b_2)+i(a_1 b_2+a_2 b_1)$$

$$e^z=e^{a+bi}=e^a \underline{e^{bi}}$$

$$=e^a(\cos b+i\sin b) \longleftrightarrow e^a\begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}$$

$$=e^a\cos b+i e^a\sin b$$

固有値

A

A 2×2

tが変数

$\varphi_A(t) = |tE - A|$ tの2次式

↑

Aの固有多項式

固有値

$|AB| = |A||B|$

$\varphi_{P^{-1}AP}(t) = |tE - P^{-1}AP|$

$= |P^{-1}(tE - A)P|$

$= |P^{-1}|\,\underbrace{|tE - A|}_{\varphi_A(t)}\,|P|$

$|P^{-1}||P| = |P^{-1}P| = |E| = 1$

$\varphi_{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}}(t)$

$= \left| tE - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right|$

$= \begin{vmatrix} t-a & 0 \\ 0 & t-b \end{vmatrix} = (t-a)(t-b)$

$$\varphi_{\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}}(t)$$

$$= \left| tE - \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \right|$$

$$= \begin{vmatrix} t-a & -1 \\ 0 & t-a \end{vmatrix}$$

$(t-$

$ax^2+bx+c=0$

根

root

solution

9月1日～11日

$$\underline{P^{-1}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} P}$$

aE

$\Big(a \quad 0$

$$\varphi_{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}(t)$$

$$= \left| tE - \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right|$$

$$= \begin{vmatrix} t-a & b \\ -b & t-a \end{vmatrix} = (t-a)^2 + b^2 = 0$$

$= t^2 - 2at + a^2 +$

$(t-a)^2 = -b^2$

$t-a = \pm b i$