

9/15(水) 3限 微積分(生物学類)

### 合成関数の微分の公式

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$g: \mathbb{R} \rightarrow \mathbb{R}$$

$$(g \circ f)'(x) = g'(f(x)) f'(x)$$

証明

$$\begin{aligned} g \circ f(x+d) &= g(f(x+d)) \\ &= g(f(x) + \underline{f'(x)d}) \quad \text{d} \\ &= g(f(x)) + \underline{g'(f(x))f'(x)d} \end{aligned}$$

### 多変数

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$g: \mathbb{R}^m \rightarrow \mathbb{R}^l$$

$$\begin{aligned} \textcircled{1} \quad g \circ f(x + a_1 d) &\leftarrow \frac{d}{dx} f(x) \circ d \\ &= g \circ f(x) + d(g \circ f)(x)(a_1) d \end{aligned}$$

$$\begin{aligned} g \circ f(x + a_1 d) &= g(f(x + a_1 d)) \\ &= g(\underline{f(x)} + \underline{df(x)(a_1) d}) \\ &\quad \downarrow \text{この点で微分 } \mathbb{R}^m \\ &= g(f(x)) + d(g(f(x))(df(x)(a_1)) d) \end{aligned}$$

結論  $d(g \circ f) = \underline{dg(f(x))} \circ \underline{df(x)}$

$l \times m$

$m \times n$

← 線型関数の合成と

行列の掛け算に対応している

線型代数を使わない  $l=m=n=2$

$$\frac{\partial z_1}{\partial x_1} = \frac{\partial z_1}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2} \frac{\partial y_2}{\partial x_1}$$

$$l=n=2 \quad m=3$$

$$\frac{\partial z_1}{\partial x_1} = \frac{\partial z_1}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2} \frac{\partial y_2}{\partial x_1} + \frac{\partial z_1}{\partial y_3} \frac{\partial y_3}{\partial x_1}$$

高階の微分

1変数

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

RからRへの線型写像

微分すると  $f'(x)$  (微分係数) が定まる

(x) (f-p)

比例関数

$$f': \mathbb{R} \rightarrow \mathbb{R}$$

微分すると  $f''(x)$  が定まる

$$f'': \mathbb{R} \rightarrow \mathbb{R}$$

⋮

$$x \in \mathbb{R}^n, f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$f'(x) = df(x)$$

R^nからR^mへの線型写像

 $L(\mathbb{R}^n; \mathbb{R}^m)$ : 線型空間

$\mathbb{R}^n$ から $\mathbb{R}^m$ の  
線型写像全体

$$\underbrace{\begin{pmatrix} & & \\ & \ddots & \\ & & \end{pmatrix}}_{\text{n個}}_m$$

(m×n)次元

$$f: \mathbb{R} \rightarrow \frac{L(\mathbb{R}; \mathbb{R})}{\mathbb{R}}$$

$$f: \mathbb{R}^n \rightarrow L(\mathbb{R}^n; \mathbb{R}^m)$$

 $n=2$   
 $m=1$ 

$$\psi: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$a, b \in \mathbb{R}^2, \alpha, \beta \in \mathbb{R}$$

2重線型写像

$$\begin{cases} \psi(a_1 + a_2, b) = \psi(a_1, b) + \psi(a_2, b) \\ \psi(\alpha a_1, b) = \alpha \psi(a_1, b) \\ \psi(a_1, b_1 + b_2) = \psi(a_1, b_1) + \psi(a_1, b_2) \\ \psi(a_1, \beta b) = \beta \psi(a_1, b) \end{cases}$$

内積  $a \cdot b$ 行列式  $\det(a \ b)$ 

$$\begin{vmatrix} a & b \end{vmatrix}$$

$$|a_1 + a_2, b| = |a_1, b| + |a_2, b|$$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2$$

$$\begin{aligned}\varphi(\mathbf{a}, \mathbf{b}) &= \varphi(a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2, b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2) \\ &= a_1 b_1 \varphi(\mathbf{e}_1, \mathbf{e}_1) + a_1 b_2 \varphi(\mathbf{e}_1, \mathbf{e}_2) + a_2 b_1 \varphi(\mathbf{e}_2, \mathbf{e}_1) + a_2 b_2 \varphi(\mathbf{e}_2, \mathbf{e}_2)\end{aligned}$$

$$(a_1, a_2) \begin{pmatrix} \varphi(\mathbf{e}_1, \mathbf{e}_1) & \varphi(\mathbf{e}_1, \mathbf{e}_2) \\ \varphi(\mathbf{e}_2, \mathbf{e}_1) & \varphi(\mathbf{e}_2, \mathbf{e}_2) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 1 \times 1 \text{ の行列}$$

2×2の行列

内積

$$\varphi(\mathbf{a}, \mathbf{b}) = \varphi(\mathbf{b}, \mathbf{a}) \text{ 対称} \quad \xleftrightarrow{\substack{\text{I} \rightarrow \\ \text{I}' \rightarrow}} \varphi(\mathbf{e}_1, \mathbf{e}_2) = \varphi(\mathbf{e}_2, \mathbf{e}_1)$$

行列式

$$\varphi(\mathbf{a}, \mathbf{b}) = -\varphi(\mathbf{b}, \mathbf{a}) \text{ 反対称} \quad \xleftrightarrow{\substack{\text{I} \rightarrow \\ \text{I}' \rightarrow}} \begin{cases} \varphi(\mathbf{e}_1, \mathbf{e}_2) = -\varphi(\mathbf{e}_2, \mathbf{e}_1) \\ \varphi(\mathbf{e}_1, \mathbf{e}_1) = \varphi(\mathbf{e}_2, \mathbf{e}_2) = 0 \end{cases}$$

一般の2重線型写像

$$\begin{aligned}\varphi: \mathbb{R}^2 \times \mathbb{R}^2 &\rightarrow \mathbb{R} \\ x \in \mathbb{R}^2 &\mapsto \varphi(x, \cdot) \in L(\mathbb{R}^2; \mathbb{R})\end{aligned}$$

$$\begin{aligned}y \in \mathbb{R}^2 &\mapsto \varphi(x, y) \in \mathbb{R} \\ &\text{線型写像}\end{aligned}$$

$$\begin{cases} \varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y) \\ \varphi(\alpha x, y) = \alpha \varphi(x, y) \end{cases}$$

$$\mathbb{R}^2 \rightarrow L(\mathbb{R}^2; \mathbb{R}) \Rightarrow \text{線型}$$

結合論

$$\begin{aligned}L(\mathbb{R}^2; L(\mathbb{R}^2; \mathbb{R}))_{4 \times 2 \times 1} &\xrightarrow{\text{I} \rightarrow \text{II}} \\ &= L(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})_{4 \times 2 \times 1} \\ &\mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \wedge \text{の } 2\text{-重線型写像の全体}\end{aligned}$$

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$

$f'(x) = df(x) \in L(\mathbb{R}^2; \mathbb{R})$ ,  $d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = d$

$f'': \mathbb{R}^2 \rightarrow L(\mathbb{R}^2; \mathbb{R})$  一般には線型ではない

$f''(x) \in L(\mathbb{R}^2; L(\mathbb{R}^2; \mathbb{R}))$

$L(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$

2×2の行列

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} \end{pmatrix}$$

$$(d, d)\psi = (d, d)\psi \leftarrow \text{左側} \quad (d, d)\psi = (d, d)\psi$$

$$0 = (d, d)\psi = (d, d)\psi$$

$$(d, k)\psi + (k, d)\psi = (d+k, k)\psi$$

$$k\psi \in L(\mathbb{R}^2)$$

$$L(\mathbb{R}^2, \mathbb{R}^2) = L(\mathbb{R}^2, L(\mathbb{R}^2; \mathbb{R}))$$

$$= L(\mathbb{R}^2, \mathbb{R}^2)$$

合成関数の微分9公式

$$f: \mathbb{R} \rightarrow \mathbb{R} \quad g: \mathbb{R} \rightarrow \mathbb{R}$$

$$(g \circ f)(x) = g(f(x))$$

証明

$$|x|$$

$$g \circ f(x+d) = g(f(x+d))$$

$$= g(f(x) + \underbrace{f'(x)d}_{\in D})$$

$$\text{算型式} \quad \frac{\partial z_1}{\partial x_1} = \frac{\partial z_1}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2} \frac{\partial y_2}{\partial x_1}$$

$$l=m=n=2$$

多変数  $\alpha, x \in \mathbb{R}^n$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$g: \mathbb{R}^m \rightarrow \mathbb{R}^l$$

$$(g \circ f)(x + \alpha d)$$

$$= g(f(x) + \underbrace{d(g \circ f)(x)}_{\text{等価}}(\alpha) d)$$

$\frac{\partial z_1}{\partial x_1} = \frac{\partial z_1}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2} \frac{\partial y_2}{\partial x_1}$

$$l=n=2 \quad m=3$$

$$\frac{\partial z_1}{\partial x_1} = \frac{\partial z_1}{\partial y_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2} \frac{\partial y_2}{\partial x_1} + \frac{\partial z_1}{\partial y_3} \frac{\partial y_3}{\partial x_1} +$$

$\in \mathbb{R}^n$

$\in \mathbb{R}^m$

$\in \mathbb{R}^l$

$d)$

$$g \circ f(x + \alpha d) = g(f(x + \alpha d))$$

$$= g(f(x) + \underbrace{df(x)(\alpha)}_{\mathbb{R}^m} d)$$

$$(g \circ f)(x)(\alpha) d$$

$$= g(f(x)) +$$

$$\frac{\partial z_1}{\partial x_1} \frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial x_2} \frac{\partial y_2}{\partial x_1} + \dots + \frac{\partial z_1}{\partial x_n} \frac{\partial y_n}{\partial x_1}$$

$$d(g \circ f)(x) = \underbrace{dg(f(x))}_{m \times n} \circ \underbrace{df(x)}_{l \times m}$$

$$\begin{aligned}
 g \circ f(x + \alpha d) &= g(f(x + \alpha d)) \\
 &= g\left(\underbrace{f(x)}_{\mathbb{R}^m} + \underbrace{\frac{df(x)(\alpha)}{\mathbb{R}^m} d\right) \\
 &= g(f(x)) + \\
 &\quad \underbrace{dg(f(x))}_{\text{differential}} \left( df(x)(\alpha) \right) d \\
 &= \underbrace{dg(f(x))}_{\text{differential}} \circ \underbrace{df(x)}_{m \times n} \\
 &\quad \xrightarrow{\text{differential}} \xrightarrow{\text{differential}}
 \end{aligned}$$

高階の微分

1変数

$f: \mathbb{R} \rightarrow \mathbb{R}$

$f'(x) = \frac{df(x)}{dx}$   $\mathbb{R}$  が  $\mathbb{R}$  の線型写像

$f': \mathbb{R} \rightarrow \mathbb{R}$  比例関数

$f'': \mathbb{R} \rightarrow \mathbb{R}$   $\langle (\mathbb{R}; \mathbb{R}) \rangle$  二階微分

$f': \mathbb{R} \rightarrow \frac{\langle (\mathbb{R}; \mathbb{R}) \rangle}{\mathbb{R}}$

$$\mathbb{R}^n \rightarrow \mathbb{R}^m \quad L(\mathbb{R}^n; \mathbb{R}^m) \quad \text{線型空間} \quad m^n$$

$\varphi(x) = df(x)$   $\mathbb{R}^n$  が  $\mathbb{R}^m$  の  
 線型写像の  
 全体

行列で表す  
 $m \times n$  の  
 $n$  個

$\underbrace{\begin{pmatrix} & & \\ & \ddots & \\ & & \end{pmatrix}}_{m \times n}$

$f: \mathbb{R}^n \rightarrow L(\mathbb{R}^n; \mathbb{R}^m)$   $m \times n$  次元

$$n=2 \quad \left\{ \begin{array}{l} 9 \text{ 場合} \\ m=1 \end{array} \right. \quad \varphi(a, b) \quad \text{内積}$$

$\varphi: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$   $a, b \in \mathbb{R}^2$   $\alpha, \beta \in \mathbb{R}$

2重線型写像  $\frac{\alpha \cdot b}{\det \begin{pmatrix} a & b \\ \alpha & b \end{pmatrix}}$

$\varphi(a_1 + a_2, b) = \varphi(a_1, b) + \varphi(a_2, b)$

$\varphi(\alpha a, b) = \alpha \varphi(a, b)$

$\varphi(a_1 b_1 + a_2 b_2, b) = \varphi(a_1 b_1, b) + \varphi(a_2 b_2, b)$

$\varphi(a, \beta b) = \beta \varphi(a, b)$

積

$a \cdot b$

行列式

$$\det \begin{pmatrix} a & b \\ a & b \end{pmatrix}$$

$|a_1 + a_2, |b| = |a_1, b| + |a_2, b|$

$b_2)$

$$\begin{aligned}
 C_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & C_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
 a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= a_1 C_1 + a_2 C_2 & b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &= b_1 C_1 + b_2 C_2 \\
 \varphi(a, b) &= \varphi(a_1 C_1 + a_2 C_2, b_1 C_1 + b_2 C_2) \\
 &= a_1 b_1 \varphi(C_1, C_1) + a_1 b_2 \varphi(C_1, C_2) \\
 &\quad + a_2 b_1 \varphi(C_2, C_1) + a_2 b_2 \varphi(C_2, C_2)
 \end{aligned}$$

$2 \times 2$  矩阵

内積

外積

I

$$\varphi(a, b) = \varphi(b, a) \text{ 反称} \iff \varphi(C_1, C_2) = \varphi(C_2, C_1)$$

行列式

$$\varphi(a, b) = -\varphi(b, a) \text{ 反称} \stackrel{(2)}{\iff} \left\{ \begin{array}{l} \varphi(C_1, C_2) = -\varphi(C_2, C_1) \\ \varphi(C_1, C_1) = 0 \end{array} \right.$$

取の2重線型写像

$$\varphi: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$x \in \mathbb{R}^2 \mapsto \varphi(x, \cdot)$$

$$\begin{cases} \varphi(\mathbf{e}_1, \mathbf{e}_2) = \varphi(\mathbf{e}_2, \mathbf{e}_1) \\ \varphi(\mathbf{e}_1, \mathbf{e}_1) = \varphi(\mathbf{e}_2, \mathbf{e}_2) = 0 \end{cases} \quad \begin{cases} \varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y) \\ \varphi(\alpha x, y) = \alpha \varphi(x, y) \end{cases}$$

$$\mathbb{R}^2 \rightarrow \mathcal{L}(\mathbb{R}^2; \mathbb{R})$$

取の2重線型写像

$$\varphi: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \quad \in \mathcal{L}(\mathbb{R}^2; \mathbb{R})$$

$$x \in \mathbb{R}^2 \mapsto (\varphi(x, \cdot)) \quad \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\begin{cases} \varphi(\mathbf{e}_1, \mathbf{e}_1) = 0 \\ \varphi(\mathbf{e}_2, \mathbf{e}_2) = 0 \end{cases} \quad \begin{cases} \varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y) \\ \varphi(\alpha x, y) = \alpha \varphi(x, y) \end{cases}$$

$$\mathbb{R}^2 \rightarrow \mathcal{L}(\mathbb{R}^2; \mathbb{R}) \quad \text{対応写像}$$