レポートのしめきり 10/17(月)

高階の微分

$f : \mathbb{R} \to \mathbb{R}$

$x \in \mathbb{R}$

$f'(x_0)$ 微分係数

x_0を動かす

~~$f(x$~~ $f' : \mathbb{R} \to \mathbb{R}$ 1階の微分

$x_0 \in \mathbb{R}$

$f''(x_0)$ 2階の微分

$f'' : \mathbb{R} \to \mathbb{R}$

続けられる

ここまでは高校の話

$f : \mathbb{R}^n \to \mathbb{R}^m$

$x_0 \in \mathbb{R}^n$

微分できる

$f'(x_0)$: $\mathbb{R}^n$から$\mathbb{R}^m$への線型写像 の全体を、

セミコロン

$\in \; L(\mathbb{R}^n ; \mathbb{R}^m) = \mathbb{R}^{mn}$

$m \times n$ の行列で表すことができる。

$m\left(\underbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}_{n}\right.$ $m \times n$ 個の実数

ここで、

x_0を動かす

高校の場合

$n=m=1$

$L(\mathbb{R};\mathbb{R})$

1×1の行列 [ ]

$\parallel$

$\mathbb{R}$

$$f' : \mathbb{R}^n \to \underline{L(\mathbb{R}^n ; \mathbb{R}^m)}$$

$\mathbb{R}^{mn}$と同一視

f'を

$x_0 \in \mathbb{R}$で微分

$f'(x_0)$：$\mathbb{R}^n$から$L(\mathbb{R}^n;\mathbb{R}^m)$への線型写像

φ：$\mathbb{R}^n$から$L(\mathbb{R}^n;\mathbb{R}^m)$への線型写像

$a \in \mathbb{R}$

$\varphi(a) \in L(\mathbb{R}^n, \mathbb{R}^m)$

$b \in \mathbb{R}^n$

$\varphi(a)(b) \in \mathbb{R}^m$

$\tilde{\varphi}(a, b)$:

$$\tilde{\varphi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$$

$$\tilde{\varphi}(\alpha a, b) = (\alpha \varphi(a))(b) = \alpha \varphi(a)(b)$$
$$= \alpha \tilde{\varphi}(a, b)$$

$$\tilde{\varphi}(a, \alpha b) = \varphi(a)(\alpha b)$$
$$= \alpha \varphi(a)(b)$$
$$= \alpha \tilde{\varphi}(a, b)$$

$$\tilde{\varphi}(\mathbf{a}_1 + \mathbf{a}_2, \mathbf{b}) = \varphi(\mathbf{a}_1 + \mathbf{a}_2)(\mathbf{b})$$

$$= \varphi(\mathbf{a}_1)(\mathbf{b}) + \varphi(\mathbf{a}_2)(\mathbf{b})$$

$$= \tilde{\varphi}(\mathbf{a}_1, \mathbf{b}) + \tilde{\varphi}(\mathbf{a}_2, \mathbf{b})$$

$$\tilde{\varphi}(\mathbf{a}, \mathbf{b}_1 + \mathbf{b}_2) = \varphi(\mathbf{a})(\mathbf{b}_1 + \mathbf{b}_2)$$

$$\uparrow$$

$$L(\mathbb{R}^n ; \mathbb{R}^m)$$

$$= \varphi(\mathbf{a})(\mathbf{b}_1) + \varphi(\mathbf{a})(\mathbf{b}_2)$$

$$= \tilde{\varphi}(\mathbf{a}, \mathbf{b}_1) + \tilde{\varphi}(\mathbf{a}, \mathbf{b}_2)$$

$\tilde{\varphi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$

二重線型性があるという。

$n = 2 \quad m = 1$

内積 $\quad \mathbf{a}, \mathbf{b} \in \mathbb{R}$

二重線型性を満たす

行列式 $\quad 2 \times 2$

$|\mathbf{a} \ \mathbf{b}|$

$f : \mathbb{R}^2 \to \mathbb{R} \quad f''(x) \quad \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ 　二重線型写像

$$\begin{cases} \psi(\mathbb{a}_1 + \mathbb{a}_2, \mathbb{b}) = \psi(\mathbb{a}_1, \mathbb{b}) + \psi(\mathbb{a}_2, \mathbb{b}) \text{。} \\ \psi(\mathbb{a}, \mathbb{b}_1 + \mathbb{b}_2) = \psi(\mathbb{a}, \mathbb{b}_1) + \psi(\mathbb{a}, \mathbb{b}_2) \text{。} \\ \psi(\alpha \mathbb{a}, \mathbb{b}) = \alpha \psi(\mathbb{a}, \mathbb{b}) \text{。} \\ \psi(\mathbb{a}, \beta \mathbb{b}) = \beta \psi(\mathbb{a}, \mathbb{b}) \text{。} \end{cases}$$

$$\mathbb{a} \in \mathbb{R} \mapsto \psi(\mathbb{a}, \_) : \mathbb{R}^n \to \mathbb{R}^m$$

$$\cap$$

$$L(\mathbb{R}^n ; \mathbb{R}^m)$$

$\mathbb{a}_1 + \mathbb{a}_2$

$\mathbb{R}^n$から $L(\mathbb{R}^n ; \mathbb{R}^m)$ への線型写像の全体

$$L(\mathbb{R}^n , \mathbb{R}^n ; \mathbb{R}^m)$$

コンマ　　セミコロン

$$= L(\mathbb{R}^n ; L(\mathbb{R}^n ; \mathbb{R}^m))$$

$\varphi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ ＝重線型写像

$$\varphi\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = \varphi(a_1 e_1 + a_2 e_2, b_1 e_1 + b_2 e_2)$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 e_1 + a_2 e_2 \qquad \left(e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b_1 e_1 + b_2 e_2$$

$$= a_1 b_1 \underline{\varphi(e_1, e_1)} + a_1 b_2 \underline{\varphi(e_1, e_2)} + a_2 b_1 \underline{\varphi(e_2, e_1)} + a_2 b_2 \underline{\varphi(e_2, e_2)}$$

— の 4つの数が与えられれば決まる

$f : \mathbb{R}^2 \to \mathbb{R}$: 線型

$$f\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right) = f(a_1 e_1 + a_2 e_2)$$

$$= a_1 f(e_1) + a_2 f(e_2)$$

$$\overset{1\times 2}{[\, f(e_1), f(e_2) \,]} \overset{2\times 1}{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}$$

$$\overset{1\times 2}{(a_1, a_2)} \begin{bmatrix} \varphi(e_1, e_1) & \varphi(e_1, e_2) \\ \varphi(e_2, e_1) & \varphi(e_2, e_2) \end{bmatrix} \overset{2\times 1}{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}$$

内積 $\varphi(\mathbf{a}, \mathbf{b}) = \varphi(\mathbf{b}, \mathbf{a})$ 対称性

$$\varphi(\mathbf{e}_1, \mathbf{e}_2) = \varphi(\mathbf{e}_2, \mathbf{e}_1)$$

$$\begin{bmatrix} \cancel{\varphi(\mathbf{e}_1, \mathbf{e}_1)} & \cancel{\varphi(\mathbf{e}_1, \mathbf{e}_2)} \\ \cancel{\varphi(\mathbf{e}_2, \mathbf{e}_1)} & \cancel{\varphi(\mathbf{e}_2, \mathbf{e}_2)} \end{bmatrix}$$

行列式 $\varphi(\mathbf{a}, \mathbf{b}) = -\varphi(\mathbf{b}, \mathbf{a})$ 反対称性

$$\varphi(\mathbf{e}_1, \mathbf{e}_2) = -\varphi(\mathbf{e}_2, \mathbf{e}_1)$$

$$\varphi(\mathbf{e}_1, \mathbf{e}_1) = \varphi(\mathbf{e}_2, \mathbf{e}_2) = 0$$

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

$$f' : \mathbb{R}^n \to L(\mathbb{R}^n ; \mathbb{R}^m)$$

$$x_0 \in \mathbb{R}^m$$

$$f''(x_0) \in L(\mathbb{R}^n, \mathbb{R}^n ; \mathbb{R}^m)$$

$$f''(x_0)(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^m$$

$\boxed{\text{定理}}$ $f''(x_0)(\mathbf{a}, \mathbf{b}) = f''(x_0)(\mathbf{b}, \mathbf{a})$

($f''(x_0)$ は対称)

これから証明する

$d_1, d_2 \in D$

$$f'(x_0)(\mathbb{a})d = f(x_0 + \mathbb{a}d) - f(x_0)$$

$$\underbrace{f''(x_0)(\mathbb{a}d_1, \mathbb{b}d_2)}_{f''(x_0)(\mathbb{a},\mathbb{b})d_1d_2}$$

$$= \underline{f''(x_0)(\mathbb{a}d_1)}(\mathbb{b}d_2)$$

$$= \{f'(x_0 + \mathbb{a}d_1) - f'(x_0)\}(\mathbb{b}d_2)$$

$$= f'(x_0 + \mathbb{a}d_1)(\mathbb{b}d_2) - f'(x_0)(\mathbb{b}d_2)$$

$$= f(x_0 + \mathbb{a}d_1 + \mathbb{b}d_2) - f(x_0 + \mathbb{a}d_1) - \{f(x_0 + \mathbb{b}d_2) - f(x_0)\}$$

$$= f(x_0 + \mathbb{a}d_1 + \mathbb{b}d_2) - f(x_0 + \mathbb{a}d_1) - f(x_0 + \mathbb{b}d_2) + f(x_0)$$

$$f''(x_0)(\mathbb{b}d_2, \mathbb{a}d_1) = f''(x_0)(\mathbb{b},\mathbb{a})d_1d_2$$

2階の微分は対称線形写像

10月11日(月)

高階の微分

$f: \mathbb{R} \to \mathbb{R}$

$x_0 \in \mathbb{R}$

$f'(x_0)$ 微分係数

x_0 を動かす

$f': \mathbb{R} \to \mathbb{R}$ 1階の

$x_0 \in \mathbb{R}$

$f''(x_0)$ 2階の微分

$f'': \mathbb{R} \to \mathbb{R}$ 高校

続けられる

$\to \mathbb{R}^m$

f'

$x_0 \in \mathbb{R}^n$

微分

$f'(x_0)$: $\mathbb{R}^n$ から $\mathbb{R}^m$ への線型写像の全体

$\in L(\mathbb{R}^n ; \mathbb{R}^m)$ セミコロン

$m \times n$ 行列の全体

$m \times n$ 個の実数

$$m\begin{array}{c} n \\ \overbrace{\left[\quad\quad\right]} \end{array}$$

$\mathbb{R}^{mn}$

x_0を動かす　$\mathbb{R}$　$n=m=1$　1×1

$\underline{\mathcal{L}(\mathbb{R};\mathbb{R})}$　[ ]

$$f' : \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$$

$$\mathbb{R}^{mn}$$

) 同一視

$x_0 \in \mathbb{R}$ で微分

$f'(x_0)$: $\underline{\mathbb{R}^n \text{ から}}$ $\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$ への線型写像

φ : $\mathbb{R}^n$ から $\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$ への線型写像

$a \in \mathbb{R}$

$\varphi(a) \in \mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$

$b \in \mathbb{R}^n$　$\underline{\varphi(a)(b)} \in \mathbb{R}^m$

$\sim$

$\tilde{\varphi}(a, b)$

$\tilde{\varphi} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\tilde{\varphi}(\alpha a, b) = (\alpha$ … $\varphi(a)($

$$\tilde{\varphi}(a, \alpha b) = \underline{\varphi(a)}(\alpha b)$$
$$= \alpha \varphi(a)(b)$$
$$= \alpha \tilde{\varphi}(a, b)$$

$$\tilde{\varphi}(a_1 + a_2, b) = \varphi(a_1 + a_2)(b)$$
$$= \varphi(a_1)(b) + \varphi(a_2)(b)$$
$$= \tilde{\varphi}(a_1, b) + \tilde{\varphi}(a_2, b)$$

内積

$a \cdot b \in \mathbb{R}$

行列式 2×2

$|a \ b|$

$f : \mathbb{R}^2 \to \mathbb{R}$

$$\tilde{\varphi}(a, b_1 + b_2) = \underline{\varphi(a)}(b_1 + b_2)$$
$$\varphi(a) \in L(\mathbb{R}^n, \mathbb{R}^m)$$

$\boxed{n = 2 \quad m = 1}$

$$= \varphi(a)(b_1) + \varphi(a)(b_2)$$
$$= \tilde{\varphi}(a, b_1) + \tilde{\varphi}(a, b_2)$$

$\tilde{\varphi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$

二重線型性

$f''(x)$

$\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ 二重線型写像

$$\begin{cases} \psi(a_1 + a_2, b) = \psi(a_1, b) + \psi(a_2, b) & \circ \\ \psi(a, b_1 + b_2) = \psi(a, b_1) + \psi(a, b_2) & \circ \\ \psi(\alpha a, b) = \alpha \psi(a, b) & \circ \\ \psi(a, \beta b) = \beta \psi(a, b) & \circ \end{cases}$$

$$\in \mathcal{L}(\mathbb{R}^n ; \mathbb{R}^m)$$

$$a \in \mathbb{R} \longmapsto \psi(a, \_) : \mathbb{R}^n \to \mathbb{R}^m$$

$a_1 + a_2$

$\mathbb{R}^n$ から $\mathcal{L}(\mathbb{R}^n ; \mathbb{R}^m)$ への線型写像の全体

$$\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n ; \mathbb{R}^m)$$

$$= \mathcal{L}(\mathbb{R}^n ; \mathcal{L}($$

$\varphi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ 二重線型写像

$f : \mathbb{R}^2 \to \mathbb{R}$ 線型　$\varphi(e_1, e_1) = -\varphi(e_1, e_1)$

$$f\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right) = f(a_1 e_1 + a_2 e_2) = a_1 f(e_1) + a_2 f(e_2)$$

$$\underset{1\times 2}{[f(e_1), f(e_2)]} \underset{2\times 1}{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}} \quad 1\times 1$$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\varphi\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = \varphi(a_1 e_1 + a_2 e_2, b_1 e_1 + b_2 e_2)$$

$$= a_1 b_1 \varphi(e_1, e_1) + a_1 b_2 \varphi(e_1, e_2) + a_2 b_1 \varphi(e_2, e_1) + a_2 b_2 \varphi(e_2, e_2)$$

$$\underset{1\times 2}{(a_1, a_2)} \underset{2\times 2}{\begin{bmatrix} \varphi(e_1, e_1) & \varphi(e_1, e_2) \\ \varphi(e_2, e_1) & \varphi(e_2, e_2) \end{bmatrix}} \underset{2\times 1}{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 e_1 + a_2 e_2$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b_1 e_1 + b_2 e_2$$

内積　$\varphi(a, b) = \varphi(b, a)$　対称性　$\Longrightarrow$　$\varphi(e_1, e_2) = \varphi(e_2, e_1)$

行列式　$\varphi(a, b) = -\varphi(b, a)$　反対称性　$\Longleftarrow$　$\varphi(e_1, e_2) = -\varphi(e_2, e_1)$, $\varphi(e_1, e_1) = \varphi(e_2, e_2) = 0$

$f : \mathbb{R}^n \to \mathbb{R}^m$

$f' : \mathbb{R}^n \to L(\mathbb{R}^n ; \mathbb{R}^m)$

$x_0 \in \mathbb{R}^n$

$f''(x_0) \in L(\mathbb{R}^n, \mathbb{R}^n ; \mathbb{R}^m)$

$f''(x_0)(a, b) \in \mathbb{R}^m$

定理 $f''(x_0)(a, b) = f''(x_0)(b, a)$

($f''(x_0)$は対称)

これから証明する

$d_1, d_2 \in D$

$$f'(x_0)(a)d = f(x_0 + ad) - f(x_0)$$

$$f''(x_0)(a, b)d_1 d_2$$

$$\shortparallel$$

$$f''(x_0)(ad_1, bd_2)$$

$$= \underline{f''(x_0)(ad_1)}\,(bd_2)$$

$$= \{f'(x_0 + ad_1) - f'(x_0)\}(bd_2)$$

$$f''(x_0)(bd_2, a$$

$$(b, a)d_1 d_2$$

$$= \underline{f'(x_0 + ad_1)(bd_2)} - f'(x_0)(bd_2)$$

$$= f(x_0 + ad_1 + bd_2) - f(x_0 + ad_1)$$

$$- \{f(x_0 + bd_2) - f(x_0)\}$$

$$= f(x_0 + ad_1 + bd_2) - f(x_0 + ad_1) - f(x_0 + bd_2)$$

$$+ f(x_0)$$