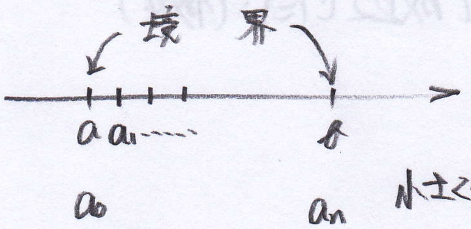
微積分学の基本定理（一次元）

↓ 高次元へ一般化

stokesの定理　　（二次元）

1次元での積分　0次元での積分

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$



小さく分割

$a_{i+1} - a_i \in D \quad a_{i+1} - a_i = d_i$

$$\int_a^b f'(x)\,dx = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f'(x)\,dx = \sum_{i=0}^{n-1} d_i f'(a)$$

$$= \sum_{i=0}^{n-1} d_i f'(0)$$

$$= \sum_{i=0}^{n-1} \{ f(a_{i+1}) - f(a_i) \}$$

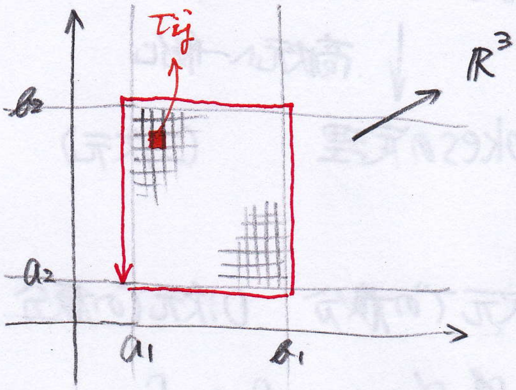
" $d_i f'(a) = f(a_{i+1}) - f(a_i)$ " 無限小のlevelでの微積分学の基本定理

$$
\begin{array}{rl}
 & f(a_n) - \cancel{f(a_{n-1})} \\
 & \cancel{f(a_{n-1})} - \cancel{f(a_{n-2})} \\
 & \vdots \\
+) & \cancel{f(a_1)} - f(a_0) \\
\hline
 & f(b) - f(a)
\end{array}
$$

$a_0 = a, \quad a_n = b$

曲面

$\tau : [a_1, b_1] \times [a_2, b_2] \to R^3$



1次の微分形式

$\omega = f\,dx + g\,dy + h\,dz$

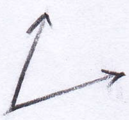
$\int_\tau d\omega = \int_{\partial\tau} \omega$ 　 τ_{ij} → 無限小のlevelで成立させたい(仮定)

$$\int_\tau \omega = \sum_{ij} \int_{\tau_{ij}} d\omega$$

$$= \sum_{ij} \int_{\partial\tau_{ij}} \omega \quad \text{仮定}$$

$$= \int_{\partial\tau} \omega$$

～前回のReport問題～



τ：無限小の平行四辺形

$\int_{\partial\tau} \omega$ を計算すると…

$$\begin{pmatrix} \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \end{pmatrix} \cdot a \times b \; d_1 d_2$$

$$= \int_\tau \underbrace{\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy}_{d\omega}$$

ベクトル解析 $\underbrace{f\mathbb{e}_1+g\mathbb{e}_2+h\mathbb{e}_3}_{\mathbb{f}} \to \left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\mathbb{e}_1+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\mathbb{e}_2+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\mathbb{e}_3$

$\updownarrow$

微分形式 $f\,dx+g\,dy+h\,dz \xrightarrow[\text{微分}]{d} dw$

ベクトル解析 rot(ation)

$$\mathrm{rot}\,\mathbb{f} = \nabla \times \mathbb{f}$$

ナブラ nabla

※ $\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$ ← 作用素のベクトル（擬似ベクトル）

$$= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} f \\ g \\ h \end{pmatrix} \qquad \mathbb{f} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

$$= \begin{pmatrix} \begin{vmatrix} \frac{\partial}{\partial y} & g \\ \frac{\partial}{\partial z} & h \end{vmatrix} \\ \begin{vmatrix} \frac{\partial}{\partial z} & h \\ \frac{\partial}{\partial x} & f \end{vmatrix} \\ \begin{vmatrix} \frac{\partial}{\partial x} & f \\ \frac{\partial}{\partial y} & g \end{vmatrix} \end{pmatrix} = \begin{pmatrix} \frac{\partial h}{\partial y}-\frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial z}-\frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial x}-\frac{\partial f}{\partial y} \end{pmatrix}$$

$\mathrm{grad}\,\varphi = \nabla\varphi$ スカラー場

$$= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial z} \end{pmatrix}$$

$\mathbb{r} = (x, y, z)$

回転

$$\mathrm{rot}\,\mathbb{r} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underbrace{\int \mathbb{r}\, d\mathbb{r}}_{\text{線積分}} = \underbrace{\int \overset{0}{\overset{\parallel}{(\mathrm{rot}\,\mathbb{r})}} \cdot dS}_{\text{面積分}}$$

$x^2 + y^2 = a^2 \quad z = 0$ (曲面) (2学期の Report)

御利益

保存力 $\mathbb{F}$

仕事 0

$\mathbb{F} = \mathrm{grad}\,\varphi$

$$\left.\begin{aligned} \frac{\partial \varphi}{\partial x} &= f \\ \frac{\partial \varphi}{\partial y} &= g \\ \frac{\partial \varphi}{\partial z} &= h \end{aligned}\right\} \text{偏微分方程式}$$

$\mathbb{F}$が保存力であるかどうかの判別法 $\Longleftrightarrow$ rot $\mathbb{F}$が消える(定理)

(物理学的証明)

任意の閉曲線に対してそれを境界とする曲面が存在する

$$\underbrace{\int \mathrm{rot}\,\mathbb{F} = 0}_{\text{(面積分)}} \qquad \mathrm{rot}\,\mathbb{F} = 0$$

$$\mathbb{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix},\quad \mathbb{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

(線積分の計算)

$= \cdots$

$$=\{f'(x)(\mathbb{a})b_1+g'(x)(\mathbb{a})b_2+h'(x)(\mathbb{a})b_3\}\,d_1d_2$$
$$-\{f'(x)(\mathbb{b})a_1+g'(x)(\mathbb{b})a_2+h'(x)(\mathbb{b})a_3\}\,d_1d_2$$

ただし

$$f'(x)=\frac{\partial f}{\partial x}(x)dx+\frac{\partial f}{\partial y}(x)dy+\frac{\partial f}{\partial z}(x)dz$$

$g'(x)=$

$h'(x)=$

(人足計算)

別解：プロの計算

$= \cdots$

$$=\underbrace{\left\{\begin{pmatrix}f'(x)(\mathbb{a})\\g'(x)(\mathbb{a})\\h'(x)(\mathbb{a})\end{pmatrix}\cdot\mathbb{b}-\begin{pmatrix}f'(x)(\mathbb{b})\\g'(x)(\mathbb{b})\\h'(x)(\mathbb{b})\end{pmatrix}\cdot\mathbb{a}\right\}}_{\varphi(\mathbb{a},\mathbb{b})}d_1d_2$$

$\varphi(\mathbb{a},\mathbb{b}) : \mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$

$$\begin{cases}\varphi(\mathbb{a}_1+\mathbb{a}_2,\mathbb{b})=\varphi(\mathbb{a}_1,\mathbb{b})+\varphi(\mathbb{a}_2,\mathbb{b})\\ \varphi(\alpha\mathbb{a},\mathbb{b})=\alpha\varphi(\mathbb{a},\mathbb{b})\end{cases}$$

$(\alpha\in\mathbb{R})$

∴ $\mathbb{a}$について線型

同様に $\mathbb{b}$について線形

$\Rightarrow$ 2重線型，交代

φは交代形式

$$\varphi = \alpha_1\, dy\wedge dz + \alpha_2\, dz\wedge dx + \alpha_3\, dx\wedge dy \qquad a_i \in \mathbb{R}$$

$\alpha_1,\ \alpha_2,\ \alpha_3$を決定すればよい.

$\rightarrow\ \mathbb{a} = \mathbb{e}_2,\ \mathbb{b} = \mathbb{e}_3$ （α_1が決定できる）

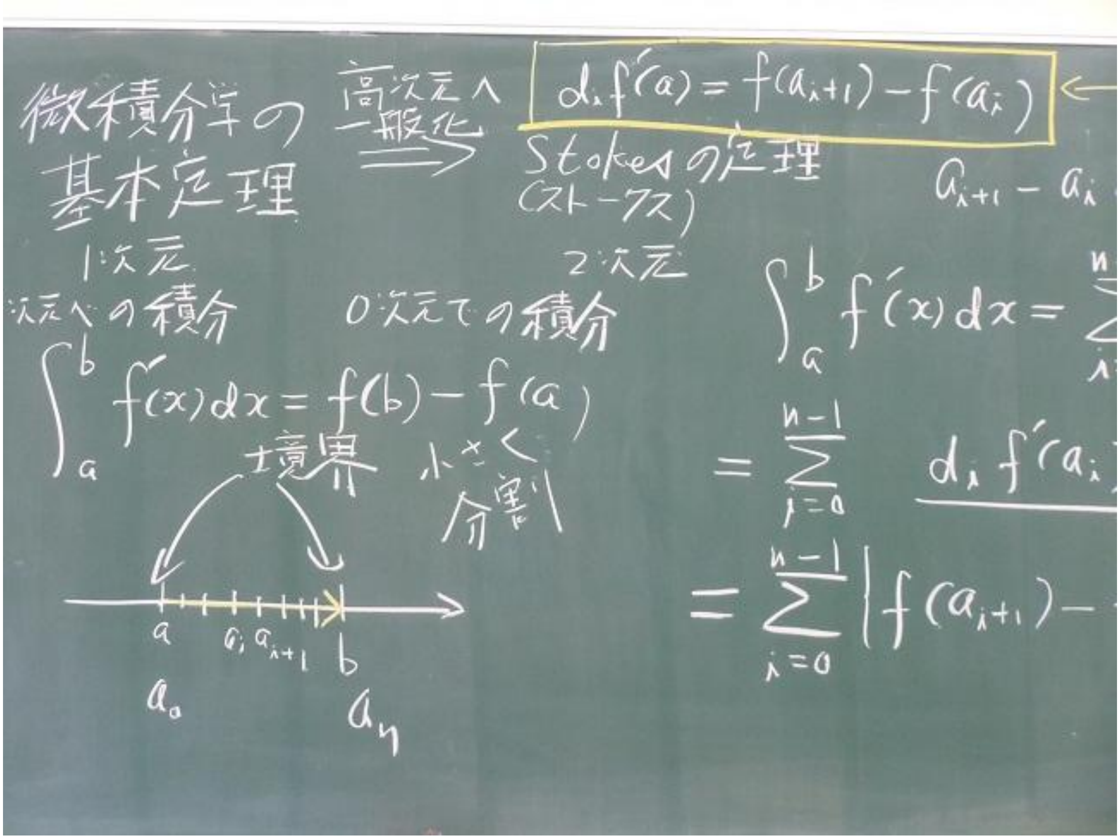
$$(dy\wedge dz)(\mathbb{e}_2, \mathbb{e}_3) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

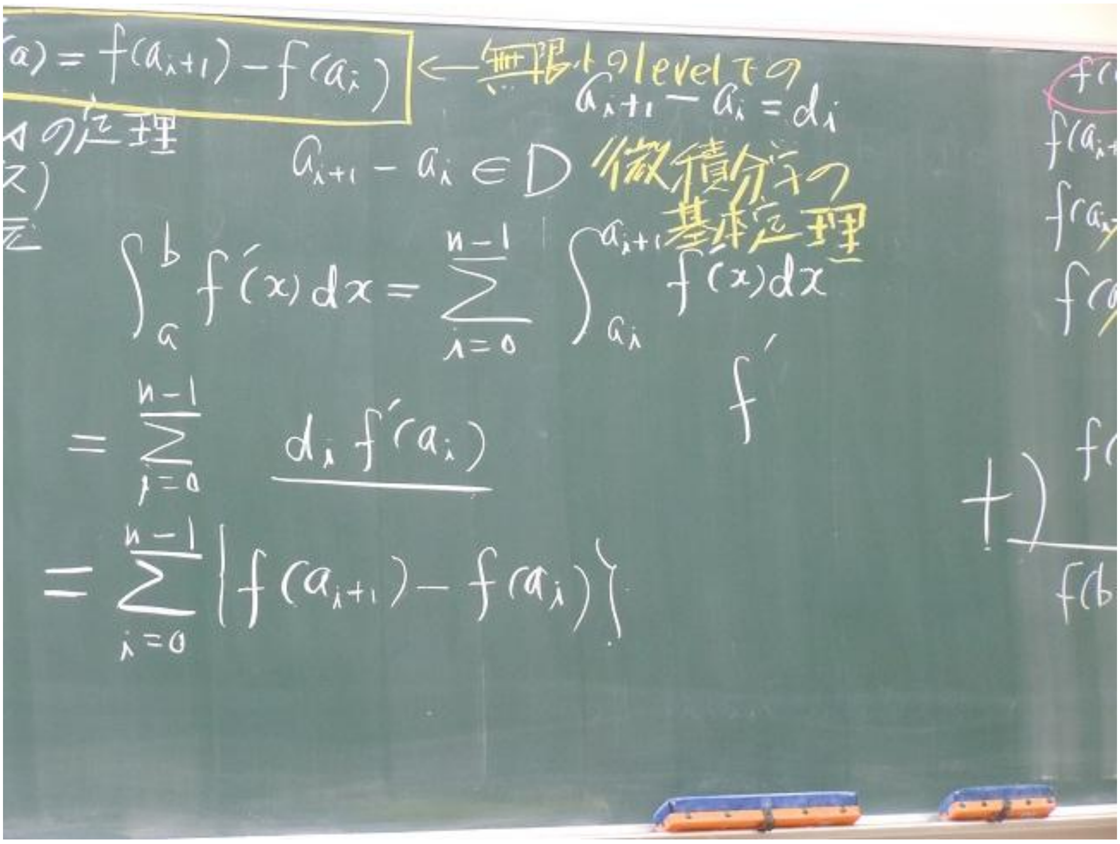
$$(dz\wedge dx)(\mathbb{e}_2, \mathbb{e}_3) = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

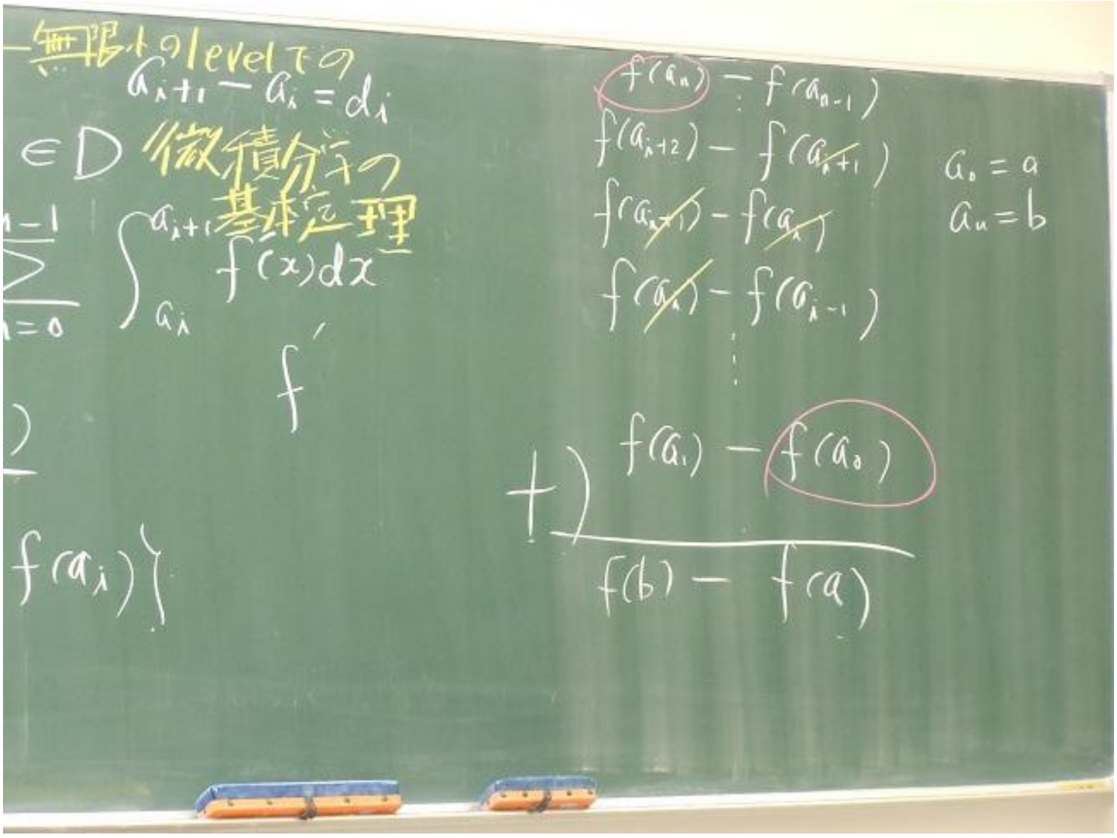
$$(dx\wedge dy)(\mathbb{e}_2, \mathbb{e}_3) = 0$$

$\rightarrow\ \mathbb{a} = \mathbb{e}_3\quad \mathbb{b} = \mathbb{e}_1\quad (\alpha_2)$

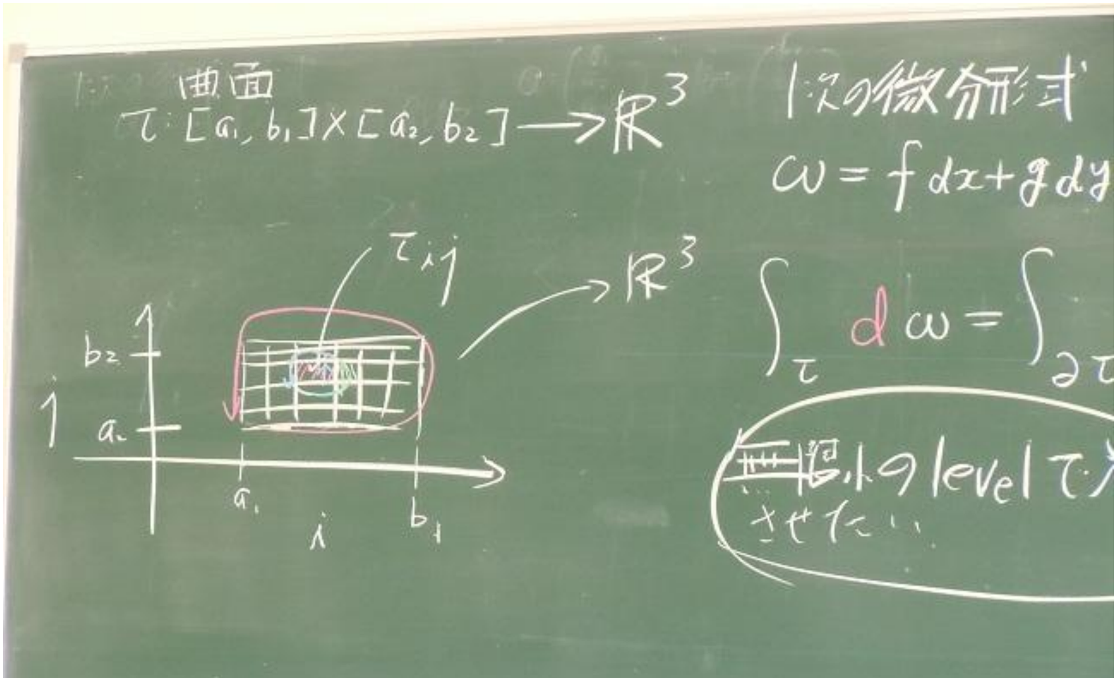
$\rightarrow\ \mathbb{a} = \mathbb{e}_1\quad \mathbb{b} = \mathbb{e}_2\quad (\alpha_3)$


微積分学の
基本定理
高次元へ
一般化
$d_i f'(a) = f(a_{i+1}) - f(a_i)$
Stokesの定理
(ストークス)
1次元
2次元
0次元での積分
$\int_a^b f'(x)dx = f(b) - f(a)$
境界
小さく
分割
a
a_i a_{i+1}
b
a_0
a_n
$a_{i+1} - a_i$
$\int_a^b f'(x)dx =$
$= \sum_{i=0}^{n-1} d_i f'(a_i)$
$= \sum_{i=0}^{n-1} \{f(a_{i+1}) -$


$(a) = f(a_{i+1}) - f(a_i)$
← 無限小のlevelでの
$a_{i+1} - a_i = d_i$
の定理
$a_{i+1} - a_i \in D$
微積分学の
基本定理
$\int_a^b f'(x)dx = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f'(x)dx$
f'
$= \sum_{i=0}^{n-1} d_i f'(a_i)$
$= \sum_{i=0}^{n-1} \{f(a_{i+1}) - f(a_i)\}$



無限小のlevelでの
$a_{i+1} - a_i = d_i$
$\in D$ 微積分の基本定理
$\sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(x)dx$
f'
$f(a_i)\}$
$f(a_n) - f(a_{n-1})$
$f(a_{i+2}) - f(\cancel{a_{i+1}})$
$f(\cancel{a_{i+1}}) - f(\cancel{a_i})$
$f(\cancel{a_i}) - f(a_{i-1})$
$a_0 = a$
$a_n = b$
$f(a_1) - f(a_0)$
$f(b) - f(a)$



曲面
$\tau : [a_1, b_1] \times [a_2, b_2] \longrightarrow \mathbb{R}^3$
1次の微分形式
$\omega = f dx + g dy$
τ_{ij}
$\mathbb{R}^3$
b_2
a_2
a_1
b_1
$\int_\tau d\omega = \int_{\partial\tau}$
無限小のlevelで
させたい

微分形式

$f\,dx + g\,dy + h\,dz$

$d\omega = \int_{\partial\tau} \omega$ τ_{ij}

のlevelで成立 仮定

$$\int_{\tau} d\omega = \sum_{i,j} \int_{\tau_{ij}} d\omega$$

$$= \sum_{i,j} \int_{\partial\tau_{ij}} \omega$$

$$= \int_{\partial\tau} \omega$$

$$d\omega = \sum_{i,j} \int_{\tau_{ij}} d\omega$$

無限小のlevelで成立(仮定)

$$= \sum_{i,j} \int_{\partial\tau_{ij}} \omega$$

$$= \int_{\partial\tau} \omega$$

前回のreport問題

τ 無限小の平行四辺形

$\int_{\partial\tau} \omega$ を計算

ベクトル解析 $f e_1 + g e_2 + h e_3 \longrightarrow \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) e_1 + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) e_2 + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) e_3$

$\updownarrow$

微分形式 $f\,dx + g\,dy + h\,dz \xrightarrow[\text{微分}]{d}$

$$\begin{pmatrix} \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \end{pmatrix} \cdot a \times b \qquad d_1 d_2$$

$$= \int_\tau \underbrace{\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy}_{d\omega}$$

ベクトル解析　rot(ation)

$$\mathrm{rot}\,\mathbf{f} = \nabla \times \mathbf{f}$$

ナブラ

nabla

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

← 作用素のベクトル

擬似ベクトル

$$\mathrm{grad}\,\varphi = \nabla \varphi$$

スカラー場

$$\mathbf{f} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \varphi = \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \\ \frac{\partial \varphi}{\partial z} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} \frac{\partial}{\partial y} & g \\ \frac{\partial}{\partial z} & h \end{vmatrix} \\ \\ \\ \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \\ \\ \\ \end{pmatrix}$$

$r = (x, y, z)$

$x^2 + y^2 = a^2$

$z = 0$

曲線

回転

$$\mathrm{rot}\, r = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \\ \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \\ \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \end{pmatrix} = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

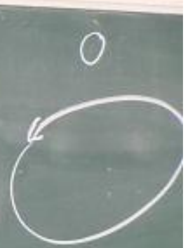
線→面

$$\int r\, dr = 0$$

$$= \int (\mathrm{rot}\, r)\, dS$$

保存力

$$f = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$



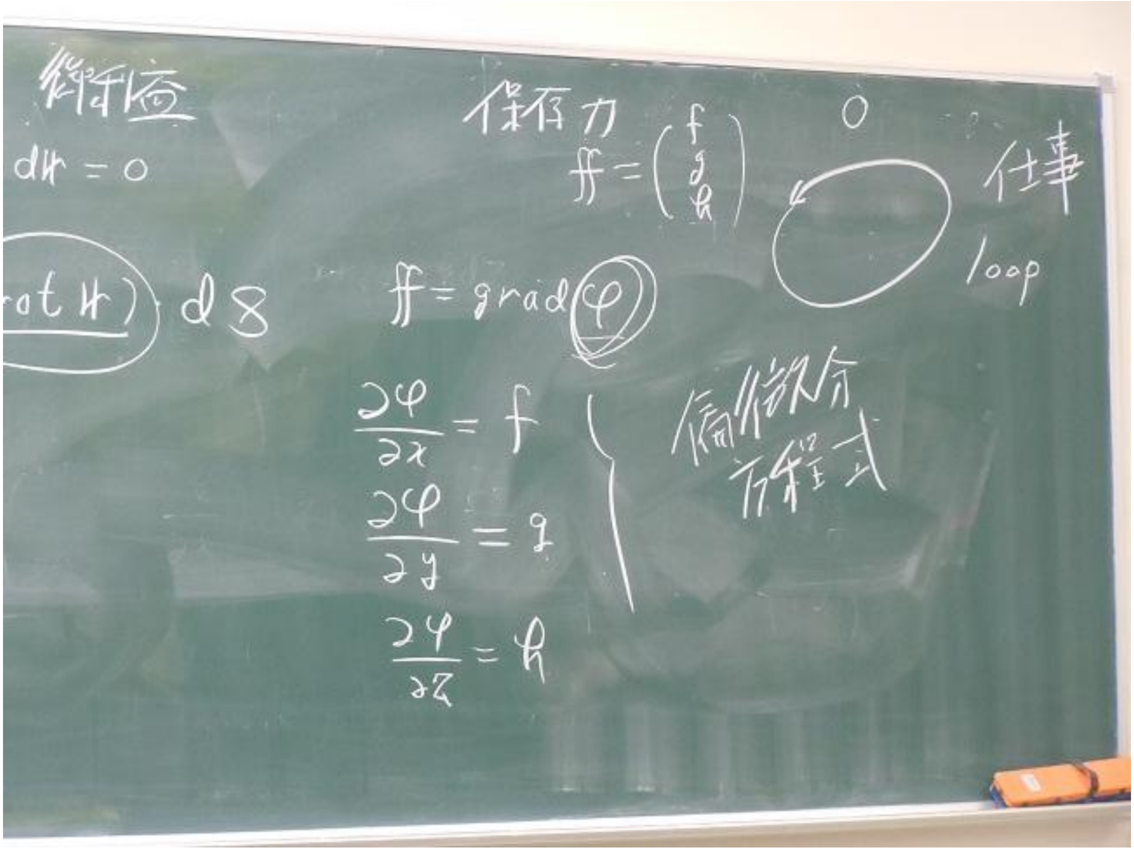
$$f = \mathrm{grad}\, \varphi$$

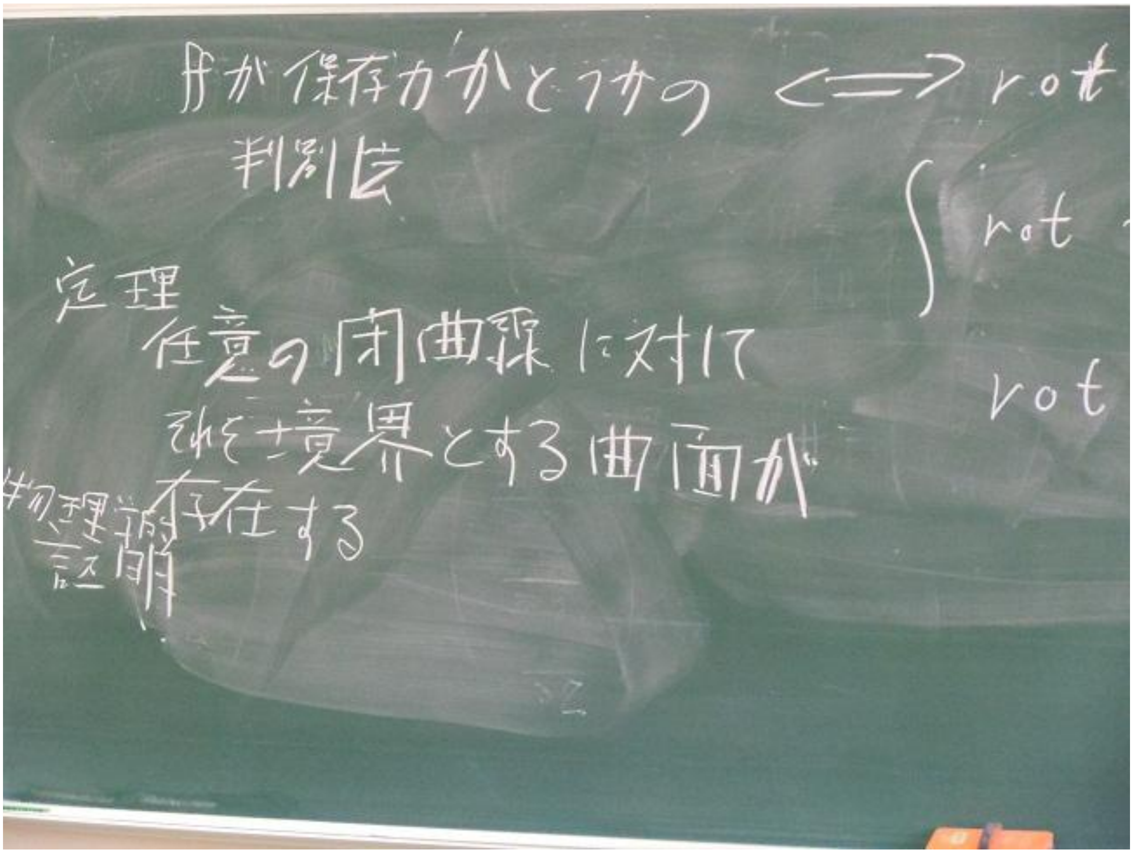
$$\frac{\partial \varphi}{\partial x} = f$$

$$\frac{\partial \varphi}{\partial y} = g$$

$$\frac{\partial \varphi}{\partial z} = h$$

偏微分方程式


dr = 0
rot r) dS
保存力
F = (f g h)
0
仕事
loop
F = grad φ
∂φ/∂x = f
∂φ/∂y = g
∂φ/∂z = h
偏微分方程式


Fが保存力かどうかの ⟺ rot
判別法
rot
rot
定理
任意の閉曲線に対して
それを境界とする曲面が
存在する
物理的証明

とつなの $\Longleftrightarrow$ rot $\mathbb{f}$ が消える

$$\int \mathrm{rot}\ \mathbb{f} = 0$$

…森に対して

…する曲面が

$$\mathrm{rot}\ \mathbb{f} = 0$$

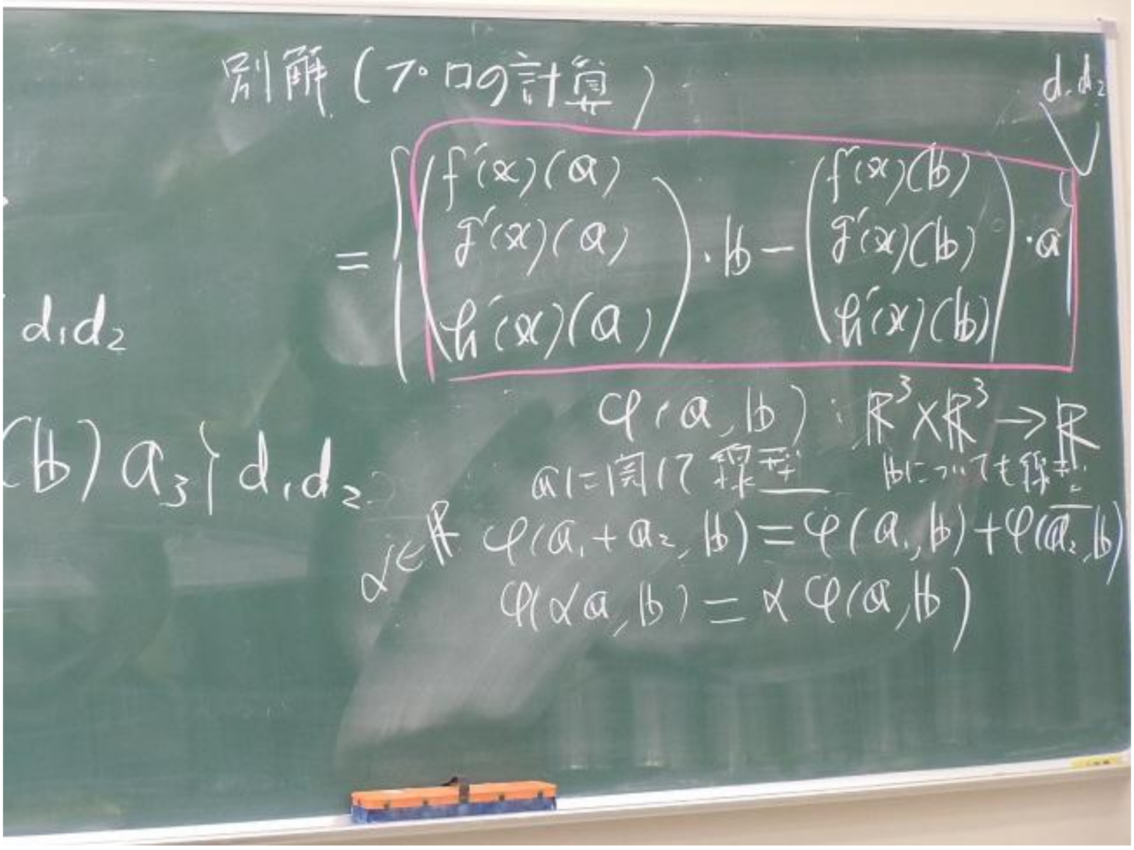
$\mathbb{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\mathbb{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ $x^2+y^2=a^2$ 人足計算

$$f'(x) = \frac{\partial f}{\partial x}(x)\,dx + \frac{\partial f}{\partial y}(x)\,dy + \frac{\partial f}{\partial z}(x)\,dz$$

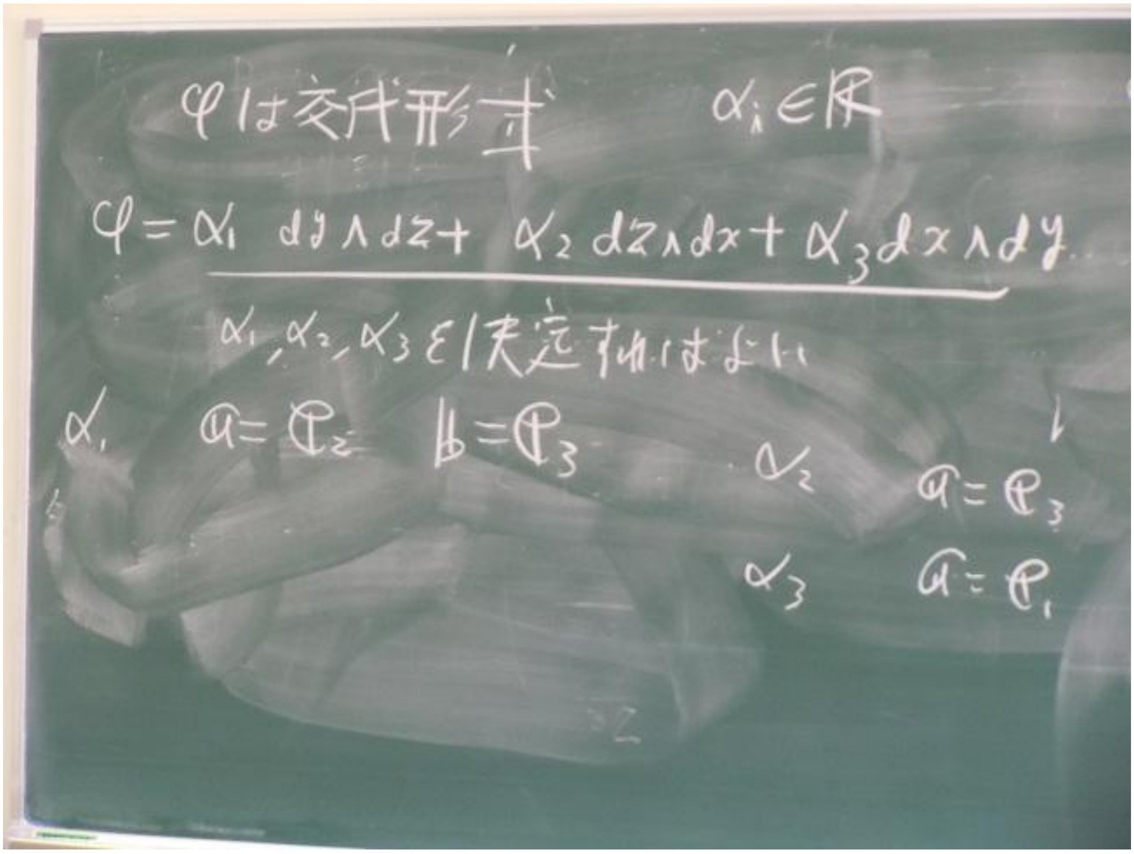
$$= \{ f'(x)(\mathbb{a})b_1 + g'(x)(\mathbb{a})b_2 + h'(x)(\mathbb{a})b_3 \} d_1 d$$

$$- \{ f'(x)(\mathbb{b})a_1 + g'(x)(\mathbb{b})a_2 + h'(x)(\mathbb{b})$$

$\left\{ \begin{array}{l} \text{2重線型} \\ \text{交代} \end{array} \right.$



別解（プロの計算）
$d_1 d_2$
$= \int \left\{ \begin{pmatrix} f'(x)(a) \\ g'(x)(a) \\ h'(x)(a) \end{pmatrix} \cdot b - \begin{pmatrix} f'(x)(b) \\ g'(x)(b) \\ h'(x)(b) \end{pmatrix} \cdot a \right\}$
$d_1 d_2$
$(b)\, a_3 \} d_1 d_2$
$\varphi(a, b) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$
aに関して線形　bについても線形
$\alpha \in \mathbb{R}$　$\varphi(a_1 + a_2, b) = \varphi(a_1, b) + \varphi(a_2, b)$
$\varphi(\alpha a, b) = \alpha \varphi(a, b)$



φは交代形式　$\alpha_i \in \mathbb{R}$
$\varphi = \alpha_1\, dy \wedge dz + \alpha_2\, dz \wedge dx + \alpha_3\, dx \wedge dy$
$\alpha_1, \alpha_2, \alpha_3$を決定すればよい
α_1　$a = e_2$　$b = e_3$
α_2　$a = e_3$
α_3　$a = e_1$

$$\varphi = \alpha_1\,dx + \alpha_2\,dy + \alpha_3\,dz \qquad e_1$$

$$\wedge dy \qquad (dy\wedge dz)(e_2, e_3) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} =$$

$$(dz\wedge dx)(e_2, e_3) = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} =$$

$= e_3 \qquad b = e_1$

$= e_1 \qquad b = e_2$

$$dy + \alpha_3\,dz \qquad e_1 \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$e_2, e_3) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$e_2, e_3) = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$$

$=$

$0 + 100$

D+ 97

2 97

3 + 97