2010. 9. 29

$$L(\mathbb{R}^2; \mathbb{R}^2; \mathbb{R}) = L(\mathbb{R}^2; L(\mathbb{R}^2; \mathbb{R}))$$

$\mathbb{R}^2 \times \mathbb{R}$ から $\mathbb{R}$ への二重線型写像の全体

$\mathbb{R}^2$ から $\mathbb{R}$ への二重線型写像全体

$\mathbb{R}^2$ から $L(\mathbb{R}^2; \mathbb{R})$ への二重線型写像の全体

線型空間 $\left\{\begin{array}{l}\text{足し算} \\ \text{スカラー倍}\end{array}\right\}$ の8つの定理

$$\varphi, \psi \in L(\mathbb{R}^2; \mathbb{R})$$

$$\begin{cases} \varphi \leftrightarrow (a_1, a_2) \\ \psi \leftrightarrow (b_1, b_2), \end{cases} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$(\varphi + \psi)(x) = \varphi(x) + \psi(x) \qquad \varphi + \psi \leftrightarrow (a_1 + b_1, a_2 + b_2)$$

$(\alpha \in \mathbb{R})$

$$(\alpha\varphi)(x) = \alpha\varphi(x) \qquad \alpha\varphi \leftrightarrow (\alpha a_1, \alpha a_2)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \underbrace{(a_1, a_2)}_{1\times 2\text{行列}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2階の微分

$f : \mathbb{R}^2 \to \mathbb{R}$

$f'(x) : \mathbb{R}^2 \to \mathbb{R}$ 線型写像

$f' : \mathbb{R}^2 \to L(\mathbb{R}^2; \mathbb{R})$

$f''(x) : \mathbb{R}^2 \to L(\mathbb{R}^2; \mathbb{R})$ 線型写像

$\in L(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$ ⟶ 二重線型写像として出てくる

対称

$a, b, x \in \mathbb{R}^2$

$f''(x)(a, b) = \partial_b \partial_a f(x)$

$f''(x)(b, a) = \partial_a \partial_b f(x)$

$d_1, d_2 \in d$

$\partial_b \partial_a f(x) d_1 d_2$

$= \{\partial_a f(x + b d_2) - \partial_a f(x)\} d_1$

$= f(x + b d_2 + a d_1) - f(x + b d_2) - f(x + a d_1) + f(x)$

$\partial_a \partial_b f(x) d_1 d_2$

$= \{\partial_b f(x + a d_1) - \partial_b f(x)\} d_2$

$= f(x + a d_1 + b d_2) - f(x + a d_1) - f(x + b d_2) + f(x)$

$\therefore \partial_b \partial_a f(x) d_1 d_2 = \partial_a \partial_b f(x) d_1 d_2$

線型代数

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$ "行列式"

$\downarrow \quad \downarrow$

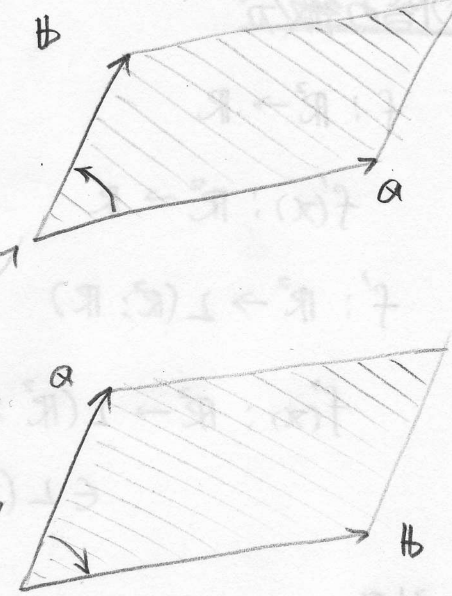
$\boldsymbol{a} \quad \boldsymbol{b}$ ： 平面の2つのベクトル

$S(\boldsymbol{a}, \boldsymbol{b}) \rightsquigarrow$ 符号つき面積

⇩

($\boldsymbol{a}$から$\boldsymbol{b}$とみたとき)

$\begin{cases} \text{反時計回り} & + \\ \text{時計回り} & - \end{cases}$

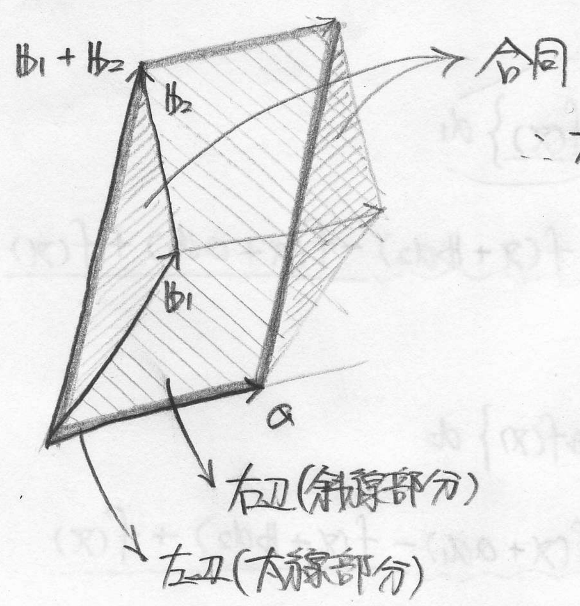


$S(\boldsymbol{a}, \boldsymbol{b}) = -S(\boldsymbol{b}, \boldsymbol{a}) \Rightarrow S(\boldsymbol{a}, \boldsymbol{a}) = 0 \qquad 2S(\boldsymbol{a}, \boldsymbol{a}) = 0$

$S(\alpha\boldsymbol{a}, \boldsymbol{b}) = \alpha S(\boldsymbol{a}, \boldsymbol{b})$

$S(\boldsymbol{a}, \beta\boldsymbol{b}) = \beta S(\boldsymbol{a}, \boldsymbol{b})$

$S(\boldsymbol{a}, \boldsymbol{b}_1 + \boldsymbol{b}_2) = S(\boldsymbol{a}, \boldsymbol{b}_1) + S(\boldsymbol{a}, \boldsymbol{b}_2)$ は成り立つか？



$S(\boldsymbol{a}_1 + \boldsymbol{a}_2, \boldsymbol{b}) = S(\boldsymbol{a}_1, \boldsymbol{b}) + S(\boldsymbol{a}_2, \boldsymbol{b})$ も同様に成り立つ

$\mathbb{a} = \mathbb{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbb{b} = \mathbb{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ のとき

$$\begin{cases} S(\mathbb{e}_1, \mathbb{e}_2) = 1 \rightarrow \text{正方形の面積} \\ S(\mathbb{e}_2, \mathbb{e}_1) = 1 \\ S(\mathbb{e}_1, \mathbb{e}_1) = 0 \\ S(\mathbb{e}_2, \mathbb{e}_2) = 0 \end{cases}$$

$$\mathbb{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 \mathbb{e}_1 + a_2 \mathbb{e}_2$$

$$\mathbb{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b_1 \mathbb{e}_1 + b_2 \mathbb{e}_2$$

$$\begin{aligned} S(\mathbb{a}, \mathbb{b}) &= S(a_1\mathbb{e}_1 + a_2\mathbb{e}_2,\ b_1\mathbb{e}_1 + b_2\mathbb{e}_2) \\ &= a_1 b_1 S(\mathbb{e}_1, \mathbb{e}_1) + a_1 b_2 S(\mathbb{e}_1, \mathbb{e}_2) \\ &\quad + a_2 b_1 S(\mathbb{e}_2, \mathbb{e}_1) + a_2 b_2 S(\mathbb{e}_2, \mathbb{e}_2) \\ &= a_1 b_2 - a_2 b_1 \quad \text{(対角成分の引き算)} \end{aligned}$$

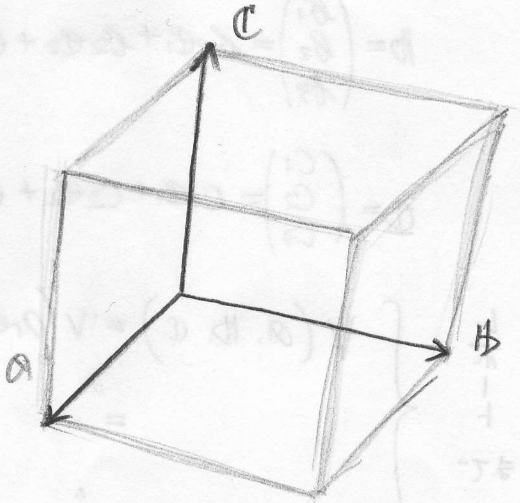
Sには二重線型性がある

(今度は空間内について考える)

$V(\mathbb{a}, \mathbb{b}, \mathbb{c})$ 〜 符号つきの $\mathbb{a}, \mathbb{b}, \mathbb{c}$ で張られる平行六面体の体積

⇓

($\mathbb{a}, \mathbb{b}$ で張られる平面の上か下か)



$\mathbb{c}$ が

$\begin{cases} \mathbb{a}\text{から}\mathbb{b}\text{へ右ねじを回すほうにある} & + \\ \quad〃\quad\text{の反対にある} & - \end{cases}$

言いかえると

($\mathbb{a}$. 親指, $\mathbb{b}$. 人差し指, $\mathbb{c}$. 中指)

$\begin{cases} \text{右手系} & + \\ \text{左手系} & - \end{cases}$

$$V(\mathbb{a},\mathbb{b},\mathbb{c}) = -V(\mathbb{b},\mathbb{a},\mathbb{c}) \Rightarrow V(\mathbb{a},\mathbb{a},\mathbb{b}) = 0$$

$$V(\alpha\mathbb{a},\mathbb{b},\mathbb{c}) = \alpha V(\mathbb{a},\mathbb{b},\mathbb{c})$$

$$V(\mathbb{a}_1+\mathbb{a}_2,\mathbb{b},\mathbb{c}) = V(\mathbb{a}_1,\mathbb{b},\mathbb{c}) + V(\mathbb{a}_2,\mathbb{b},\mathbb{c})$$

$\mathbb{a}=\mathbb{e}_1$, $\mathbb{b}=\mathbb{e}_2$, $\mathbb{c}=\mathbb{e}_3$のとき　$V(\mathbb{e}_1,\mathbb{e}_2,\mathbb{e}_3)=1$　(立方体)

$$V(\mathbb{e}_1,\mathbb{e}_2,\mathbb{e}_3)=1$$

$$V(\mathbb{e}_2,\mathbb{e}_1,\mathbb{e}_3)=-1$$

$$V(\mathbb{e}_3,\mathbb{e}_2,\mathbb{e}_1)=-1$$

$$V(\mathbb{e}_1,\mathbb{e}_3,\mathbb{e}_2)=-1$$

$$V(\mathbb{e}_1,\mathbb{e}_3,\mathbb{e}_2)=-1$$

$V(\mathbb{e}_2,\mathbb{e}_3,\mathbb{e}_1)=1$　(∵ 2回入れかえてる)

$V(\mathbb{e}_3,\mathbb{e}_1,\mathbb{e}_2)=1$　(　〃　)

$$V(\mathbb{e}_1,\mathbb{e}_1,\mathbb{e}_3)=0$$

$\vdots$

$$\mathbb{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=a_1\mathbb{e}_1+a_2\mathbb{e}_2+a_3\mathbb{e}_3$$

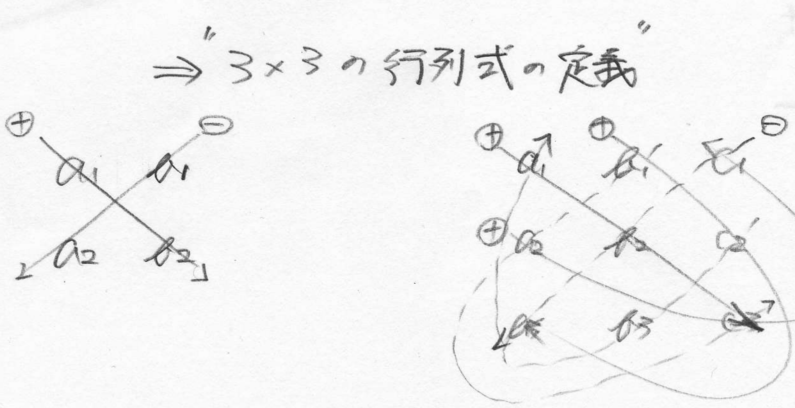
$$\mathbb{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=b_1\mathbb{e}_1+b_2\mathbb{e}_2+b_3\mathbb{e}_3$$

$$\mathbb{c}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=c_1\mathbb{e}_1+c_2\mathbb{e}_2+c_3\mathbb{e}_3$$

レポート
月まで

$$\left\{\begin{aligned} V(\mathbb{a},\mathbb{b},\mathbb{c}) &= V(a_1\mathbb{e}_1+a_2\mathbb{e}_2+a_3\mathbb{e}_3,\ b_1\mathbb{e}_1+b_2\mathbb{e}_2+b_3\mathbb{e}_3,\ c_1\mathbb{e}_1+c_2\mathbb{e}_2+c_3\mathbb{e}_3) \\ &= \cdots \end{aligned}\right.$$

⇒ "3×3 の行列式の定義"



内積（スカラー積）

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

外積（ベクトル積）

$\boldsymbol{a} \times \boldsymbol{b}$.

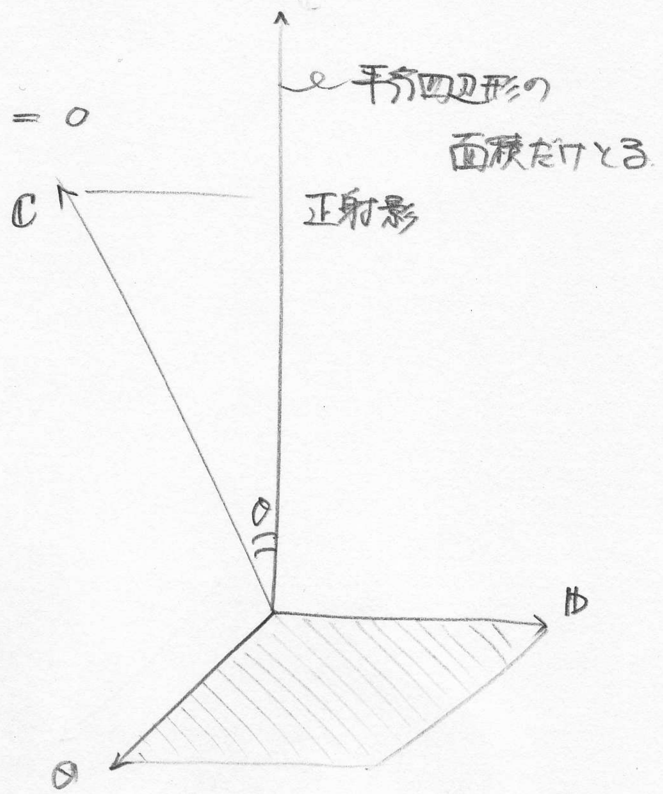
$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} \Rightarrow \boldsymbol{a} \times \boldsymbol{a} = 0$$

$$\alpha \boldsymbol{a} \times \boldsymbol{b} = \alpha (\boldsymbol{a} \times \boldsymbol{b})$$

$$\boldsymbol{a} \times \beta \boldsymbol{b} = \beta (\boldsymbol{a} \times \boldsymbol{b})$$

$$(\boldsymbol{a}_1, \boldsymbol{a}_2) \times \boldsymbol{b} = \boldsymbol{a}_1 \times \boldsymbol{b} + \boldsymbol{a}_2 \times \boldsymbol{b}$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$$
$$= V(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$$
$$= |\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}|$$



（a, bでできる平行四辺形の面積に高さをかけている）

平行六面体の体積に同じ

$$\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R}) = \mathcal{L}(\mathbb{R}^2; \mathcal{L}(\mathbb{R}^2; \mathbb{R}))$$

$\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ への
2重線型写像の
全体

$\mathbb{R}^2$から$\mathbb{R}$への
線型写像の
全体 $/\!/\ \mathbb{R}^2$

$\mathbb{R}^2$から$\mathcal{L}(\mathbb{R}^2; \mathbb{R})$への
線型写像の全体

線型空間 $\left\{ \begin{array}{l} \text{足し算} \\ \text{スカラー倍} \end{array} \right\}$ 8つの公理

$\varphi \longleftrightarrow (a_1, a_2)$

$\psi \longleftrightarrow (b_1, b_2)$

$\varphi, \psi \in \mathcal{L}(\mathbb{R}^2; \mathbb{R})$

$(\varphi + \psi)(x) \doteq \varphi(x) + \psi(x)$ $\quad \varphi + \psi \longleftrightarrow (a_1 + b_1, a_2 + b_2)$

$\alpha \in \mathbb{R}$ $\quad 2 \times 1$ $\quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

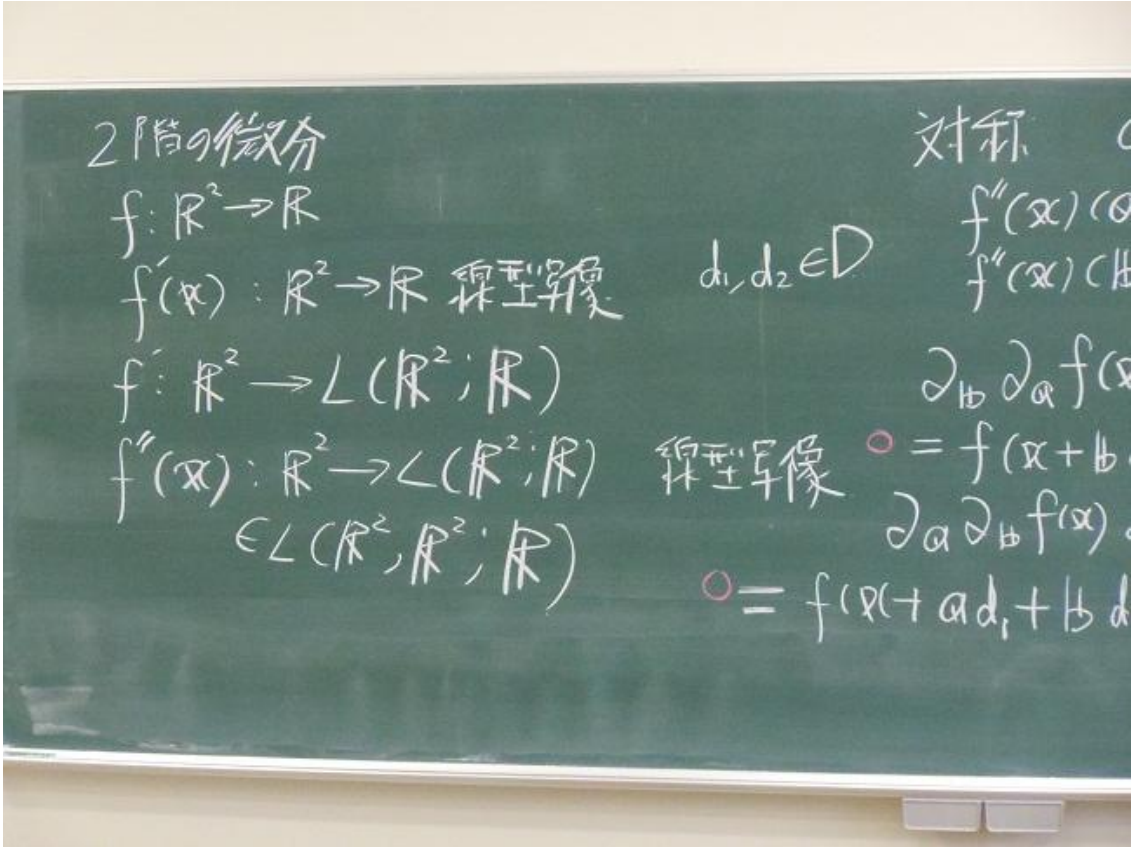
$(\alpha \varphi)(x) = \alpha \varphi(x)$

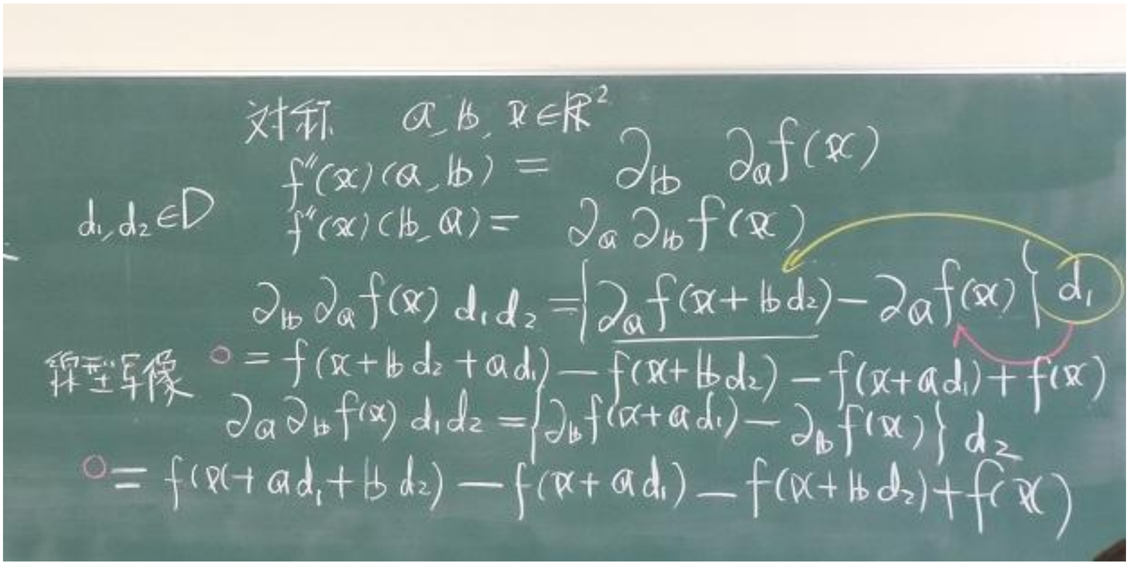
1×2 行列

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto (a_1, a_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$(a_1, a_2) + (b_1, b_2)$
$= (a_1 + b_1, a_2 + b_2)$

$\alpha(a_1, a_2) = (\alpha a_1, \alpha a_2)$

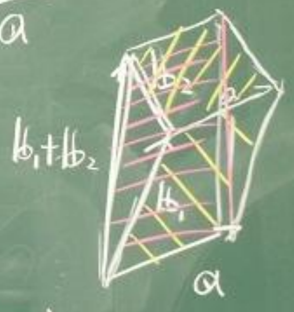

2階の微分
$f: \mathbb{R}^2 \to \mathbb{R}$
$f'(x): \mathbb{R}^2 \to \mathbb{R}$ 線型写像
$f': \mathbb{R}^2 \to L(\mathbb{R}^2; \mathbb{R})$
$f''(x): \mathbb{R}^2 \to L(\mathbb{R}^2; \mathbb{R})$ 線型写像
$\in L(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$
$d_1, d_2 \in D$
対称


対称 $a, b, x \in \mathbb{R}^2$
$f''(x)(a, b) = \partial_b \partial_a f(x)$
$d_1, d_2 \in D$
$f''(x)(b, a) = \partial_a \partial_b f(x)$
$\partial_b \partial_a f(x)\, d_1 d_2 = \{\partial_a f(x + b d_2) - \partial_a f(x)\} d_1$
線型写像
$= f(x + b d_2 + a d_1) - f(x + b d_2) - f(x + a d_1) + f(x)$
$\partial_a \partial_b f(x)\, d_1 d_2 = \{\partial_b f(x + a d_1) - \partial_b f(x)\} d_2$
$= f(x + a d_1 + b d_2) - f(x + a d_1) - f(x + b d_2) + f(x)$

線型代数　　行列式

$$\left|\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}\right| = a_{11}a_{22} - a_{21}a_{12}$$

$\mathbf{a}$　$\mathbf{b}$



$S(\mathbf{a}, \mathbf{b})$　符号のついた面積

反時計回り +

$$S(\mathbf{a}, \mathbf{b}) = -S(\mathbf{b}, \mathbf{a}) \Rightarrow S(\mathbf{a}, \mathbf{a}) = 0$$

$$2S(\mathbf{a}, \mathbf{a}) = 0$$

α, β スカラー

$$S(\alpha\mathbf{a}, \mathbf{b}) = \alpha S(\mathbf{a}, \mathbf{b})$$

$$S(\mathbf{a}, \beta\mathbf{b}) = \beta S(\mathbf{a}, \mathbf{b})$$

$(\ ), (\ )$

$$S(\mathbf{a}, \mathbf{b}_1 + \mathbf{b}_2) = S(\mathbf{a}, \mathbf{b}_1) + S(\mathbf{a}, \mathbf{b}_2)$$

$$S(\mathbf{a}_1 + \mathbf{a}_2, \mathbf{b}) = S(\mathbf{a}_1, \mathbf{b}) + S(\mathbf{a}_2, \mathbf{b})$$

$\mathbf{a} = \mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$　$\mathbf{b} = \mathbf{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$　$S(\mathbf{e}_1, \mathbf{e}_2) = 1$

$S(\mathbf{e}_1, \mathbf{e}_1) = S(\mathbf{e}_2, \mathbf{e}_2) = 0$　$S(\mathbf{e}_2, \mathbf{e}_1) = -1$

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 \qquad \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 \qquad \mathbf{e}_1 =$$

$$V(\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_3) = 0$$

$$S(\mathbf{a}, \mathbf{b}) = S(a_1\mathbf{e}_1 + a_2\mathbf{e}_2, b_1\mathbf{e}_1 + b_2\mathbf{e}_2)$$

$$= a_1b_1 \underbrace{S(\mathbf{e}_1, \mathbf{e}_1)}_{0} + a_1b_2 \underbrace{S(\mathbf{e}_1, \mathbf{e}_2)}_{1}$$

$$+ a_2b_1 \underbrace{S(\mathbf{e}_2, \mathbf{e}_1)}_{-1} + a_2b_2 \underline{S(\mathbf{e}_2, \mathbf{e}_2)}$$

$$= a_1b_2 - a_2b_1$$

$$V(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1) = 1$$

$$V(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2) = 1$$

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

$V(\mathbf{a}, \mathbf{b}, \mathbf{c}) =$ 行列式 $(\mathbf{a}, \mathbf{b}, \mathbf{c}$ の値

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

右手系 +

左手系 −

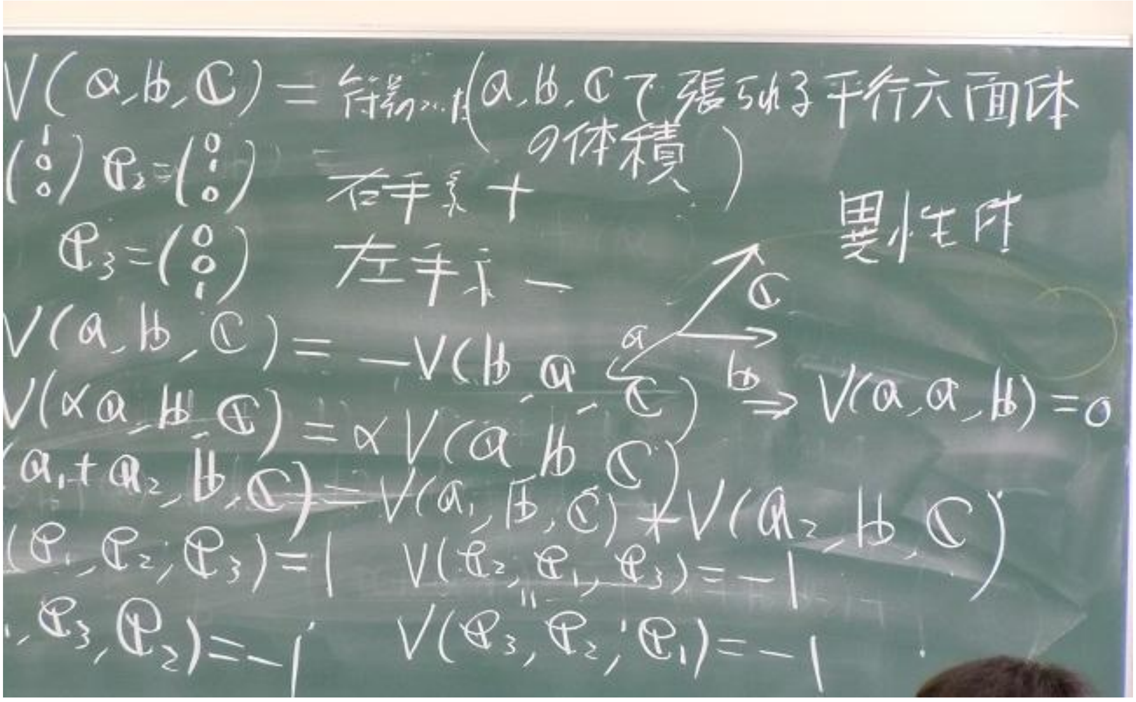
$$V(\mathbf{a}, \mathbf{b}, \mathbf{c}) = -V(\mathbf{b}, \mathbf{a}$$

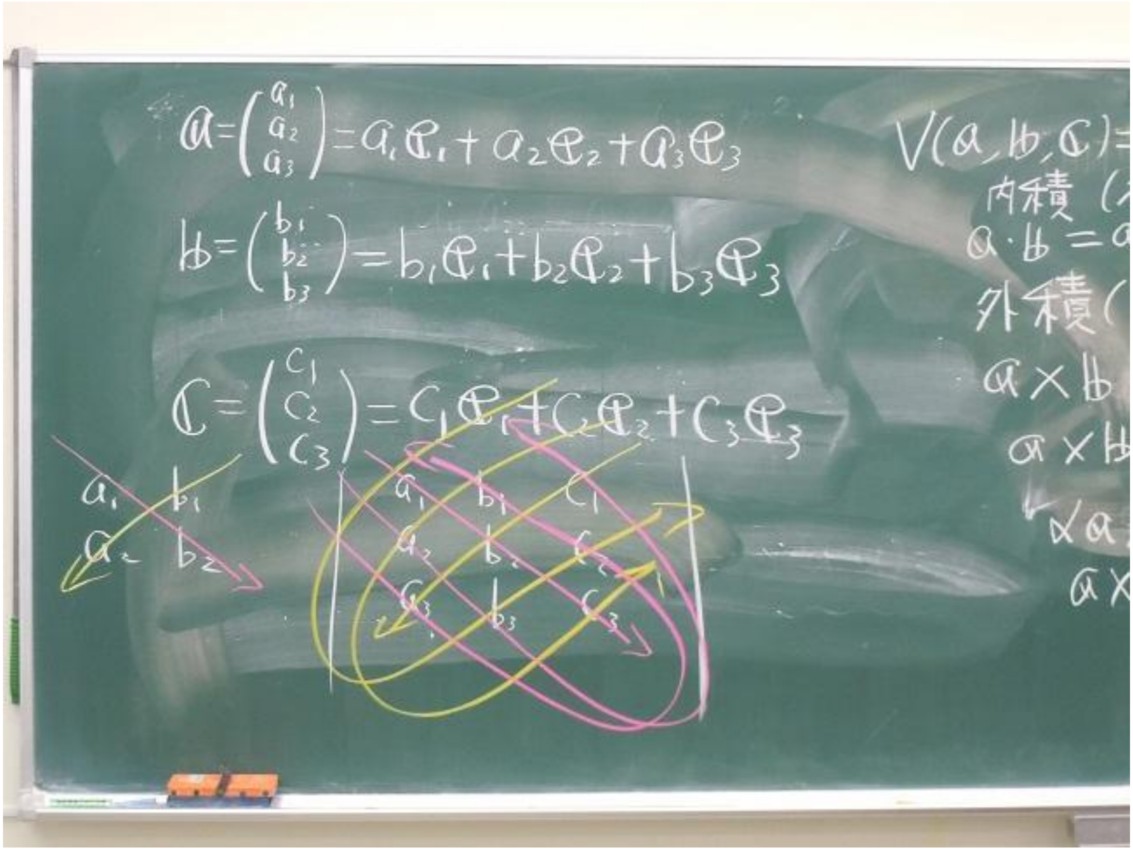
$$V(\alpha\mathbf{a}, \mathbf{b}, \mathbf{c}) = \alpha V(\mathbf{a}, \mathbf{b}$$

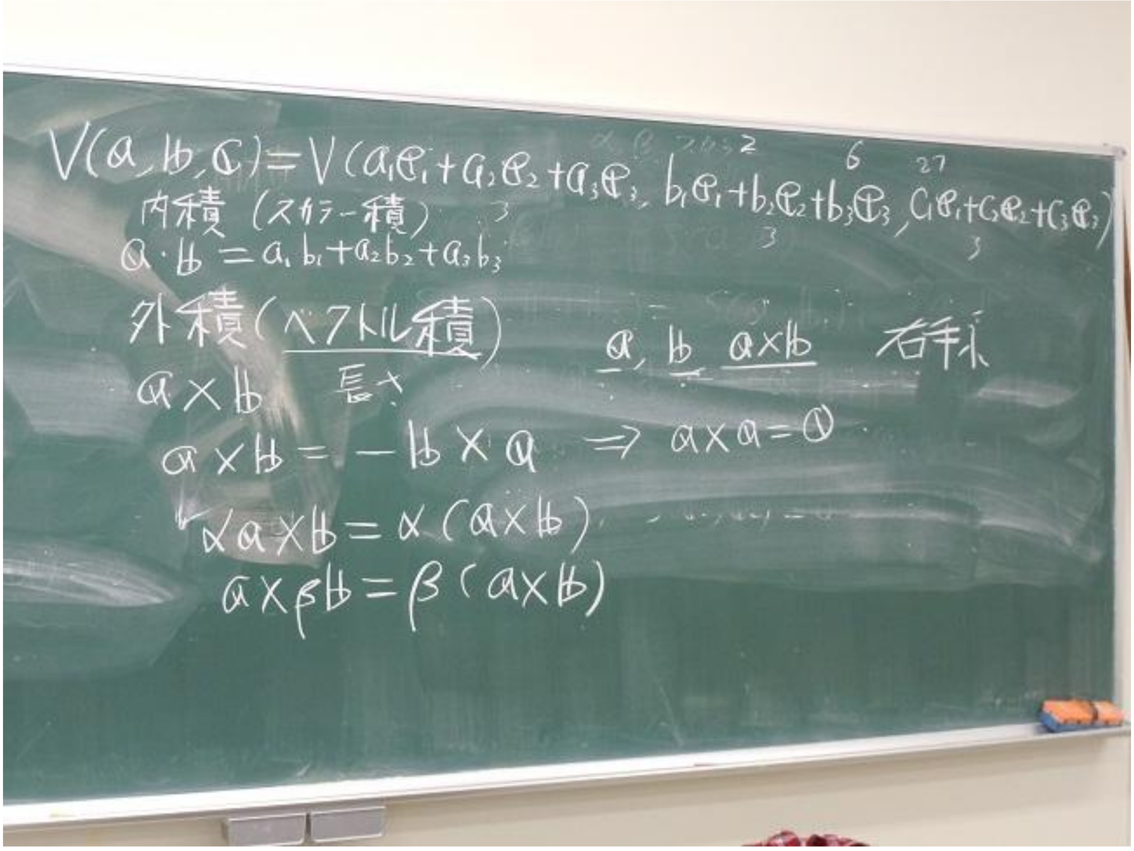
$$V(\mathbf{a}_1 + \mathbf{a}_2, \mathbf{b}, \mathbf{c}) = V(\mathbf{a}_1, \mathbf{b}, \mathbf{c}$$

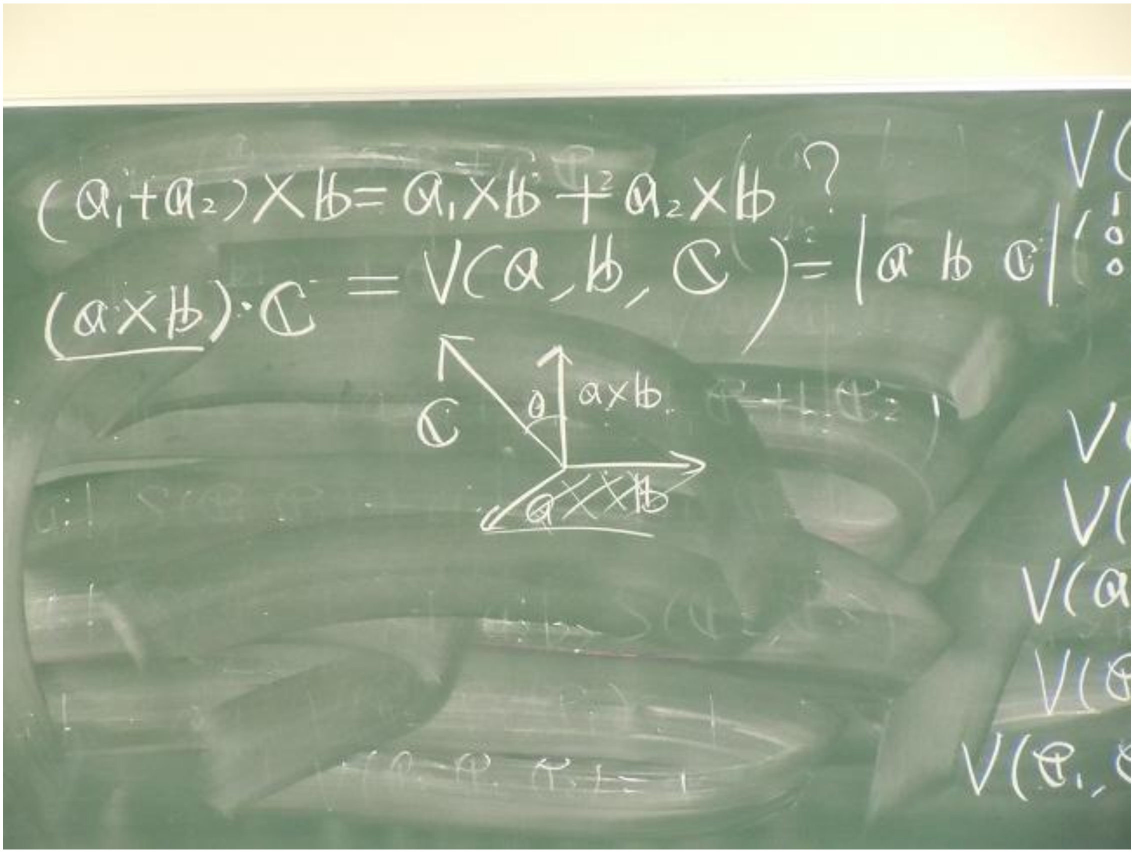
$$V(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1 \qquad V(\mathbf{e}_2, \mathbf{e}_1$$

$$V(\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2) = -1 \qquad V(\mathbf{e}_3, \mathbf{e}$$


V(a, b, c) = 符号つき (a, b, c で張られる平行六面体の体積)
e2 = (0 1 0)
e3 = (0 0 1)
右手系 +
左手系 −
V(a, b, c) = −V(b, a, c)
V(a, a, b) = 0
V(αa, b, c) = αV(a, b, c)
V(a1 + a2, b, c) = V(a1, b, c) + V(a2, b, c)
V(e1, e2, e3) = 1
V(e2, e1, e3) = −1
V(e3, e2, e1) = −1


a = (a1 a2 a3) = a1e1 + a2e2 + a3e3
b = (b1 b2 b3) = b1e1 + b2e2 + b3e3
c = (c1 c2 c3) = c1e1 + c2e2 + c3e3
V(a, b, c) =
内積
外積
a × b


V(a, b, c) = V(a₁e₁+a₂e₂+a₃e₃, b₁e₁+b₂e₂+b₃e₃, c₁e₁+c₂e₂+c₃e₃)
内積（スカラー積）
a·b = a₁b₁+a₂b₂+a₃b₃
外積（ベクトル積）
a, b a×b 右手系
a×b 長さ
a×b = −b×a ⇒ a×a = 0
αa×b = α(a×b)
a×βb = β(a×b)


(a₁+a₂)×b = a₁×b + a₂×b ?
(a×b)·c = V(a, b, c) = |a b c|
c
θ
a×b
a×b