微積分（5コマ目）

微分方程式

$$D = \{ d \in \mathbb{R} \mid d^2 = 0 \}$$

$\alpha \in \mathbb{R}$ & $d \in D \Rightarrow \alpha d \in D$

$d_1 . d_2 \in D \not\Rightarrow d_1 + d_2 \in D$

$\left(\because (d_1 + d_2)^2 = \underset{0}{\underset{\shortparallel}{d_1^2}} + d_2^2 + 2 d_1 d_2 \right)$

$$D_1 \subseteq D_2 = \{ d \in \mathbb{R} \mid d^3 = 0 \}$$

$d_1 . d_2 \in D \Rightarrow d_1 + d_2 \in D_2$

$\left(\because (d_1 + d_2)^3 = d_1^3 + 3 d_1^2 d_2 + 3 d_1 d_2^2 + d_2^3 \right.$
$\left. = 0 \right)$

$$D_n = \{ d \in \mathbb{R} \mid d^{n+1} = 0 \}$$

命題

$d_1 \in D_m ,\ d_2 \in D_n \Rightarrow d_1 + d_2 \in D_{n+m}$

先程の場合 $n = m = 1$

証明

$(d_1 + d_2)^{n+m+1}$

2項定理

$\stackrel{\text{}}{=} \sum \bigcirc d_1^i d_2^j$　　$i + j = n + m + 1$　i, j 自然数

D_m, D_nの条件より　$i \le m$　$j \le n$　　$i + j \le m + n$　となる項しか残らない。

$i + j = n + m + 1$ となる項はない。

$= 0$

$\boxed{\text{系}\quad d_1 , \cdots , d_n \in D = D_1 \Rightarrow d_1 + \cdots + d_n \in D_n}$

この証明（帰納法）がレポートI

$d_1 , d_2 \in D \Rightarrow d_1 + d_2 \in D_2$

$d_1 + d_2 \in D_2 \quad d_3 \in D = D_1 \Rightarrow (d_1 + d_2) + d_3 \in D_3$

写像 $(d_1, \cdots, d_n) \in D^n = \underbrace{D \times \cdots \times D}_{n\text{個}} \to D_n$ 全射

$d_1 + \cdots + d_n$

$x = x(t)$

$\boxed{x' = ax}$

$x(t+d) = x(t) + \underline{x'(t)}d \quad (^\forall d \in D)$

時刻 0 位置 c

d_1 $\quad c(1+ad_1)$

$$\left(\begin{aligned} x(d_1) &= x(0) + x'(0)d_1 \\ &= c + a\underset{c}{x(0)}d_1 \\ &= c(1+ad_1) \end{aligned}\right)$$

$d_1 + d_2$

二つの法則を満たす

$$\begin{aligned} x(d_1+d_2) &= x(d_1) + x'(d_1)d_2 \\ &= x(d_1) + ax(d_1)d_2 \\ &= x(d_1)\{1+ad_2\} \\ &= c(1+ad_1)(1+ad_2) \\ &= c\{1 + a(d_1+d_2) + a^2 d_1 d_2\} \\ &= c\{1 + a(d_1+d_2) + \frac{a^2}{2}(d_1+d_2)^2\} \end{aligned}$$

$d_1 d_2 = \frac{(d_1+d_2)^2}{2}$

$d_1 + d_2 = d \in D_2$ とおく

$$= c\{1 + ad + \frac{a^2}{2}d^2\}$$

時刻 $d_i \in D \ (i = 1, 2, 3)$

$d_1 + d_2 + d_3$

$$\begin{aligned}
x(\underset{\sim\sim\sim\sim}{d_1+d_2}+d_3) &= x(d_1+d_2) + x'(d_1+d_2)d_3 \\
&= x(d_1+d_2) + ax(d_1+d_2)d_3 \\
&= x(d_1+d_2)\{1 + ad_3\} \\
&= c\{1 + a(d_1+d_2) + a^2 d_1 d_2\}\{1 + ad_3\} \\
&= c\{1 + a(d_1+d_2+d_3) + a^2(d_1d_2 + d_1d_3 + d_2d_3) + a^3 d_1d_2d_3
\end{aligned}$$

$$\left(\begin{aligned} d_1d_2 + d_1d_3 + d_2d_3 &= \frac{(d_1+d_2+d_3)^2}{2} \\ d_1d_2d_3 &= \frac{(d_1+d_2+d_3)^3}{6} \end{aligned} \right)$$

$$= c\{1 + a(d_1+d_2+d_3) + \frac{a^2}{2}(d_1+d_2+d_3)^2 + \frac{a^3}{6}(d_1+d_2+d_3)$$

$d_1 + d_2 + d_3 = d \in D_3$ とおく

$$= c\{1 + ad + \frac{a^2}{2}d^2 + \frac{a^3}{6}d^3\}$$

参 $e^{at} = 1 + at + \frac{1}{2}(at)^2 + \frac{1}{6}(at)^3 + \cdots$

Report II $d_1, \cdots, d_n \in D$

$d = d_1 + \cdots + d_n \in D_n$ 　n次の場合について位置を求める

$x = \sin t$

$x' = \cos t \qquad y = x'$

$x'' = -\sin t$

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$ 連立微分方程式

時刻	x	y
0	C_1	C_2
$d_1 \in D$	$C_1 + C_2 d_1$	$C_2 - C_1 d_1$

$$\left(\begin{aligned} x(d_1) &= x(0) + x'(0) d_1 \\ &= x(0) + y(0) d_1 = C_1 + C_2 d_1 \end{aligned} \right)$$

$$\left(\begin{aligned} y(d_1) &= y(0) + \underbrace{y'(0)}_{-x(0)} d_1 \\ &= C_2 - C_1 d_1 \end{aligned} \right)$$

$d_1 + d_2$

$$\begin{aligned} & x(d_1 + d_2) \\ &= x(d_1) + \underbrace{x'(d_1)}_{y(d_1)} d_2 \\ &= C_1 + C_2 d_1 + \{C_2 - C_1 d_1\} d_2 \\ &= C_1 + C_2 (d_1 + d_2) - C_1 d_1 d_2 \\ &= C_1 \{1 - d_1 d_2\} + C_2 (d_1 + d_2) \\ &= C_1 \left\{ 1 - \frac{(d_1 + d_2)^2}{2} \right\} + C_2 (d_1 + d_2) \end{aligned}$$

$$\begin{aligned} & y(d_1 + d_2) \\ &= y(d_1) + \underbrace{y'(d_1)}_{-x(d_1)} d_2 \\ &= \{C_2 - C_1 d_1\} - \{C_1 + C_2 d_1\} d_2 \\ &= -C_1 (d_1 + d_2) + C_2 \{1 - d_1 d_2\} \\ &= -C_1 (d_1 + d_2) + C_2 \left\{ 1 - \frac{(d_1 + d_2)^2}{2} \right\} \end{aligned}$$

時刻 $d_i \in D \ (i=1,2,3)$

$d_1 + d_2 + d_3$

$$d_1 d_2 + d_1 d_3 + d_2 d_3 = \frac{(d_1+d_2+d_3)^2}{2}$$

$$d_1 d_2 d_3 = \frac{(d_1+d_2+d_3)^3}{3!}$$

$x(d_1+d_2+d_3)$

$$= x(d_1+d_2) + \underbrace{x'(d_1+d_2)}_{y(d_1+d_2)} d_3$$

$$= c_1(1-d_1d_2) + c_2(d_1+d_2) + \{-c_1(d_1+d_2) + c_2(1-d_1d_2)\}d_3$$

$$= c_1(1-(d_1d_2+d_1d_3+d_2d_3)) + c_2\{(d_1+d_2+d_3) - d_1d_2d_3\}$$

$x = \sin t \quad x' = \cos t \quad x'' = -\sin t$

$x'' = -x$ 2階の方程式 $\begin{cases} x' = y \\ y' = -x \end{cases}$ 1階

$d_1 + d_2 + d_3 = d \in D_3$ とおく。

$$= c_1\underbrace{\left(1 - \frac{d^2}{2}\right)}_{\cos t} + c_2\underbrace{\left(d - \frac{d^3}{3!}\right)}_{\sin t}$$

Report III

$d_1 + d_2 + d_3 + d_4 \ (+d_5)$ を求める

微分方程式の解法

$$x(t+d) = x(t) + x'(t)\,d$$

$$D_1 = D = \{ d \in \mathbb{R} \mid d^2 = 0 \}$$

$$d \in D \ \& \ \alpha \in \mathbb{R} \longrightarrow \alpha d \in D$$

$$(\alpha d)^2 = \alpha^2 d^2 = 0$$

$$\left.\begin{array}{l} d_1 \in D \\ d_2 \in D \end{array}\right\} \begin{array}{l} \Rightarrow d_1 + d_2 \in D_2 \\ \not\Rightarrow d_1 + d_2 \in D \end{array}$$

$$(d_1+d_2)^2 = \underset{0}{\underset{\parallel}{d_1^2}} + \underset{0}{\underset{\parallel}{d_2^2}} + 2 d_1 d_2$$

$$D_2 = \{ d \in \mathbb{R} \mid d^3 = 0 \}$$

$$D_3 = \{ d \in \mathbb{R} \mid d^4 = 0 \}$$

$$D_n = \{ d \in \mathbb{R} \mid d^{n+1} = 0 \}$$

$$(d_1+d_2)^3 = \underline{d_1^3} + \underline{3 d_1^2 d_2} + \underline{3 d_1 d_2^2} + \underline{d_2^3} = 0$$

微分方程式

$D_1 = D = \{ d \in \mathbb{R} \mid d^2 = 0 \}$

$\alpha \in \mathbb{R}$ & $d \in D \Rightarrow \alpha d \in D$

$d_1, d_2 \in D \not\Rightarrow d_1 + d_2 \in D$

$(d_1 + d_2)^2 = \underset{\substack{\parallel \\ 0}}{d_1^2} + \underset{\substack{\parallel \\ 0}}{d_2^2} + \underline{2 d_1 d_2}$

I

$D_1 \subseteq D_2 = \{ d \in \mathbb{R} \mid d^3 = 0 \}$ $n = m = 1$

$\boxed{d_1, d_2 \in D \Rightarrow d_1 + d_2 \in D_2}$

$(d_1 + d_2)^3 = d_1^3 + 3 d_1^2 d_2 + 3 d_1 d_2^2 + d_2^3$

$= 0$

$D_n = \{ d \in \mathbb{R} \mid d^{n+1} = 0 \}$

$\boxed{\text{系} \quad d_1, \ldots, d_n \in D = D_1 \Rightarrow d_1 + \cdots + d_n \in D_n}$

$d_1, d_2 \in D \Rightarrow d_1 + d_2 \in D_2$

$d_1 + d_2 \in D_2 \quad d_3 \in D = D_1 \Rightarrow (d_1 + d_2) + d_3 \in D_3$

命題

$\boxed{d_1 \in D_m, \ d_2 \in D_n \Rightarrow d_1 + d_2 \in D_{n+m}}$

証明

$(d_1 + d_2)^{n+m+1}$

2項定理

$= \underline{\sum \bigcirc d_1^i d_2^j}$

$n = 3$

i, j 自然数

$\boxed{i + j = n + m + 1}$

$m+1$

$n+1$

$i \leq m$

$j \leq n$

$i + j \leq m + n$

写像:

$$(d_1, \dots, d_n) \in D^n = \underbrace{D \times \cdots \times D}_{n個} \longrightarrow D_n$$

$d_1 \in D$	時刻	位置
合計	0	C
$d_1 + \cdots + d_n$	d_1	$C(1 + a d_1)$
$d_1 d_2 = \frac{(d_1+d_2)^2}{2}$	$d_1 + d_2$	$C\{1 + a(d_1+d_2) + \frac{a^2}{2}(d_1+d_2)^2$

$\longrightarrow x$

$x = x(t)$

$\boxed{x' = ax}$ の法則を満たす

$$x(t+d) = x(t) + \underline{x'(t)}\, d \quad (\forall d \in D)$$

$$x(d_1) = x(0) + \underline{x'(0)}\, d_1$$

$$= C + a x(0) d \ldots$$

$$x(\underline{d_1} + d_2) = x(d_1) + x'(d_1) d_2$$

$$= x(d_1) + a x(d_1) d \ldots$$

時刻 $d_i \in D$ $(i=1,2,3)$

$d_1+d_2+d_3$

位置

$(d_1, d_2) \in D^2 \Rightarrow d_1+d_2 \in D_2$

$C\,e^{at} = 1 + at + \frac{1}{2}(at)^2 + \frac{1}{6}(at)^3 + \cdots$

$d_1d_2 + d_1d_3 + d_2d_3 = \dfrac{(d_1+d_2+d_3)^2}{2}$

$d_1d_2d_3 = \dfrac{(d_1+d_2+d_3)^3}{6}$

$t = d \in D$

$t = d \in D_2$

$t = d \in D_3$

$d_1+d_2+d_3 = d \in D_3$

$x(d_1+d_2+d_3) = x(d_1+d_2) + x'(d_1+d_2)\,d_3$

$= x(d_1+d_2) + a\,x(d_1+d_2)\,d_3$

$= x(d_1+d_2)\{1 + a d_3\}$ 2'

$= C\{1 + a(d_1+d_2) + a^2 d_1 d_2\}\{1 + a d_3\}$

$= C\{1 + a(d_1+d_2+d_3) + a^2(d_1d_2 + d_1d_3 + d_2d_3) + a^3 d_1d_2d_3\}$

特殊関数

$= C\{1 + ad + \frac{a^2}{2}d^2 + \frac{a^3}{6}d^3\}$ 3'

$d_1, \ldots, d_n \in D$

II $d = d_1 + \cdots + d_n \in D_n$

$= C\{1 + a(d_1+d_2+d_3) + \frac{a^2}{2}(d_1+d_2+d_3)^2 + \frac{a^3}{6}(d_1+d_2+d_3)^3\}$

$x(t)$

$x = \sin t$

$x' = \cos t$

$x'' = -\sin t$

2変数

$y = x'$

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$ ← 連立微分方程式

時刻	x	y
0	C_1	C_2
$d_1 \in D$	$C_1 + C_2 d_1$	$C_2 - C_1 d_1$

$$x(d_1) = x(0) + \underline{x'(0)}\, d_1$$

$$= x(0) + y(0)\, d_1 = C_1 + C_2 d_1$$

$$y(d_1) = y(0) + \underline{y'(0)}\, d_1$$

時刻

d_1+d_2

$$d_1 d_2 = \frac{(d_1+d_2)^2}{2}$$

$$x(\underline{d_1}+d_2) = x(d_1) + \underset{y(d_1)}{\underline{x'(d_1)}}\, d_2$$

$$= C_1 + C_2 d_1 + \{C_2 - C_1 d_1\} d_2$$

$$= C_1 + C_2(d_1+d_2) - C_1 d_1 d_2$$

$$= C_1\{1 - d_1 d_2\} + C_2(d_1+d_2)$$

$$= C_1\left\{1 - \frac{(d_1+d_2)^2}{2}\right\} + C_2(d_1+d_2)$$

$$y(d_1+d_2) = y(d_1) + \underset{-x(d_1)}{\underline{y'(d_1)}}\, d_2$$

$$= \{C_2 - C_1 d_1\} - \{C_1 + C_2 d_1\} d_2$$

$$= -C_1(d_1+d_2) + C_2\{1 - d_1 d_2\}$$

$$= -C_1(d_1+d_2) + C_2\left\{1 - \frac{(d_1+d_2)^2}{2}\right\}$$

$d_1+d_2+d_3$ $\quad d_i \in D\ (i=1,2,3)$

$d_1+d_2+d_3 = d \in D_3$

$d_1d_2+d_1d_3+d_2d_3 = \dfrac{(d_1+d_2+d_3)^2}{2}$

$$x(d_1+d_2+d_3) = x(d_1+d_2) + \underbrace{x'(d_1+d_2)}_{y(d_1+d_2)} d_3,$$

$\overline{d_1+d_2+d_3+d_4+d_5}$ III

$d_1d_2d_3 = \dfrac{(d_1+d_2+d_3)^3}{3!}$

$$= C_1(1-d_1d_2) + C_2(d_1+d_2)$$

$$+\{-C_1(d_1+d_2) + C_2(1-d_1d_2)\}d_3$$

$$= C_1\{1-(d_1d_2+d_1d_3+d_2d_3)\} + C_2\{(d_1+d_2+d_3) - d_1d_2d_3\}$$

$$C_1\underbrace{\left(1-\frac{d^2}{2}\right)}_{\cos t} + C_2\underbrace{\left(d-\frac{d^3}{3!}\right)}_{\sin t}$$

$x = \cos t$

$x = \sin t$

$x = \sin t$

$x' = \cos t$

$x'' = -\sin t$

$x' = y$ ← 連立

$\begin{cases} x' = y \\ y' = -x \end{cases}$ 1階

$x'' = -x$

2階の方程式