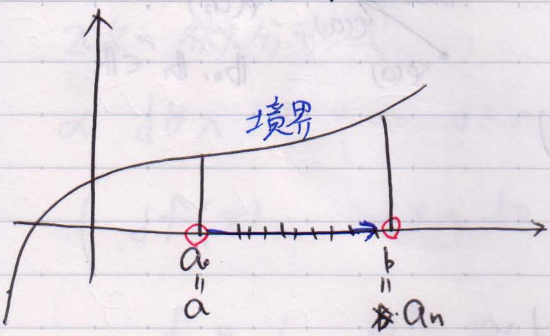
微積分学の基本定理



$$\int_a^b f'(x)\,dx = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f'(x)\,dx = f'(a_i)\,d_i = f(a_{i+1}) - f(a_i)$$

そもそも我々の微分の定義

無限小の区間での微積分学の基本定理

⇓

自動的

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$a_{i+1} - a_i \stackrel{def}{=} d_i$$

$$\int_{a_i}^{a_{i+1}} f'(x)\,dx \left(= f'(a_i)\,d_i\right) = f(a_{i+1}) - \cancel{f(a_i)}$$

$$\int_{a_{i-1}}^{a_i} f'(x)\,dx = \cancel{f(a_i)} - \cancel{f(a_{i-1})}$$

$$\vdots$$

$$+ \big) \qquad\qquad - f(a_0) = f(a)$$

$$\int_a^b f'(x)\,dx \qquad f(b) - f(a)$$

$\mathbb{R}^3$（空間）

0次の微分形式 $\xrightarrow{d}$ 1次の微分形式 $\xrightarrow{d}$ 2次の微分形式 $\xrightarrow{d}$ 3次の微分形式

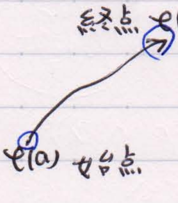
スカラー場

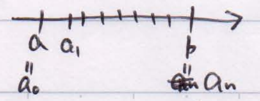
$f : \mathbb{R}^3 \to \mathbb{R} \quad d \in D$

曲線 $\varphi : [a, b] \to \mathbb{R}^3$

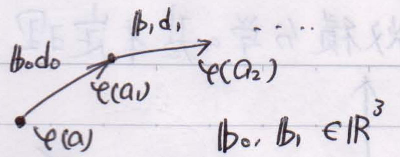
$$\int_\varphi df = \int_{\partial\varphi} f = f(\varphi(b)) - f(\varphi(a))$$

1次の微分形式





$a_{i+1} - a_i = d_i \in D$



$\mathbb{b}_0, \mathbb{b}_1 \in \mathbb{R}^3$

ω は1次の微分形式 on $\mathbb{R}^3$

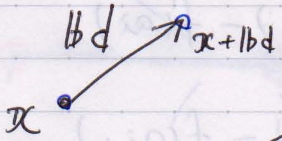
$$\int_\varphi \omega = \sum_{i=0}^{n-1} \underbrace{\omega(\varphi(a_i))}_{\substack{\text{1次の交代形式} \\ \mathbb{R}^3 \to \mathbb{R}\ \text{線型}}} (\mathbb{b}_i d_i)$$

$$= \sum_{i=0}^{n-1} \omega(\varphi(a_i))(\mathbb{b}_i) d_i$$

stokesの定理

無限小のlevel 成り立たせる

df $\quad \mathbb{b} \in \mathbb{R}^3 \quad d \in D$



$f(x+\mathbb{b}d) - f(x)$

$=$

$df(x)(\mathbb{b})d$

$df(x)$

1次の微分形式 ω on $\mathbb{R}^3$

$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

$\alpha_1 dx + \alpha_2 dy + \alpha_3 dz$ $\quad \left(dx : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto x_1 \right)$

$f e_1 + g e_2 + h e_3$

$f, g, h : \mathbb{R}^3 \to \mathbb{R}$

$f(x,y,z)\, e_1 + g(x,y,z)\, e_2 + h(x,y,z)\, e_3$

$f(x,y,z)\, dx + g(x,y,z)\, dy + h(x,y,z)\, dz$

2次の微分形式

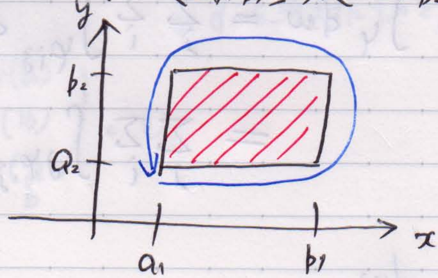
$\alpha_1\, dy\wedge dz + \alpha_2\, dz\wedge dx + \alpha_3\, dx\wedge dy$

$\left(df(x) \qquad dy\wedge dz\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\right) = \begin{vmatrix}x_2 & y_2\\ x_3 & y_3\end{vmatrix} \right)$

$f, g, h : \mathbb{R}^3 \to \mathbb{R}$

曲面

$\varphi : [a_1, b_1]\times[a_2, b_2] \to \mathbb{R}^3$



1次の微分形式 ω on $\mathbb{R}^3$

$$\int_\varphi \underset{2次}{\underline{d\omega}} = \int_{\partial\varphi} \omega$$

φの境界

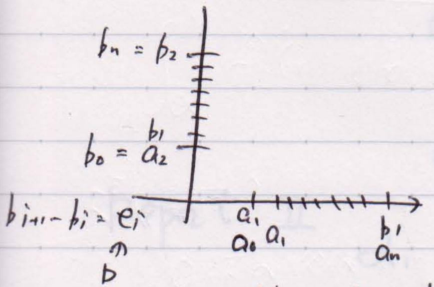
+

曲線

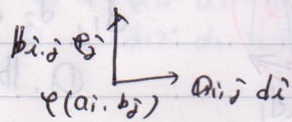
φは曲面

2次の微分形式 τ の φ 上での積分

$$\int_\varphi \tau = \sum_{j=0}^{m-1}\sum_{i=0}^{n-1} \tau(\varphi(a_i, b_j))(\mathbf{a}_{i,j} d_i, \mathbf{b}_{i,j}, e_j)$$



$a_{i+1} - a_i = d_i \in D$



$$= \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} \tau(\varphi(a_i, b_j))(\mathbf{a}_{i,j}, \mathbf{b}_{i,j})\, \cancel{d_i b_j}\; d_i e_j$$

$$\int_\varphi \tau = \int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} \tau(\varphi(s,t))\left(\frac{\partial\varphi}{\partial s}(s,t), \frac{\partial\varphi}{\partial t}(s,t)\right) ds\right) dt$$

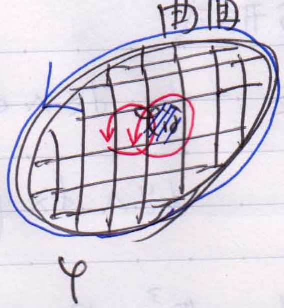
Stokesの定理

$$\boxed{\int_\varphi d\omega = \int_{\partial\varphi} \omega}$$

φ 曲面

ω 1次の微分形式

無限小のlevel



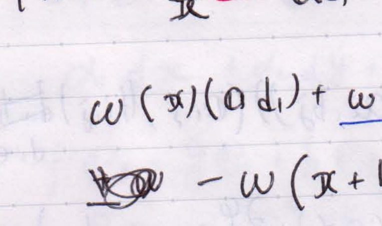
$$\int_\varphi d\omega = \sum_j \sum_i \int_{\varphi_{ij}} d\omega$$

$$= \sum_j \sum_i \int_{\partial\varphi_{ij}} \omega$$

$d\omega$

$\omega = f dx + g dy + h dz$

$f, g, h : \mathbb{R}^3 \to \mathbb{R}$



$d_1, d_2 \in D$

$\mathbb{a}, \mathbb{b} \in \mathbb{R}^3$

$$\omega(\mathbb{x})(\mathbb{a}d_1) + \underline{\omega(\mathbb{x}+\mathbb{a}d_1)(\mathbb{b}d_2)}$$

$$- \omega(\mathbb{x}+\mathbb{b}d_2)(\mathbb{a}d_1) - \underline{\omega(\mathbb{x})(\mathbb{b}d_2)}$$

$$= \{\omega(\mathbb{x}+\mathbb{a}d_1)(\mathbb{b}) - \omega(\mathbb{x})(\mathbb{b})\} d_2$$

$$- \{\omega(\mathbb{x}+\mathbb{b}d_2)(\mathbb{a}) - \omega(\mathbb{x})(\mathbb{a})\} d_1$$

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= \Big[\{ f(x + \mathbf{a} d_1) - f(x) \} b_1 + \{ g(x + \mathbf{a} d_1) - g(x) \} b_2 + \{ h(x + \mathbf{a} d_1) - h(x) \} b_3 \Big] d_2$$

$$= \Big[df(x)(\mathbf{a}) d_1 b_1 + dg(x)(\mathbf{a}) d_1 b_2 + dh(x)(\mathbf{a}) d_1 b_3 \Big] d_2$$

$$- \Big[df(x)(\mathbf{b}) d_2 a_1 + dg(x)(\mathbf{b}) d_2 a_2 + dh(x)(\mathbf{b}) d_2 a_3 \Big] d_1$$

$d_1 d_2$を落としたものを考える。

$$\mathbb{R}^3 \times \mathbb{R}^3 \ni (\mathbf{a}, \mathbf{b}) \longmapsto \begin{pmatrix} df(x)(\mathbf{a}) \\ dg(x)(\mathbf{a}) \\ dh(x)(\mathbf{a}) \end{pmatrix} \cdot \cancel{\mathbf{a}}\ \mathbf{b} - \begin{pmatrix} df(x)(\mathbf{b}) \\ dg(x)(\mathbf{b}) \\ dh(x)(\cancel{\mathbf{a}}\ \mathbf{b}) \end{pmatrix} \cdot \mathbf{a}$$

Report I　二重線型，交代であることを確認

$$\alpha_1 \underbrace{dy \wedge dz}_{0} + \alpha_2 \underbrace{dz \wedge dx}_{0} + \alpha_3 \underbrace{dx \wedge dy}_{1}$$

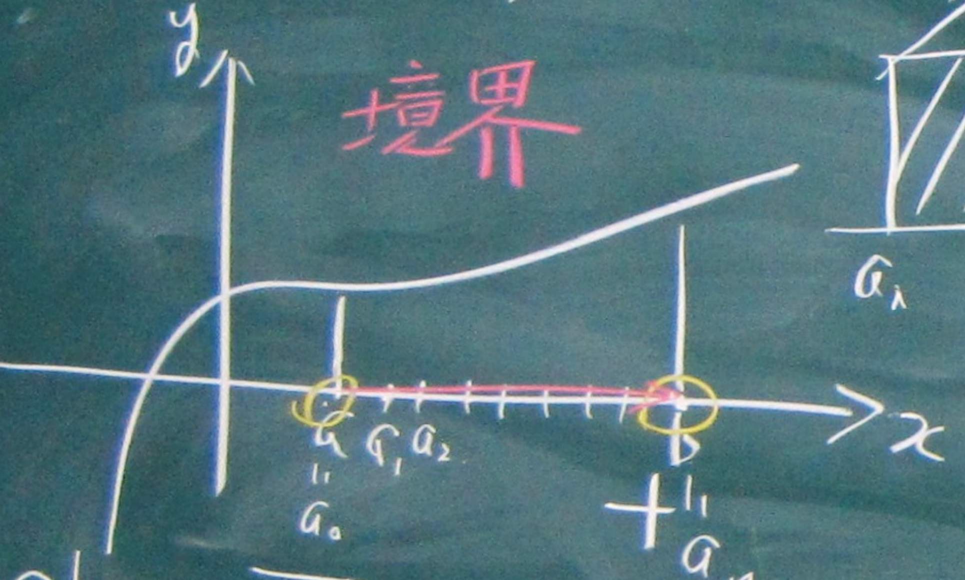
$\mathbf{a} = e_1$　$\mathbf{b} = e_2$ とすれば α_3 の係数がでる

$\mathbf{a} = e_2$, $\mathbf{b} = e_3$ とすれば α_1 の係数がでる

$\mathbf{a} = e_3$, $\mathbf{b} = e_1$ とすれば α_2 の係数がでる

Report II　α_1, α_2, α_3 をまとめる。

微積分学の基本定理



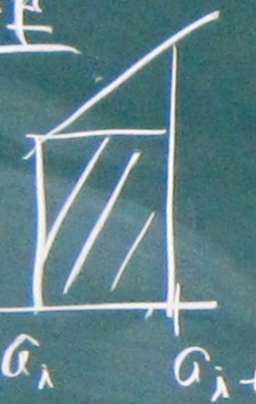
$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$a_{i+1} - a_i \overset{\text{def}}{=} d_i \in D$$

0 に

a b 1次元

種明かし



$$\int_a^b f'(x)\,dx = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f'(x)\,dx$$

$$= f'(a_i)\,d_i = f(a_{i+1}) - f(a_i)$$

自動的

そもそも我々の微分の定義

無限小の区間での微積分学の基本定理

$$f(b) = f(a_n)$$

$$\int_{a_{i+1}}^{a_{i+2}} f'(x)\,dx = f(a_{i+2}) - f(\cancel{a_{i+1}})$$

$$\int_{a_i}^{a_{i+1}} f'(x)\,dx = f'(a_i)\,d_i = f(\cancel{a_{i+1}}) - f(\cancel{a_i})$$

$$\int_{a_{i-1}}^{a_i} f'(x)\,dx = f(\cancel{a_i}) - f(a_{i-1})$$

$$+)\qquad\qquad -f(a_0) = f(a)$$

$$\int_a^b f'(x)\,dx \qquad f(b) - f(a)$$

R^3 (空間)

0次の微分形式 $\xrightarrow{d}$ 1次の微分形式 $\xrightarrow{d}$ 2次の微分形式 $\xrightarrow{d}$ 3次の微分形式

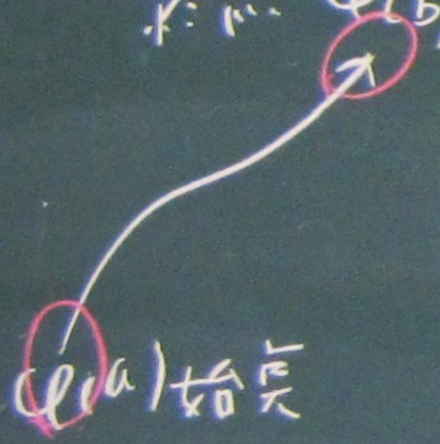
スカラー場

$f: \mathbb{R}^3 \to \mathbb{R}$ $\quad d \in D$

$\mathbb{b}_0, \mathbb{b}_1 \in \mathbb{R}^3$

曲線 $\varphi: [a, b] \longrightarrow \mathbb{R}^3$

終点 $\varphi(b)$



$\varphi(a)$ 始点

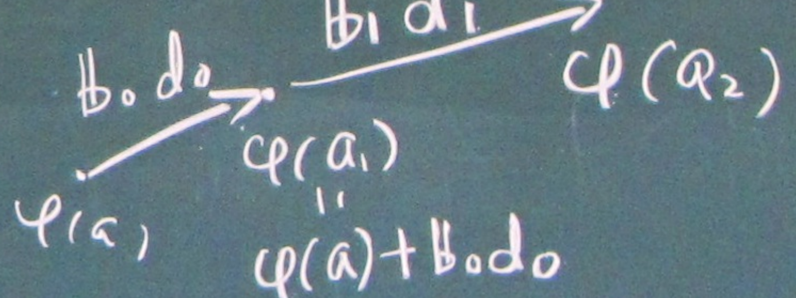
$$\int_\varphi \underset{\uparrow}{df} = \int_{\partial\varphi} f = f(\varphi(b)) - f(\varphi(a))$$

1次の微分形式

ω は1次の微分形式 on $\mathbb{R}^3$

$$\int_\varphi \omega$$

$$\int_a^b \omega(\varphi(t))(\varphi'(t))\,dt$$

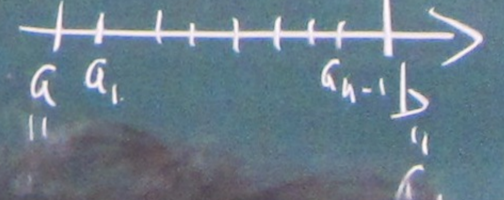


$$\int_\varphi \omega = \sum_{i=0}^{n-1} \underline{\omega(\varphi(a_i))}(\mathbb{b}_i d_i)$$

1次の交代形式

$\mathbb{R}^3 \to \mathbb{R}$ 線形

$$= \sum_{i=0}^{n-1} \omega(\varphi(a_i))(\mathbb{b}_i)\, d_i$$



Stokesの定理

無限小のlevelで成立させる

df　　　$\mathbb{b} \in \mathbb{R}^3$　$d \in D$

$\mathbb{R}^n$　$\mathbb{b}d \mapsto x + \mathbb{b}d$　　$dx \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto x_1$

x

$df(x)(\mathbb{b})d = f(x + \mathbb{b}d) - f(x)$

$df(x) \quad dy \wedge dz \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right)$

$= \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$

1次の微分形式 ω on $\mathbb{R}^3$

$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

$\alpha_1 dx + \alpha_2 dy + \alpha_3 dz$

$f e_1 + g e_2 + h e_3$

$f, g, h : \mathbb{R}^3 \to \mathbb{R}$

$f(x, y, z) e_1 + g(x, y, z) e_2 + h(x, y, z) e_3$

$f(x, y, z) dx + g(x, y, z) dy + h(x, y, z) dz$

2次の微分形式

$f \, dy \wedge dz + g \, dz \wedge dx + h \, dx \wedge dy$

$f, g, h : \mathbb{R}^3 \to \mathbb{R}$

1次の微分形式 ω on $\mathbb{R}^3$ 曲面

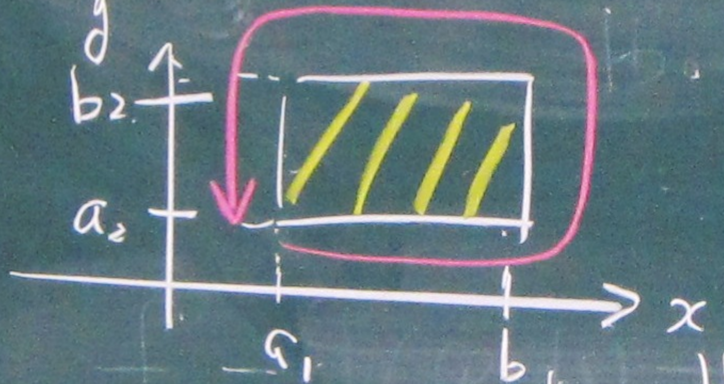
$$\int_{\varphi} \frac{d\omega}{2次} = \int_{\partial\varphi} \omega$$

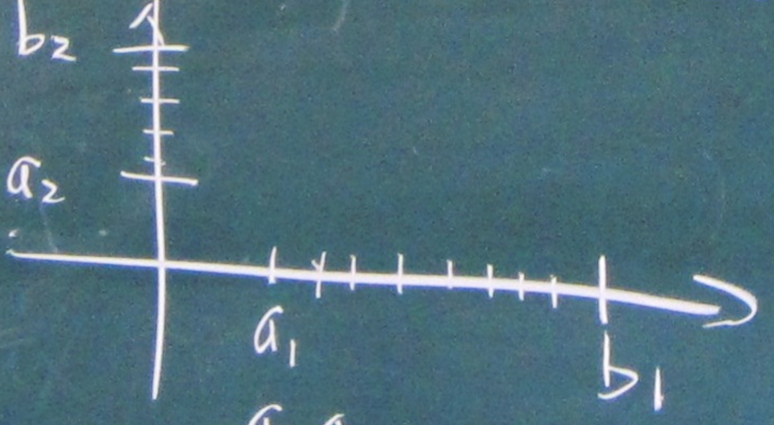
φの境界

φは曲面

曲線

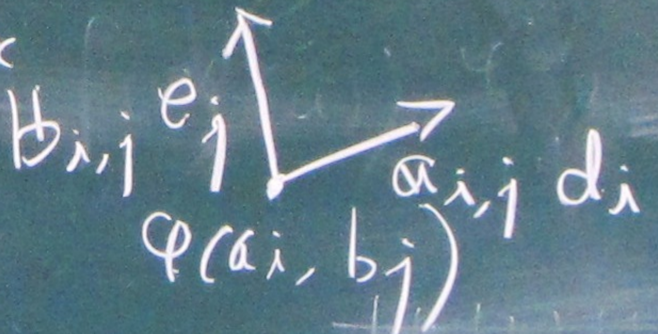
$\varphi: [a_1, b_1] \times [a_2, b_2] \longrightarrow \mathbb{R}^3$





$b_{i+1} - b_i = e_i \in D$

$a_{i+1} - a_i = d_i \in D$



2次の微分形式 τのφ上での積分

$$\int_{\varphi} \tau = \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} \tau(\varphi(a_i, b_j))(a_{i,j} d_i, b_{i,j} e_j)$$

$$= \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} \tau(\varphi(a_i, b_j))(a_{i,j}, b_{i,j}) d_i e_j$$

$$\int_{\varphi} \tau = \ldots (s, t))\left(\frac{\partial\varphi}{\partial s}(s,t), \ldots\right) ds) dt$$

Stokesの定理

$$\boxed{\int_{\varphi} d\omega = \int_{\partial\varphi} \omega}$$

φ 曲面
ω 1次の微分形式

無限小のlevel

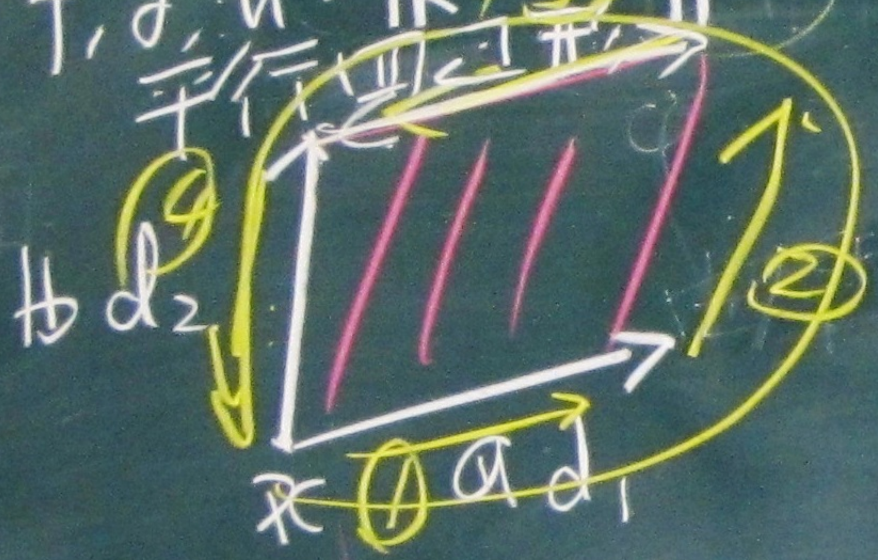
$$\int_{\varphi} d\omega = \sum_{j} \sum_{i} \int_{\varphi_{ij}} d\omega$$

$$= \sum_{j} \sum_{i} \int_{\partial\varphi_{ij}} \omega$$

$d\omega$

$\omega = f\,dx + g\,dy + h\,dz$

$f, g, h : \mathbb{R}^3 \to \mathbb{R}$



$d_1, d_2 \in D$

$\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$

$$\omega(x)(\mathbf{a}\,d_1) + \omega(x + \mathbf{a}\,d_1)(\mathbf{b}\,d_2)$$

$$- \,\omega(x + \mathbf{b}\,d_2)(\mathbf{a}\,d_1) - \omega(x)(\mathbf{b}\,d_2)$$

$$= \{\omega(x + \mathbf{a}\,d_1)(\mathbf{b}) - \omega(x)(\mathbf{b})\}$$

$$- \{\omega(x + \mathbf{b}\,d_2)(\mathbf{a}) - \omega$$

$$= \left[\{ f(x + a d_1) - f(x) \} b_1 + \{ g(x + a d_1) - g(x) \} b_2 + \{ h(x + a d_1) - h(x) \} b_3 \right] d_2$$

$$= \left[df(x)(a) d_1 b_1 + dg(x)(a) d_1 b_2 + dh(x)(a) d_1 b_3 \right] d_2 \wedge$$

$$- \left[df(x)(b) d_2 a_1 + dg(x)(b) d_2 a_2 + dh(x)(b) d_2 a_3 \right] d_1$$

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$d_1 d_2$を除いたものを考える

$$\begin{pmatrix} df(x)(a) \\ dg(x)(a) \\ dh(x)(a) \end{pmatrix} \cdot b - \begin{pmatrix} df(x)(b) \\ dg(x)(b) \\ dh(x)(b) \end{pmatrix} \cdot a$$

I 二重 交代

II $\alpha_1 \, dy \wedge dz + \alpha_2 \, dz \wedge dx + \alpha_3 \, dx \wedge dy$

$a = e_1$	e_2	e_3
$b = e_2$	e_3	e_1