9/15(水) 3限 微積分(生物学類)

合成関数の微分の公式

$f: \mathbb{R} \to \mathbb{R}$

$g: \mathbb{R} \to \mathbb{R}$ $\quad (g \circ f)'(x) = g'(f(x))f'(x)$

証明

$g \circ f(x+d) = g(f(x+d))$

$= g(\underbrace{f(x) + f'(x)d}_{\in D})$

$= g(f(x)) + g'(f(x))f'(x)d$

多変数

$f: \mathbb{R}^n \to \mathbb{R}^m$

$g: \mathbb{R}^m \to \mathbb{R}^l$

$g \circ f(x + ad)$ ← $\frac{d}{dx}f(x)$ の d

$= g \circ f(x) + d(g \circ f)(x)(a)d$

$g \circ f(x + ad) = g(f(x+ad))$

$= g(f(x) + \underbrace{df(x)(a)d}_{\in \mathbb{R}^m})$

↓この点で微分

$= g(f(x)) + dg(f(x))(df(x)(a))d$

結論 $\boxed{d(g \circ f) = \underbrace{dg(f(x))}_{l \times m} \circ \underbrace{df(x)}_{m \times n}}$

← 線型関数の合成と行列の掛け算は対応している

線型代数を使わない $l = m = n = 2$

$\frac{\partial z_1}{\partial x_1} = \frac{\partial z_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2}\frac{\partial y_2}{\partial x_1}$

$l = n = 2 \quad m = 3$

$\frac{\partial z_1}{\partial x_1} = \frac{\partial z_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2}\frac{\partial y_2}{\partial x_1} + \frac{\partial z_1}{\partial y_3}\frac{\partial y_3}{\partial x_1}$

高階の微分

1変数

$f: \mathbb{R} \to \mathbb{R}$　　$\mathbb{R}$から$\mathbb{R}$への線型写像 ←比例関数

微分すると $f'(x)$ (微分係数)が定まる

$f': \mathbb{R} \to \mathbb{R}$

微分すると $f''(x)$ が定まる

$f'': \mathbb{R} \to \mathbb{R}$

⋮

$x \in \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^m$

$f'(x) = df(x)$　　$\mathbb{R}^n$から$\mathbb{R}^m$への線型写像

$L(\mathbb{R}^n; \mathbb{R}^m)$:線型空間

$\mathbb{R}^n$から$\mathbb{R}^m$への線型写像全体

行列で表わすと m個 $\overbrace{\begin{pmatrix} & & \\ & & \end{pmatrix}}^{n個}$　$m \times n$次元

$f: \mathbb{R} \to \underline{L(\mathbb{R}; \mathbb{R})}$
$\qquad\qquad\quad \| $
$\qquad\qquad\quad \mathbb{R}$

$f: \mathbb{R}^n \to L(\mathbb{R}^n; \mathbb{R}^m)$

$\left.\begin{matrix} n=2 \\ m=1 \end{matrix}\right\}$の場合

$\varphi: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$　　$a, b \in \mathbb{R}^2$, $\alpha, \beta \in \mathbb{R}$

2重線型写像

$$\begin{cases} \varphi(a_1 + a_2, b) = \varphi(a_1, b) + \varphi(a_2, b) \\ \varphi(\alpha a, b) = \alpha \varphi(a, b) \\ \varphi(a, b_1 + b_2) = \varphi(a, b_1) + \varphi(a, b_2) \\ \varphi(a, \beta b) = \beta \varphi(a, b) \end{cases}$$

内積 $a \cdot b$

行列式 $\det(a\ b)$

$|a\ b|$

$|a_1 + a_2, b| = |a_1, b| + |a_2, b|$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 e_1 + a_2 e_2 \qquad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b_1 e_1 + b_2 e_2$

$$\varphi(a, b) = \varphi(a_1 e_1 + a_2 e_2, b_1 e_1 + b_2 e_2)$$
$$= a_1 b_1 \underline{\varphi(e_1, e_1)} + a_1 b_2 \underline{\varphi(e_1, e_2)} + a_2 b_1 \underline{\varphi(e_2, e_1)} + a_2 b_2 \underline{\varphi(e_2, e_2)}$$

$$(a_1, a_2) \underbrace{\begin{pmatrix} \varphi(e_1, e_1) & \varphi(e_1, e_2) \\ \varphi(e_2, e_1) & \varphi(e_2, e_2) \end{pmatrix}}_{2\times 2\text{の行列}} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 1\times 1 \text{ の行列}$$

内積

$\varphi(a, b) = \varphi(b, a)$ 対称 $\xleftrightarrow[\mathrm{I}(1)]{\text{レポート}}$ $\varphi(e_1, e_2) = \varphi(e_2, e_1)$

行列式

$\varphi(a, b) = -\varphi(b, a)$ 反対称 $\xleftrightarrow{\mathrm{I}(2)}$ $\begin{cases} \varphi(e_1, e_2) = -\varphi(e_2, e_1) \\ \varphi(e_1, e_1) = \varphi(e_2, e_2) = 0 \end{cases}$

一般の2重線型写像

$\varphi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$x \in \mathbb{R}^2 \longmapsto \varphi(x, \cdot) : \mathbb{R}^2 \to \mathbb{R} \quad \in L(\mathbb{R}^2; \mathbb{R})$

$\underbrace{y \in \mathbb{R}^2 \longmapsto \varphi(x, y) \in \mathbb{R}}_{\text{線型写像}}$

$$\begin{cases} \varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y) \\ \varphi(\alpha x, y) = \alpha \varphi(x, y) \end{cases}$$

$\mathbb{R}^2 \to L(\mathbb{R}^2; \mathbb{R}) \Rightarrow$ 線型

結論

$L(\mathbb{R}^2; L(\mathbb{R}^2; \mathbb{R}))$ 4次元

$= L(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$ 4次元 レポートII

$\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ への2重線型写像の全体

$f: \mathbb{R}^2 \to \mathbb{R}$

$f'(x) = df(x) \in L(\mathbb{R}^2; \mathbb{R})$

$f': \mathbb{R}^2 \to L(\mathbb{R}^2; \mathbb{R})$　一般には線型ではない

$$f''(x) \in L(\mathbb{R}^2; L(\mathbb{R}^2; \mathbb{R}))$$
$$\parallel$$
$$L(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$$

2×2の行列

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} \end{pmatrix}$$

合成関数の微分の公式

1変数の場合 (高校)

$f : \mathbb{R} \to \mathbb{R}$

$g : \mathbb{R} \to \mathbb{R}$

$$(g\circ f)'(x) = g'(f(x))f'(x)$$

証明　$|x|$

$$g\circ f\,(x+d) = g(f(x+d))$$

$$= g(f(x) + \underbrace{f'(x)\,d}_{\in D})$$

$$= g(f(x)) + g'(f(x))\,\underline{f'(x)\,d}$$

線型代数で使わない　$l=m=n=2$

$$\frac{\partial z_1}{\partial x_1} = \frac{\partial z_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2}\frac{\partial y_2}{\partial x_1}$$

多変数 $a, x \in \mathbb{R}^n$

$f : \mathbb{R}^n \to \mathbb{R}^m$

$g : \mathbb{R}^m \to \mathbb{R}^l$

$$g\circ f\,(x + a d)$$

$$= g\circ f(x) + d(g\circ f)(x)(a)\,d$$

結論

$$\boxed{d(g\circ f) = \underbrace{dg(f(x))}_{l\times m} \circ \underbrace{df(x)}_{m\times n}}$$

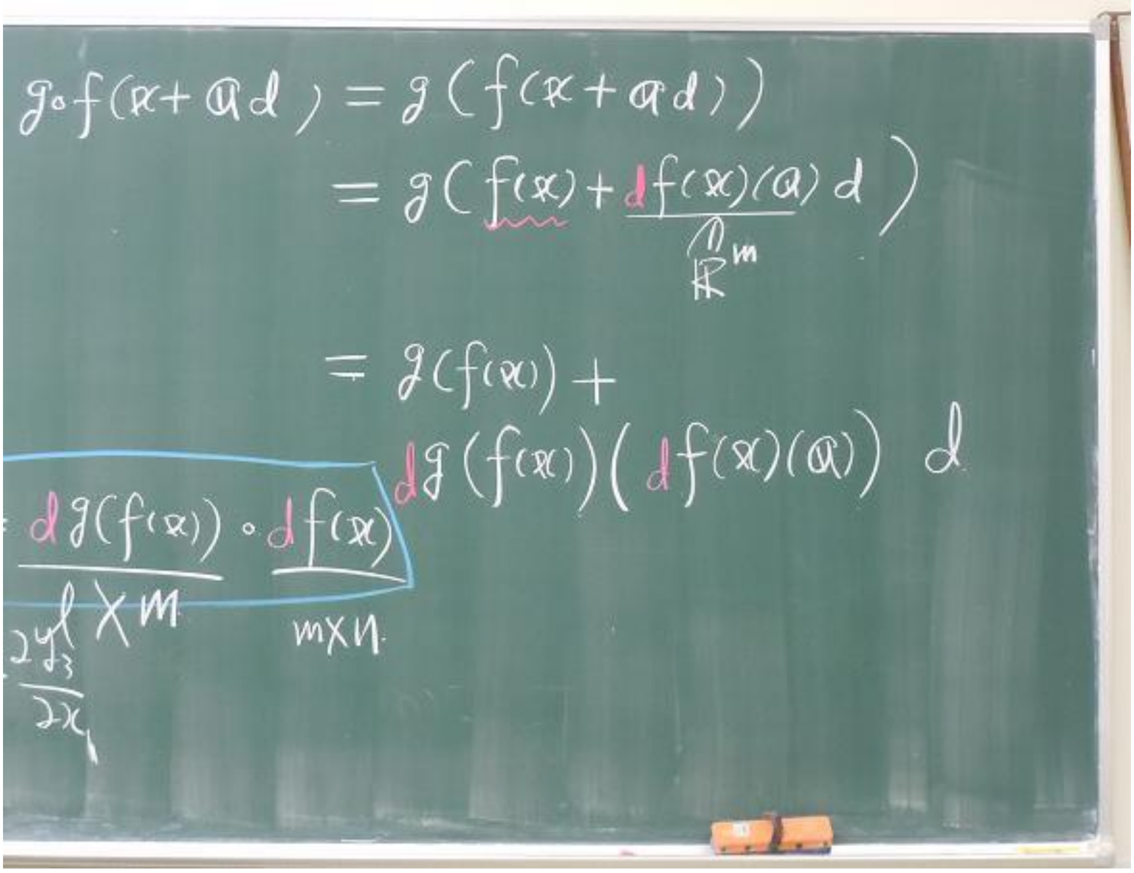
$l=n=2 \quad m=3$

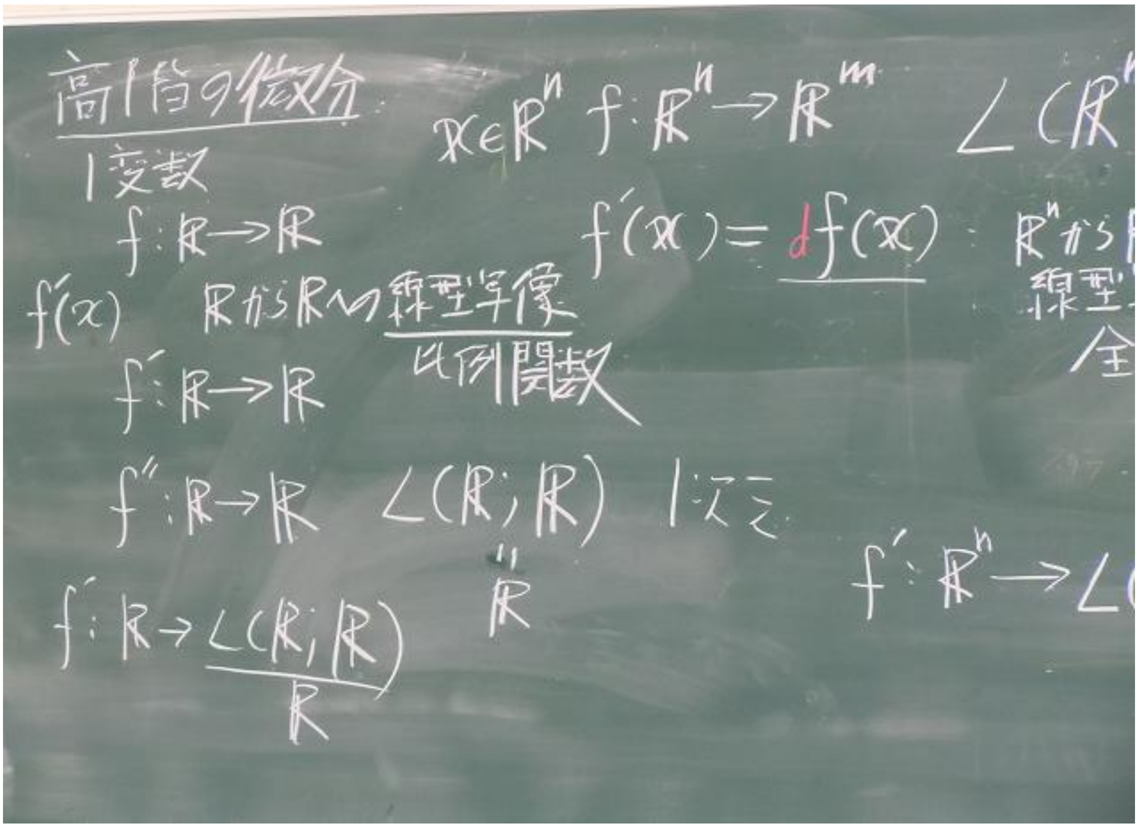
$$\frac{\partial z_1}{\partial x_1} = \frac{\partial z_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2}\frac{\partial y_2}{\partial x_1} + \frac{\partial z_1}{\partial y_3}\frac{\partial y_3}{\partial x_1}$$

$$g\circ f(x + a d) = g(f(x + a d))$$

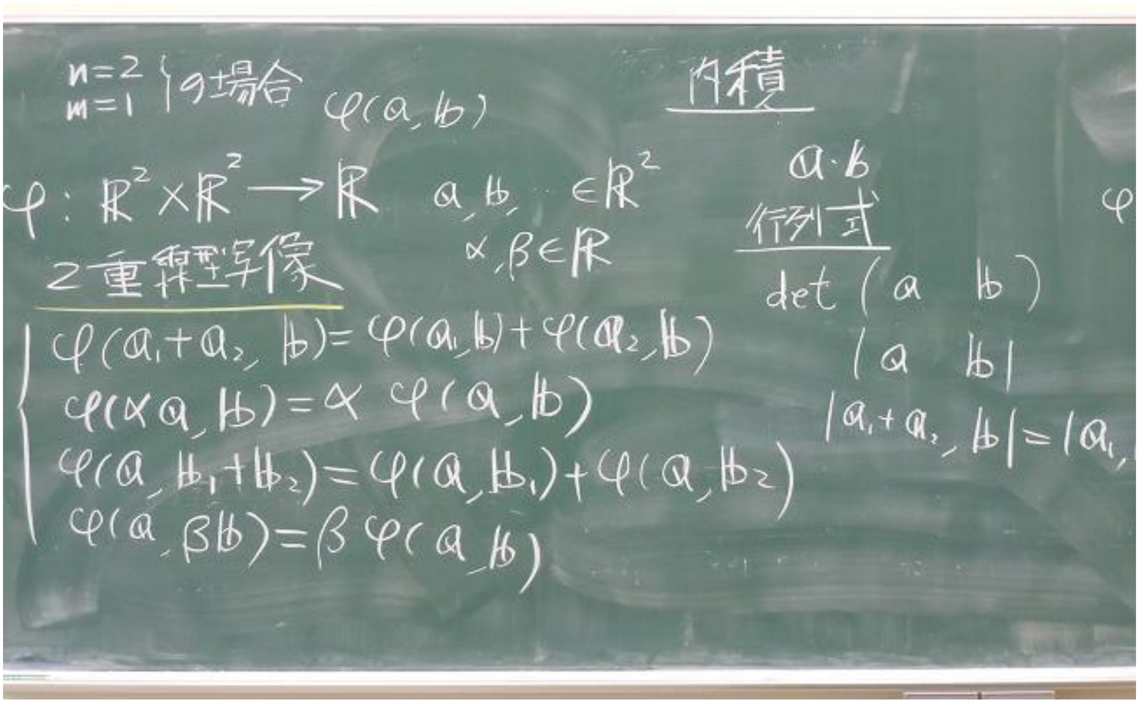
$$= g(f(x) + \underbrace{df(x)(a)\,d}_{\in \mathbb{R}^m})$$

$$= g(f(x)) + dg(f(x))(df(x)(a))$$

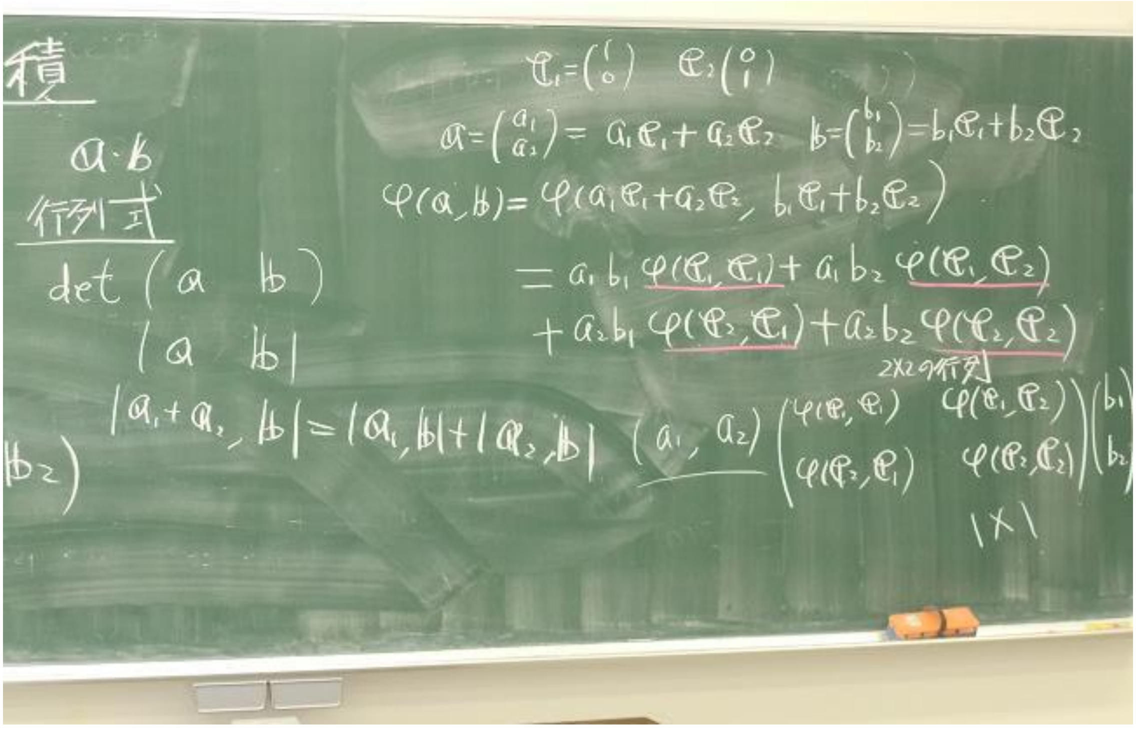

$g \circ f(x + \alpha d) = g(f(x + \alpha d))$
$= g(f(x) + \underbrace{df(x)(\alpha)}_{\in \mathbb{R}^m} d)$
$= g(f(x)) + dg(f(x))(df(x)(\alpha))\, d$
$\underbrace{dg(f(x))}_{l \times m} \circ \underbrace{df(x)}_{m \times n}$
$\frac{\partial y_3}{\partial x_1}$

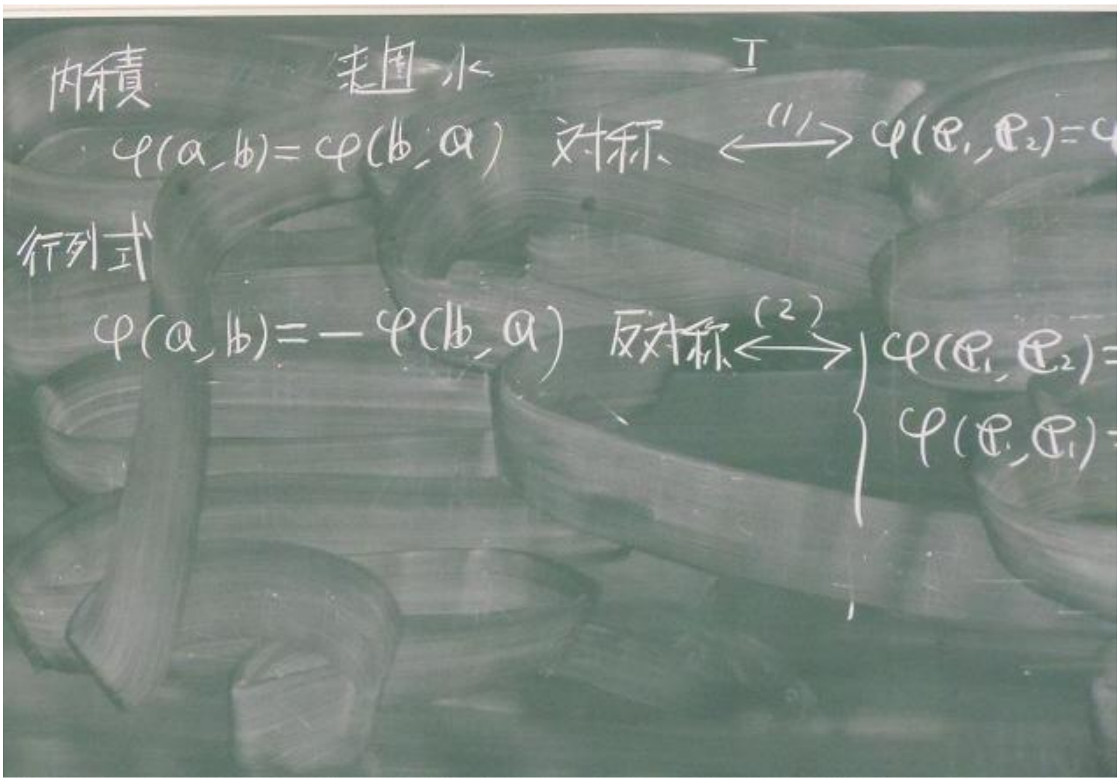

高階の微分
1変数
$f: \mathbb{R} \to \mathbb{R}$
$f'(x)$ $\mathbb{R}$から$\mathbb{R}$への線型写像
比例関数
$f': \mathbb{R} \to \mathbb{R}$
$f'': \mathbb{R} \to \mathbb{R}$ $L(\mathbb{R}; \mathbb{R})$ 1次元
$\mathbb{R}$
$f': \mathbb{R} \to \underbrace{L(\mathbb{R}; \mathbb{R})}_{\mathbb{R}}$
$x \in \mathbb{R}^n$ $f: \mathbb{R}^n \to \mathbb{R}^m$
$f'(x) = \underline{df(x)}$
$\mathbb{R}^n$から
線型
全
$f': \mathbb{R}^n \to L($

$\mathbb{R}^n \to \mathbb{R}^m$
$(x) = df(x)$
$\mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$ 線型空間
$\mathbb{R}^n$から$\mathbb{R}^m$への線型写像の全体
行列で表すと
$m \times n$
n個
m個
mn
1次元
$f' : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n;\mathbb{R}^m)$
$m \times n$次元



$n=2$, $m=1$ の場合
$\varphi(\mathbf{a},\mathbf{b})$
内積
$\varphi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$
$\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$
$\alpha, \beta \in \mathbb{R}$
$\mathbf{a} \cdot \mathbf{b}$
行列式
2重線型写像
$\det(\mathbf{a}\ \mathbf{b})$
$|\mathbf{a}\ \mathbf{b}|$
$\varphi(\mathbf{a}_1+\mathbf{a}_2, \mathbf{b}) = \varphi(\mathbf{a}_1,\mathbf{b}) + \varphi(\mathbf{a}_2,\mathbf{b})$
$\varphi(\alpha\mathbf{a}, \mathbf{b}) = \alpha\,\varphi(\mathbf{a},\mathbf{b})$
$\varphi(\mathbf{a}, \mathbf{b}_1+\mathbf{b}_2) = \varphi(\mathbf{a},\mathbf{b}_1) + \varphi(\mathbf{a},\mathbf{b}_2)$
$\varphi(\mathbf{a}, \beta\mathbf{b}) = \beta\,\varphi(\mathbf{a},\mathbf{b})$
$|\mathbf{a}_1+\mathbf{a}_2, \mathbf{b}| = |\mathbf{a}_1,$


積
$\mathbb{a}\cdot\mathbb{b}$
行列式
$\det(\mathbb{a}\ \ \mathbb{b})$
$|\mathbb{a}\ \ \mathbb{b}|$
$|\mathbb{a}_1+\mathbb{a}_2, \mathbb{b}| = |\mathbb{a}_1, \mathbb{b}| + |\mathbb{a}_2, \mathbb{b}|$
$\mathbb{e}_1=\begin{pmatrix}1\\0\end{pmatrix}$ $\mathbb{e}_2\begin{pmatrix}0\\1\end{pmatrix}$
$\mathbb{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix}=a_1\mathbb{e}_1+a_2\mathbb{e}_2$ $\mathbb{b}=\begin{pmatrix}b_1\\b_2\end{pmatrix}=b_1\mathbb{e}_1+b_2\mathbb{e}_2$
$\varphi(\mathbb{a},\mathbb{b})=\varphi(a_1\mathbb{e}_1+a_2\mathbb{e}_2, b_1\mathbb{e}_1+b_2\mathbb{e}_2)$
$=a_1b_1\varphi(\mathbb{e}_1,\mathbb{e}_1)+a_1b_2\varphi(\mathbb{e}_1,\mathbb{e}_2)$
$+a_2b_1\varphi(\mathbb{e}_2,\mathbb{e}_1)+a_2b_2\varphi(\mathbb{e}_2,\mathbb{e}_2)$
2×2の行列
$(a_1, a_2)\begin{pmatrix}\varphi(\mathbb{e}_1,\mathbb{e}_1) & \varphi(\mathbb{e}_1,\mathbb{e}_2)\\ \varphi(\mathbb{e}_2,\mathbb{e}_1) & \varphi(\mathbb{e}_2,\mathbb{e}_2)\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix}$
$|\times|$


内積
I
$\varphi(\mathbb{a},\mathbb{b})=\varphi(\mathbb{b},\mathbb{a})$ 対称 $\overset{(1)}{\longleftrightarrow}$ $\varphi(\mathbb{e}_1,\mathbb{e}_2)=$
行列式
$\varphi(\mathbb{a},\mathbb{b})=-\varphi(\mathbb{b},\mathbb{a})$ 反対称 $\overset{(2)}{\longleftrightarrow}$ $\varphi(\mathbb{e}_1,\mathbb{e}_2)=$
$\varphi(\mathbb{e}_1,\mathbb{e}_1)$

I

$\overset{(1)}{\longleftrightarrow} \varphi(e_1, e_2) = \varphi(e_2, e_1)$

$\overset{(2)}{\longleftrightarrow} \begin{cases} \varphi(e_1, e_2) = -\varphi(e_2, e_1) \\ \varphi(e_1, e_1) = \varphi(e_2, e_2) = 0 \end{cases}$

一般の2重線型写像

$\varphi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$x \in \mathbb{R}^2 \longmapsto \varphi(x,$

$\begin{cases} \varphi(x_1 + x_2, y) = \varphi(x_1, \\ \varphi(\alpha x, y) = \alpha \varphi(x \end{cases}$

$\mathbb{R}^2 \to L(\mathbb{R}^2; \mathbb{R})$

一般の2重線型写像

$\varphi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$x \in \mathbb{R}^2 \longmapsto \varphi(x, \cdot) \in L(\mathbb{R}^2; \mathbb{R})$

$\varphi(x, \cdot) : \mathbb{R}^2 \to \mathbb{R}$

$y \in \mathbb{R}^2 \longmapsto \varphi(x, y) \in \mathbb{R}$ 線型写像

$\begin{cases} \varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y) \\ \varphi(\alpha x, y) = \alpha \varphi(x, y) \end{cases}$

$\mathbb{R}^2 \to L(\mathbb{R}^2; \mathbb{R})$ 線型