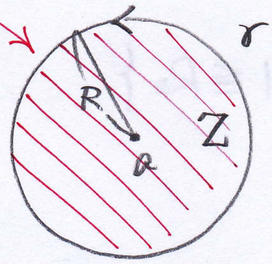
Causyの積分公式.

2011. 2. 16. Tue.

微積分

f : 解析関数

$$f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$



z : 円板の内部の点

$$\frac{f(w)}{w-z}=\frac{f(w)}{(w-a)-(z-a)}$$

$$=\frac{1}{w-a}\cdot\frac{1}{1-\frac{z-a}{w-a}}$$

$$\left(\begin{array}{l}\text{等比級数}\quad |r|<1\\ 1+r+r^2+r^3+\cdots\cdots=\frac{1}{1-r}\end{array}\right)$$

$$=\frac{1}{w-a}\cdot\left\{1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\cdots\right\}\ (\text{無限級数})$$

$$=\frac{f(w)}{w-a}+\frac{f(w)}{(w-a)^2}(z-a)+\frac{f(w)}{(w-a)^3}(z-a)^2+\frac{f(w)}{(w-a)^4}(z-a)^3+\cdots$$

$$f(z)=\frac{1}{2\pi i}\left\{\int_\gamma\frac{f(w)}{w-a}dw+(z-a)\int_\gamma\frac{f(w)}{(w-a)^2}dw+(z-a)^2\int_\gamma\frac{f(w)}{(w-a)^3}dw+\cdots\right\}$$

$$f(z)=a_0+a_1(z-a)+a_2(z-a)^2+a_3(z-a)^3+\cdots\cdots$$

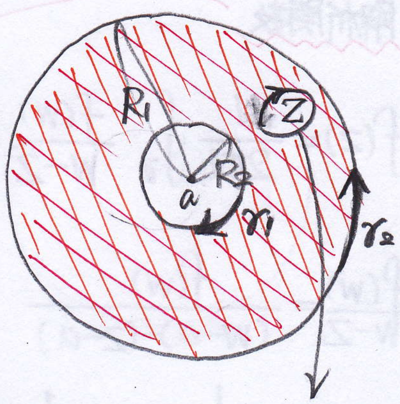
巾級数(無限次の多項式)

$$\left(a_i=\frac{1}{2\pi i}\int_\gamma\frac{f(w)}{(w-a)^{i+1}}dw\right)$$

f: 解析関数

$R_1 < R_2$

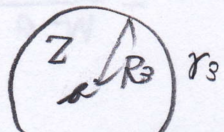
$\{ z \in \mathbb{C} \mid R_1 \leqq |z-a| \leqq R_2 \}$



$f(z)$

(stokesの定理)

$$\int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)\,dz = 0$$



R_3を小さくとる.

$$\int_{\gamma_2} f(z)dz - \int_{\gamma_1} f(z)dz = \int_{\gamma_3} F(z)dz.$$

$\gamma_1 \cup \gamma_2 \cup \gamma_3$

fの代わりに

$$\frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{(w-a)-(z-a)}dw$$

$$= \frac{1}{2\pi i}\int_{\gamma_2} \frac{1}{1-\frac{z-a}{w-a}}dw.$$

$$= \frac{1}{2\pi i}\int_{\gamma_2} f(w)\left\{\frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(z-a)^3} + \frac{(z-a)^3}{(z-a)^4} + \cdots\right\}dw$$

$$= \frac{1}{2\pi i}\left\{\int_{\gamma_2} \frac{f(w)}{w-a}dw + (z-a)\int_{\gamma_2}\frac{f(w)}{(w-a)^2}dw + (z-a)^2\int_{\gamma_2}\frac{f(w)}{(w-a)^3}dw + \cdots\right\}$$

$$\text{“}\sum_{n=0}^{\infty} a_n (z-a)^n\text{”}$$

$$\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{(w-a)-(z-a)}dw \qquad |w-a|<|z-a|$$

$$= \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{\frac{w-a}{z-a}-1}\cdot\frac{1}{z-a}\,dw$$

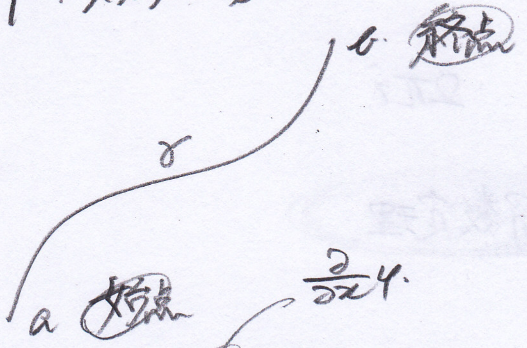
$$= \frac{-1}{2\pi i}\,\frac{1}{z-a}\left\{\int_{\gamma_1} f(w)\,dw + \frac{1}{z-a}\int_{\gamma_1} f(w)(w-a)\,dw + \frac{1}{(z-a)^2}\int_{\gamma_1} f(w)(w-a)^2\,dw + \cdots\right\}$$

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_3}\frac{f(w)}{w-z}\,dw \qquad \text{“}\sum_{n=-1}^{-\infty} a_n (z-a)^n\text{”}$$

$$\sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

保存力

φ：スカラー場



$\frac{\partial}{\partial x}\varphi$

$$\int_\gamma \varphi_x\,dx + \varphi_y\,dy + \varphi_z\,dz = \varphi(b) - (a)$$

Causy-Riemannの微分方程式

$f(x+iy) + ig(x+iy)$ 解析関数

$f : \mathbb{C} \to \mathbb{R}$

$g : \mathbb{C} \to \mathbb{R}$

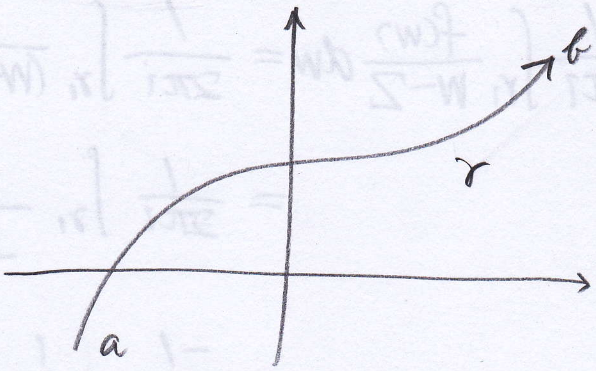
$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$\{f + ig\}' = f_x + ig_x$

↓

原始関数

複素平面で



$$\int_\gamma \{f_x + ig_x\}(dx + i\,dy)$$

$$= \int_\gamma (f_x\,dx - g_x\,dy) + i\int_\gamma (f_x\,dy + g_x\,dx)$$

$$= \int_\gamma (f_x\,dx + f_y\,dy) + i\int_\gamma (g_y\,dy + g_x\,dx)$$

$$= f(b) - f(a) + i\{g(b) - g(a)\}$$

$$= f(b) + ig(b) - (f(a) + ig(a))$$

$$\therefore \int_\gamma f(z)\,dz = \sum_{n=-\infty}^{\infty} a_n \int_\gamma (z-a)^n\,dz$$

$\left(\frac{1}{z-a}\right.$ は原始関数をもたない…… $\left.\right)$

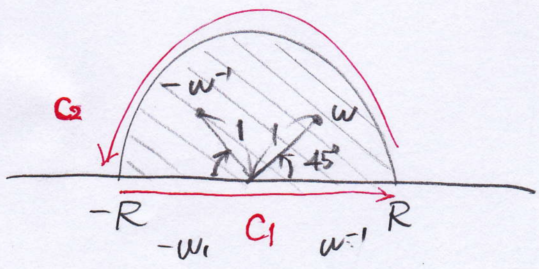
$$= a_{-1}\int_\gamma \frac{dz}{z-a} \qquad 2\pi i$$

留数定理

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a} \quad (0 < a < 1)$$

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$



$$\frac{1}{1+z^4} = \underline{\qquad} + \underline{\qquad} + \underline{\qquad} + \underline{\qquad}$$

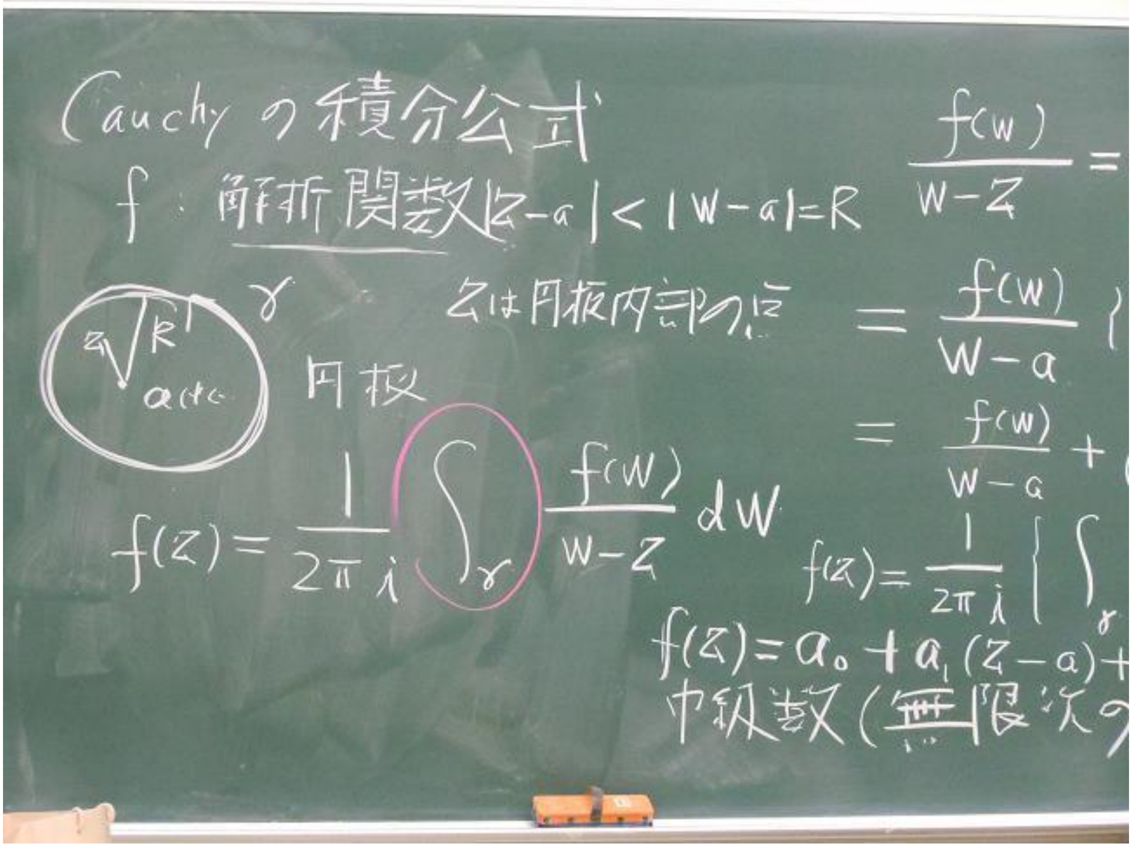
0になる部分：極.

$f(z)$の極は $z = w^{\pm 1}, -w^{\pm 1}$ $\left(w = e^{\frac{i\pi}{4}}\right.$

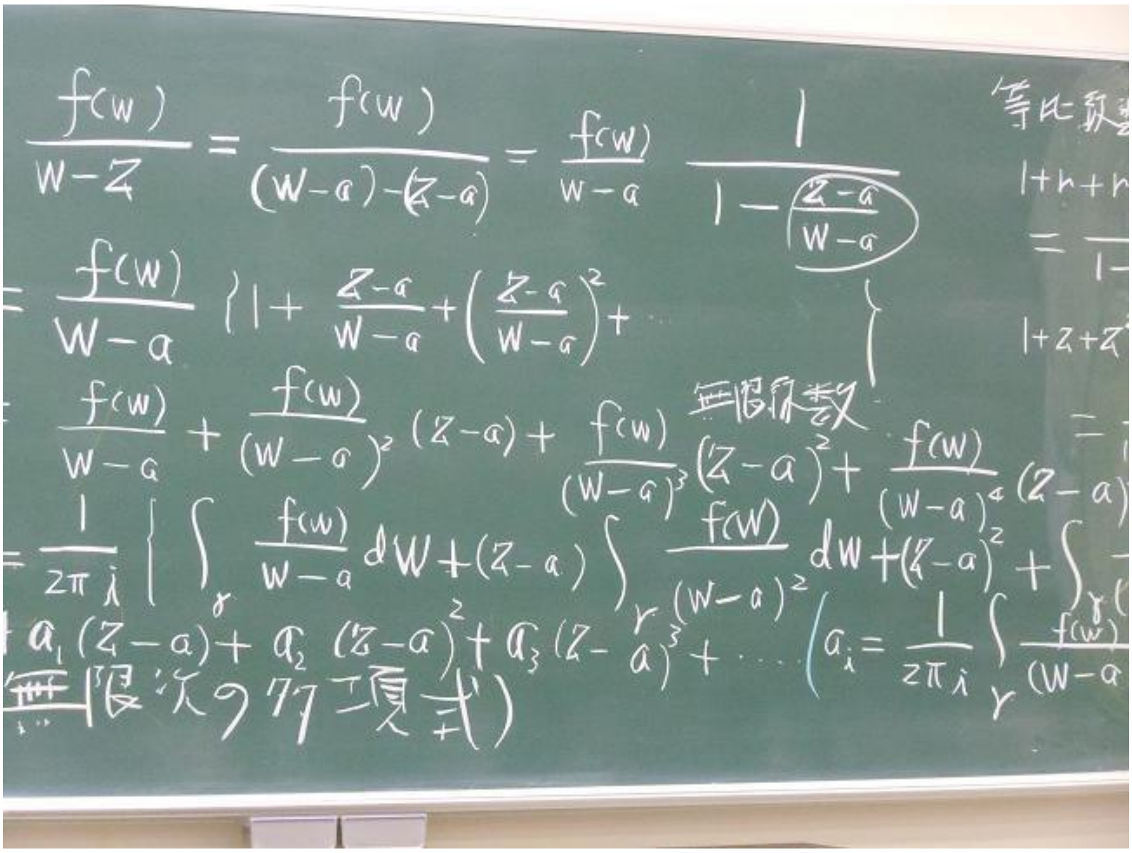
$$\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$1 + z^4 = (z^2 + 1)$$

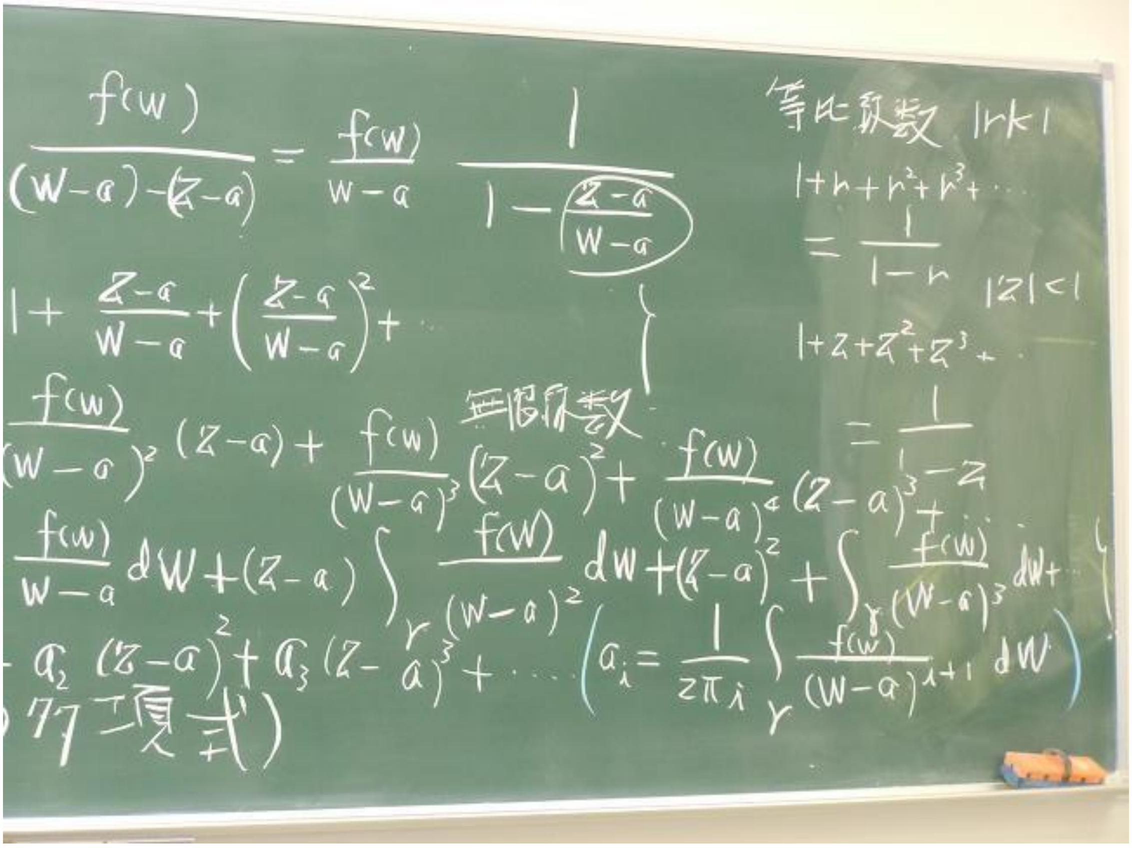
$$= (z - w)(z - w^{-1})(z + w)(z + w^{-1})$$



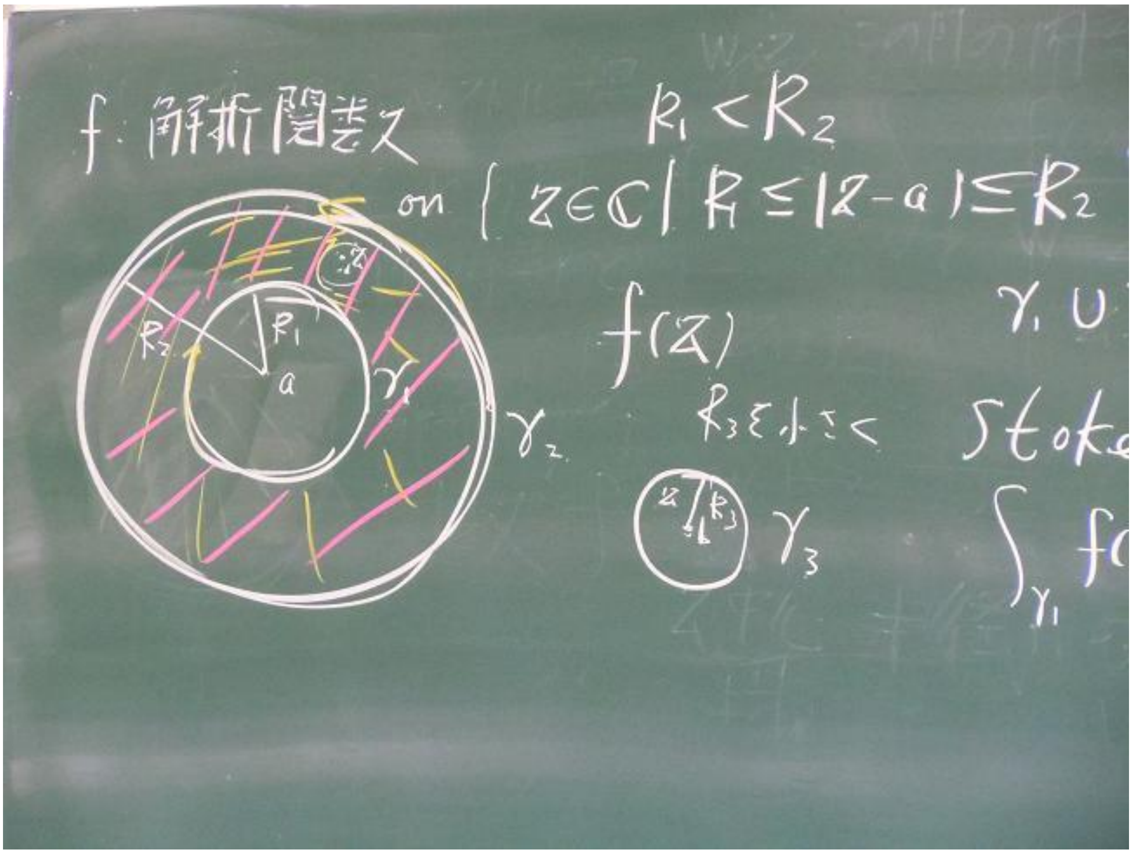
Cauchyの積分公式
f：解析関数 $|z-a| < |w-a| = R$
γ
zは円板内部の点
R
a中心
円板
$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$
$\frac{f(w)}{w-z} =$
$= \frac{f(w)}{w-a}\{$
$= \frac{f(w)}{w-a} +$
$f(z) = \frac{1}{2\pi i}\{\int_\gamma$
$f(z) = a_0 + a_1(z-a) +$
巾級数（無限次の



$\frac{f(w)}{w-z} = \frac{f(w)}{(w-a)-(z-a)} = \frac{f(w)}{w-a}\,\frac{1}{1-\frac{z-a}{w-a}}$
$= \frac{f(w)}{w-a}\left\{1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots\right\}$
無限級数
$= \frac{f(w)}{w-a} + \frac{f(w)}{(w-a)^2}(z-a) + \frac{f(w)}{(w-a)^3}(z-a)^2 + \frac{f(w)}{(w-a)^4}(z-a)$
$= \frac{1}{2\pi i}\left\{\int_\gamma \frac{f(w)}{w-a}\,dw + (z-a)\int_\gamma \frac{f(w)}{(w-a)^2}\,dw + (z-a)^2 + \int_\gamma\right.$
$a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \cdots \quad \left(a_i = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a}\right.$
無限次の多項式)
等比級数
$1+r+r$
$= \frac{}{1-}$
$1+z+z$



f(w) / ((w−a)−(z−a)) = f(w)/(w−a) · 1 / (1 − (z−a)/(w−a))
等比級数 |r|<1
1+r+r²+r³+… = 1/(1−r)
|z|<1
1+z+z²+z³+… = 1/(1−z)
1 + (z−a)/(w−a) + ((z−a)/(w−a))² + …
無限級数
f(w)/(w−a)² (z−a) + f(w)/(w−a)³ (z−a)² + f(w)/(w−a)⁴ (z−a)³ + …
f(w)/(w−a) dw + (z−a) ∫γ f(w)/(w−a)² dw + (z−a)² + ∫γ f(w)/(w−a)³ dw + …
a₂ (z−a)² + a₃ (z−a)³ + … (aᵢ = 1/(2πi) ∫γ f(w)/(w−a)^(i+1) dw)
二項式)



f 解析関数
R₁ < R₂
on { z∈ℂ | R₁ ≤ |z−a| ≤ R₂
R₂
R₁
a
γ₁
γ₂
f(z)
R₃を小さく
z
R₃
γ₃
γ₁ ∪

γ_2) $\qquad \int_{\gamma_2} f(z)\,dz - \int_{\gamma_1} f(z$

$\cup\, \gamma_2 \cup \gamma_3$

kesの定理

$f(z)\,dz - \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz =$

$$\int_{\gamma_2} f(z)\,dz - \int_{\gamma_1} f(z)\,dz = \int_{\gamma_3} f(z)\,dz$$

$f(z)\,dz + \int_{\gamma_3} f(z)\,dz = 0$

fの代りに

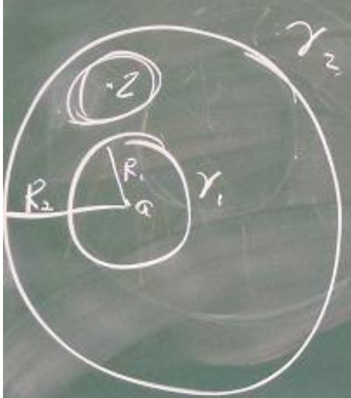
$$\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)}{(w-a)-(z-a)}\,dw = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)}{(w-a)}\frac{1}{1-\frac{z-a}{w-a}}\,dw$$

$$= \frac{1}{2\pi i}\int_{\gamma_2} f(w)\left\{\frac{1}{w-a}+\frac{z-a}{(w-a)^2}+\frac{(z-a)^2}{(w-a)^3}+\cdots\right\}dw$$

$$= \frac{1}{2\pi i}\left\{\int_{\gamma_2}\frac{f(w)}{w-a}\,dw + (z-a)\int_{\gamma_2}\frac{f(w)}{(w-a)^2}\,dw + (z-a)^2\int_{\gamma_2}\frac{f(w)}{(w-a)^3}\,dz + \cdots\right\}$$

$$|w-a|<|z-a|$$

$$\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{(w-a)-(z-a)}\,dw$$



$$= \frac{-1}{2\pi i}\,\frac{1}{z-a}\left\{\int_{\gamma_1} f(w)\,dw + \right.$$

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_3}\frac{f(w)}{w-z}\,dw$$

$$\frac{f(w)}{(w-a)-(z-a)}\,dw = \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{\frac{w-a}{z-a}-1}\quad\frac{1}{z-}$$

$$\left\{\int_{\gamma_1} f(w)\,dw + \frac{1}{z-a}\int_{\gamma_1} f(w)(w-a)\,dw + \right.$$

$$\int_{\gamma_3}\frac{f(w)}{w-z}\,dw \qquad \sum_{n=-1}^{-\infty} a_n (z-a)^n$$

$$\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{\frac{w-a}{z-a}-1}\,\frac{1}{z-a}\,dw$$

$$\frac{1}{z-a}\int_{\gamma_1} f(w)(w-a)\,dw+\frac{1}{(z-a)^2}\int_{\gamma_1} f(w)(w-a)^2\,dw$$

$$\sum_{n=-1}^{-\infty} a_n(z-a)^n$$

$$\sum_{n=-\infty}^{\infty} a_n(z-a)^n$$

$$\frac{\partial\varphi}{\partial x}$$

$$\int_{\gamma}\varphi_x\,dx+\varphi_y\,dy=\varphi(b)-\varphi(a)$$

保存力

φ スカラー場

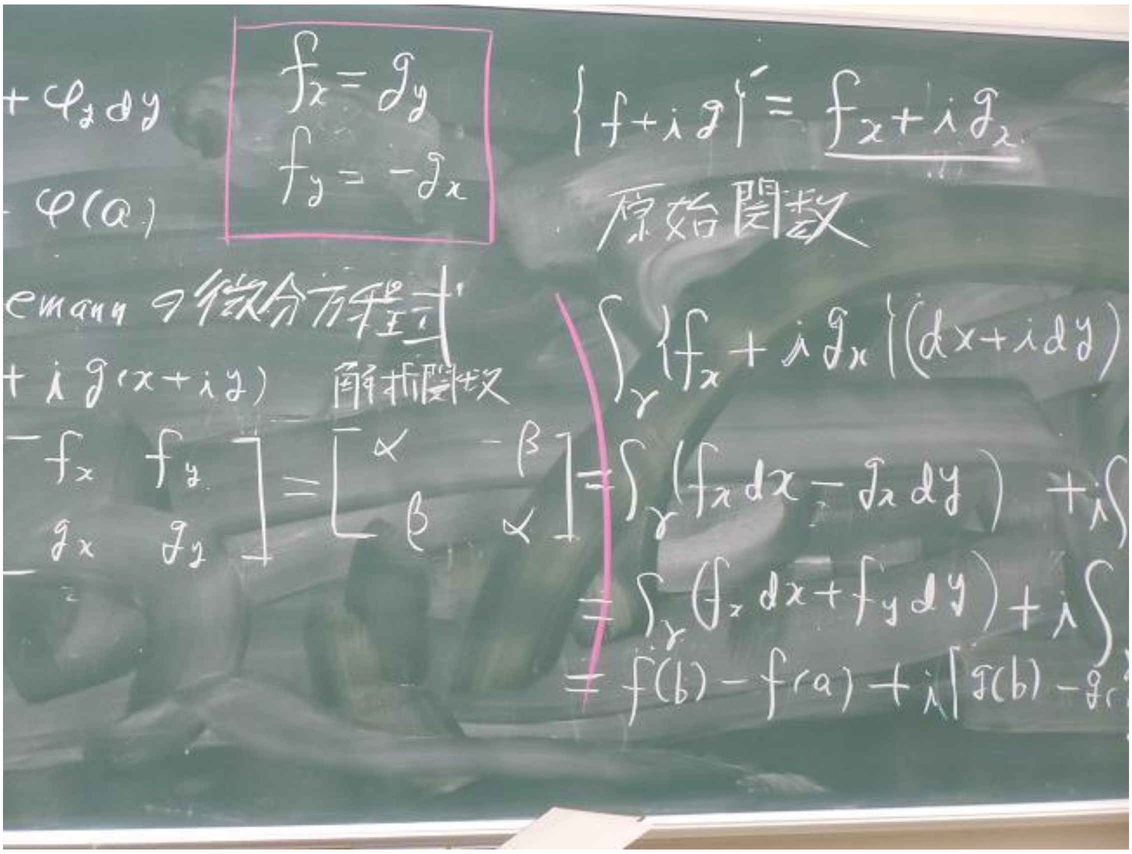
Cauchy-Riemann の微分

$f(x+iy)+i\,g(x+iy)$

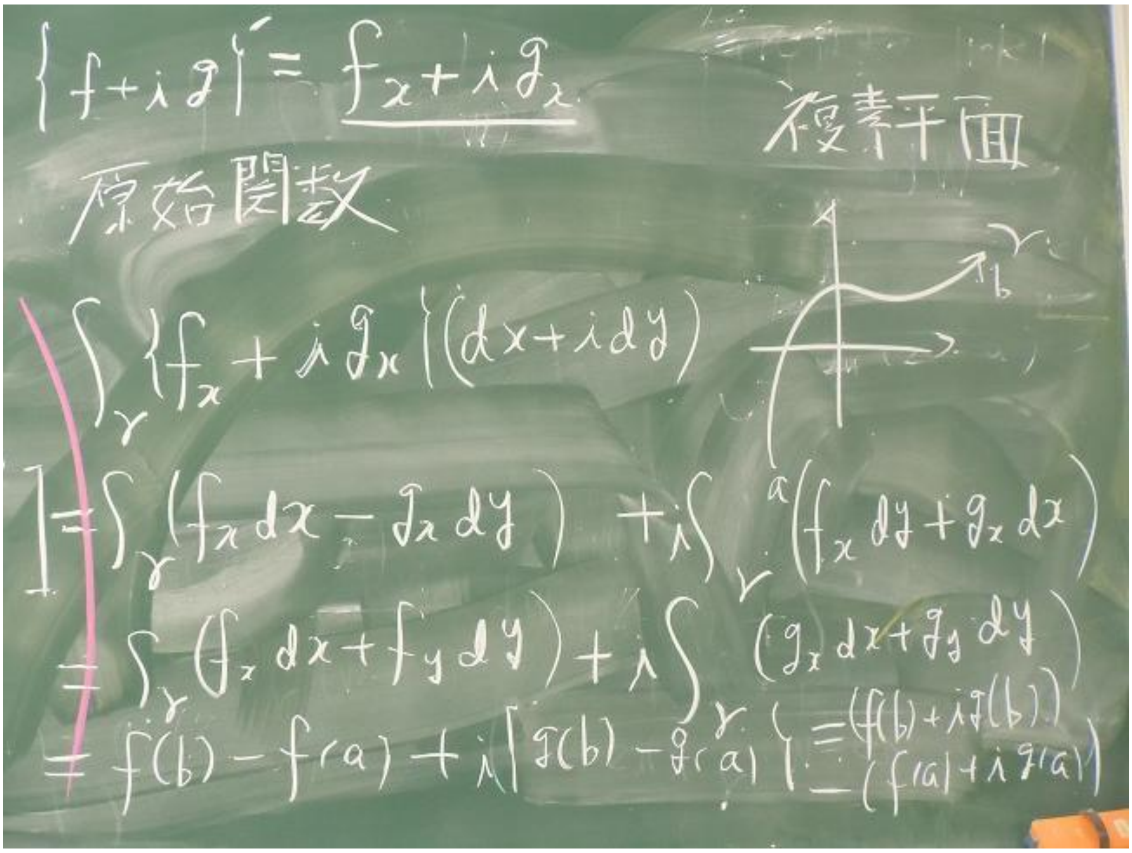
$f:\mathbb{C}\to\mathbb{R}$

$g:\mathbb{C}\to\mathbb{R}$

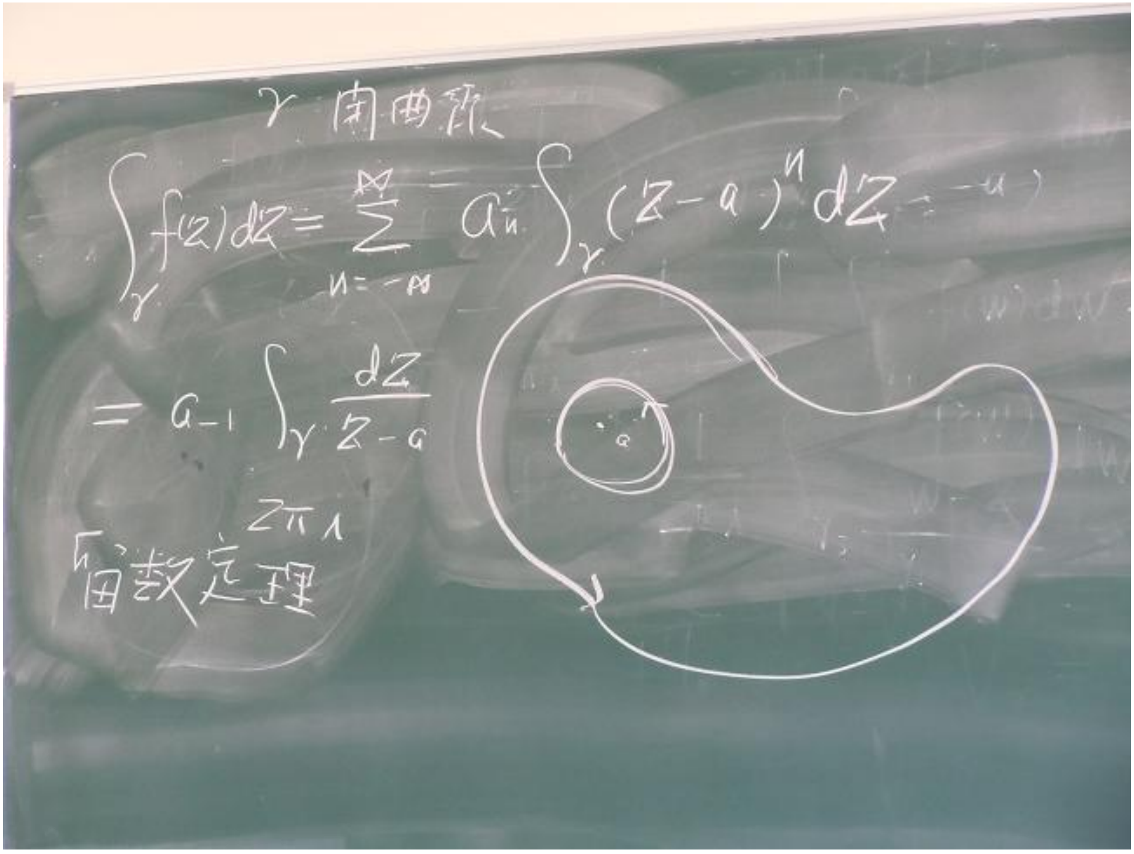
$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} =$$

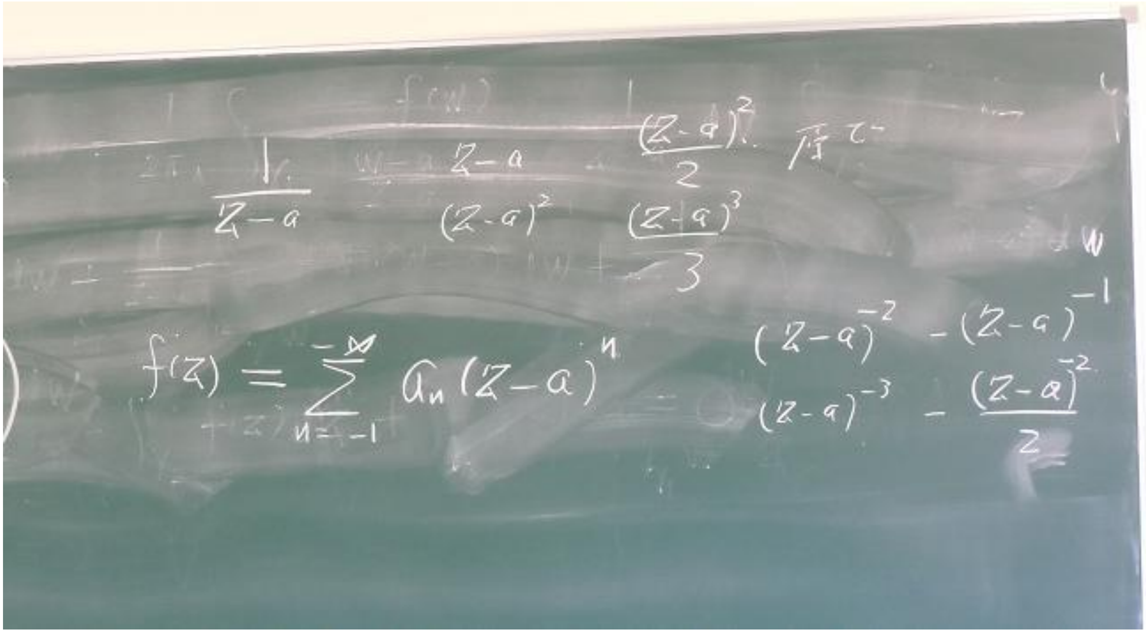


$+ \varphi_y dy$
$\varphi(a)$
$f_x = g_y$
$f_y = -g_x$
$\{f+ig\}' = f_x + i g_x$
原始関数
emann の微分方程式
$+ i g(x+iy)$ 解析関数
$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$
$\int_\gamma \{f_x + i g_x\}(dx + i dy)$
$= \int_\gamma (f_x dx - g_x dy) + i\int$
$= \int_\gamma (f_x dx + f_y dy) + i\int$
$= f(b) - f(a) + i\{g(b) - g($



$\{f+ig\}' = f_x + i g_x$
複素平面
原始関数
$\int_\gamma \{f_x + i g_x\}(dx + i dy)$
$= \int_\gamma (f_x dx - g_x dy) + i\int_\gamma (f_x dy + g_x dx)$
$= \int_\gamma (f_x dx + f_y dy) + i\int_\gamma (g_x dx + g_y dy)$
$= f(b) - f(a) + i\{g(b) - g(a)\} = (f(b) + i g(b)) - (f(a) + i g(a))$
a
b
γ

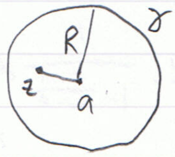

γ 闭曲线
∫γ f(z)dz = Σ n=−∞..∞ aₙ ∫γ (z−a)ⁿ dz
= a₋₁ ∫γ dz/(z−a)
2πi
留数定理


1/(z−a)   z−a   (z−a)²/2
(z−a)²   (z−a)³/3
f(z) = Σ n=−1..−∞ aₙ (z−a)ⁿ
(z−a)⁻²   −(z−a)⁻¹
(z−a)⁻³   −(z−a)⁻²/2

2/16(水) 3限　微積分(生物学類

Cauchyの積分公式

f：解析関数



$|z-a| < |w-a| = R$

zは円板内部の点

円板

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

$$\frac{f(w)}{w-z} = \frac{f(w)}{(w-a)-(z-a)} = \frac{f(w)}{w-a}\cdot\frac{1}{1-\frac{z-a}{w-a}}$$

$$= \frac{f(w)}{w-a}\left\{1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots\right\}$$

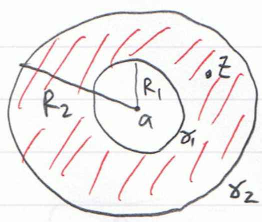
$$= \frac{f(w)}{w-a} + \frac{f(w)}{(w-a)^2}(z-a) + \frac{f(w)}{(w-a)^3}(z-a)^2 + \frac{f(w)}{(w-a)^4}(z-a)^3 + \cdots \quad \text{(無限級数)}$$

$$f(z) = \frac{1}{2\pi i}\left\{\int_\gamma \frac{f(w)}{w-a}\,dw + (z-a)\int_\gamma \frac{f(w)}{(w-a)^2}\,dw + (z-a)^2\int_\gamma \frac{f(w)}{(w-a)^3}\,dw + \cdots\right\}$$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \cdots \quad \left(a_i = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{i+1}}\,dw\right)$$

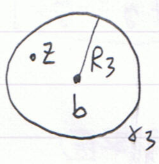
$f(z)$を巾(ベキ)級数(無限次の多項式)として表わせた

f：解析関数



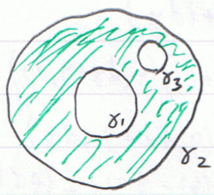
$R_1 < R_2$

$\{z \in \mathbb{C} \mid R_1 \leq |z-a| \leq R_2\}$



zが微小な半径R_3の円板に含まれるように点bをとる

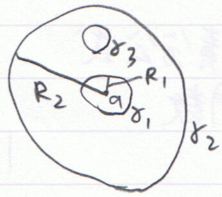
$\gamma_1 \cup \gamma_2 \cup \gamma_3$



Stokesの定理

$$\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz = 0$$

$$\int_{\gamma_2} f(z)dz - \int_{\gamma_1} f(z)dz = \int_{\gamma_3} f(z)dz$$



fの代わりに

$$\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)}{(w-a)-(z-a)}dw = \frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)}{(w-a)}\cdot\frac{1}{1-\frac{z-a}{w-a}}dw$$

$$= \frac{1}{2\pi i}\int_{\gamma_2} f(w)\left\{\frac{1}{w-a}+\frac{z-a}{(w-a)^2}+\frac{(z-a)^2}{(w-a)^3}+\cdots\right\}dw$$

$$= \frac{1}{2\pi i}\left\{\int_{\gamma_2}\frac{f(w)}{w-a}dw + (z-a)\int_{\gamma_2}\frac{f(w)}{(w-a)^2}dw + (z-a)^2\int_{\gamma_2}\frac{f(w)}{(w-a)^3}dw + \cdots\right\}$$

$|w-a|<|z-a|$

$$\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{(w-a)-(z-a)}dw = \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{\frac{w-a}{z-a}-1}\cdot\frac{1}{z-a}dw$$

$$= \frac{-1}{2\pi i}\cdot\frac{1}{z-a}\left\{\int_{\gamma_1} f(w)dw + \frac{1}{z-a}\int_{\gamma_1} f(w)(w-a)dw + \frac{1}{(z-a)^2}\int_{\gamma_1} f(w)(w-a)^2 dw \cdots\right\}$$

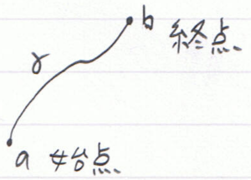
$$f(z) = \frac{1}{2\pi i}\int_{\gamma_3}\frac{f(w)}{w-z}dw$$

$$\sum_{n=-1}^{-\infty} a_n (z-a)^n$$

$\sum_{n=-\infty}^{\infty} a_n (z-a)^n$ の形で表わせる

保存力

φ；スカラー場



$$\int_\gamma \underbrace{\varphi_x}_{\frac{\partial\varphi}{\partial x}} dx + \underbrace{\varphi_y}_{\frac{\partial\varphi}{\partial y}} dy + \underbrace{\varphi_z}_{\frac{\partial\varphi}{\partial z}} dz$$

$$= \varphi(b) - \varphi(a)$$

Cauchy-Riemann の微分方程式

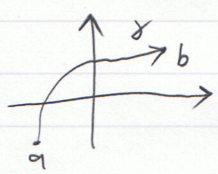
$f(x+iy)+i\,g(x+iy)$：解析関数

$$\boxed{\begin{matrix} f_x = g_y \\ f_y = -g_x \end{matrix}}$$

$f: \mathbb{C}\to\mathbb{R}$
$g: \mathbb{C}\to\mathbb{R}$

$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

複素平面



$$\{f+ig\}' = \underbrace{f_x + i g_x}_{\text{原始関数}}$$

$$\int_\gamma \{f_x + i g_x\}(dx+idy)$$

$$= \int_\gamma (f_x dx - g_x dy) + i\int_\gamma (f_x dy + g_x dy)$$

$$= \int_\gamma (f_x dx + f_y dy) + i\int_\gamma (g_x dx + g_y dy)$$

$$= f(b) - f(a) + i\{g(b) - g(a)\} = (f(b)+ig(b)) - (f(a)+ig(a))$$