9/8(水) 3限　微積分(生物学類)

多変数の微分

Kock-Lawvere の公理

$f : D \to \mathbb{R}$

$(\exists! a \in \mathbb{R})(\forall d \in D)$

$(f(d) = f(0) + ad)$

1次式

⇓ 一般化

$f : D \to \mathbb{R}^n$

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

$f_i : D \to \mathbb{R}$

$f_i(d) = f_i(0) + a_i d$

$$f(d) = \begin{pmatrix} f_1(d) \\ \vdots \\ f_n(d) \end{pmatrix} = \begin{pmatrix} f_1(0) + a_1 d \\ \vdots \\ f_n(0) + a_n d \end{pmatrix}$$

$$= \begin{pmatrix} f_1(0) \\ \vdots \\ f_n(0) \end{pmatrix} + d \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a \text{ (ベクトル)}$$

$f : D \to \mathbb{R}^n$

$(\exists! a \in \mathbb{R}^n)(\forall d \in D)$

$(f(d) = f(0) + ad)$

$f : \mathbb{R}^n \to \mathbb{R}^n$

$x_0, a \in \mathbb{R}^n$

$d \in D \longmapsto f(x_0 + ad) \in \mathbb{R}^m$

$f(x + ad) = f(x_0) \overset{\exists!}{+} b\, d$

$b \in \mathbb{R}^m$

$b = \partial_a f(x_0)$ … 関数fの点x_0におけるa方向の微分

(1) $\partial_{a_1+a_2} f(x_0) = \partial_{a_1} f(x_0) + \partial_{a_2} f(x_0) \quad a_1, a_2 \in \mathbb{R}^n$

(2) $\partial_{\alpha a} f(x_0) = \alpha \partial_a f(x_0) \quad \alpha \in \mathbb{R}$

(1)の証明

$f(x_0 + (a_1 + a_2)d)$

$= f(\underline{x_0 + a_1 d} + a_2 d)$

↑この点において微分

$= \underline{f(x_0 + a_1 d)} + \underline{\partial_{a_2}} f(\underline{x_0} + a_1 d)\, d \qquad \partial_{a_2} f : \mathbb{R}^n \to \mathbb{R}^m$

$= \{f(x_0) + \partial_{a_1} f(x_0) d\} + \{\partial_{a_2} f(x_0) + \partial_{a_1} \partial_{a_2} f(x_0)\, \underline{d}\}\underline{d} \qquad d^2 = 0$

$= f(x_0) + \partial_{a_1} f(x_0) d + \partial_{a_2} f(x_0) d$

$= f(x_0) + \{\underline{\partial_{a_1} f(x_0) + \partial_{a_2} f(x_0)}\} d$

(2)の証明

$f(x_0 + (\alpha a) d) = f(x_0 + a(\alpha \underset{\in D}{\underline{d}}))$

$= f(x_0) + \partial_a f(x_0) \alpha d$

$= f(x_0) + (\underline{\alpha \partial_a f(x_0)}) d$

$df(x_0)$ ← このdは$\frac{df}{dx}$のような時のd

$\left.\begin{array}{l} f : \mathbb{R}^n \to \mathbb{R}^m \\ x_0 \in \mathbb{R}^n \end{array}\right\}$ 固定

$a \in \mathbb{R}^n \longmapsto \partial_a f(x_0) \in \mathbb{R}^m$ 線型関数

一般に $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ (線型)

$$\varphi\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = \varphi(x_1 e_1 + \cdots + x_n e_n) \qquad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= x_1 \varphi(e_1) + \cdots + x_n \varphi(e_n)$$

$$\mathbb{R}^m \ni \varphi(e_1) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \varphi(e_2) = \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, \varphi(e_n) = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

$$\varphi\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

線型写像 $df(x_0)$ (微分) を $m \times n$ の行列で表わす

$\partial_{e_1} f(x_0)$ $\qquad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad x_0 = \begin{pmatrix} x_0^1 \\ \vdots \\ x_0^n \end{pmatrix}$

$$f(x_0 + e_1 d) = f\left(x_0 + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} d\right)$$
$$= f(x_0) + \frac{\partial f}{\partial x_1}(x_0)\, d$$

偏微分

e_2 について同様のことをすると

$\frac{\partial f}{\partial x_2}(x_0)$

$\vdots$

e_n について

$\frac{\partial f}{\partial x_n}(x_0)$

$x = (x_1, \cdots, x_n)$

$$f(x) = \begin{pmatrix} y_1 = y_1(x_1, \cdots, x_n) \\ \vdots \\ y_m = y_m(x_1, \cdots, x_n) \end{pmatrix}$$

$m \times n$ 行列

$$\left(\frac{\partial f}{\partial x_1}(x_0), \cdots, \frac{\partial f}{\partial x_n}(x_0)\right) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}$$

合成関数の微分

$f : \mathbb{R} \to \mathbb{R}$

$g : \mathbb{R} \to \mathbb{R}$

$g \circ f$: 合成関数

x_0 で微分

$$g \circ f(x_0 + d) = g(f(x_0 + d))$$
$$= g(f(x_0) + \underbrace{f'(x_0)}_{\text{スカラー}} d) \quad \in D$$
$$= g(f(x_0)) + g'(f(x_0))\, f'(x_0)\, d$$

$$(g \circ f)' = g'(\underbrace{f(x_0)}_{y_0})\, f'(x_0)$$

NO.

DATE

多変数になるとどうなるか

1変数の場合　$[g'(y_0)][f'(x_0)]$　$[1\times1$行列$]$同士の掛け算

$f: \mathbb{R}^n \to \mathbb{R}^m$　　$x_0 \in \mathbb{R}^n$

$g: \mathbb{R}^m \to \mathbb{R}^l$　　$y_0 = f(x_0) \in \mathbb{R}^m$

$$\underbrace{\begin{pmatrix} \frac{\partial z_1}{\partial y_1}(y_0) & \cdots & \frac{\partial z_1}{\partial y_m}(y_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial z_l}{\partial y_1}(y_0) & \cdots & \frac{\partial z_l}{\partial y_m}(y_0) \end{pmatrix}}_{l\times m} \underbrace{\begin{pmatrix} \frac{\partial y_1}{\partial x_1}(x_0) & \cdots & \frac{\partial y_1}{\partial x_n}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1}(x_0) & \cdots & \frac{\partial y_m}{\partial x_n}(x_0) \end{pmatrix}}_{m\times n} = l\times n$$

線型写像の合成

$$d(g\circ f)(x_0) = dg(y_0)\circ df(x_0)$$

線型代数を使わないとどうなるか？

$l=m=n=2$

$z_1 = z_1(y_1, y_2)$　　$y_1 = y_1(x_1, x_2)$

$z_2 = z_2(y_1, y_2)$　　$y_2 = y_2(x_1, x_2)$

$$\frac{\partial z_1}{\partial x_1} = \frac{\partial z_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2}\frac{\partial y_2}{\partial x_1}$$

多変数の微分

Kock-Lawvereの公理

$$f : D \to \mathbb{R}$$

$$(\exists!\, a \in \mathbb{R})(\forall d \in D)\Big(\underbrace{f(d) = f(0) + ad}_{1次式}\Big)$$

$$f : D \to \mathbb{R}^n$$

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

$$f_i : D \to \mathbb{R}$$

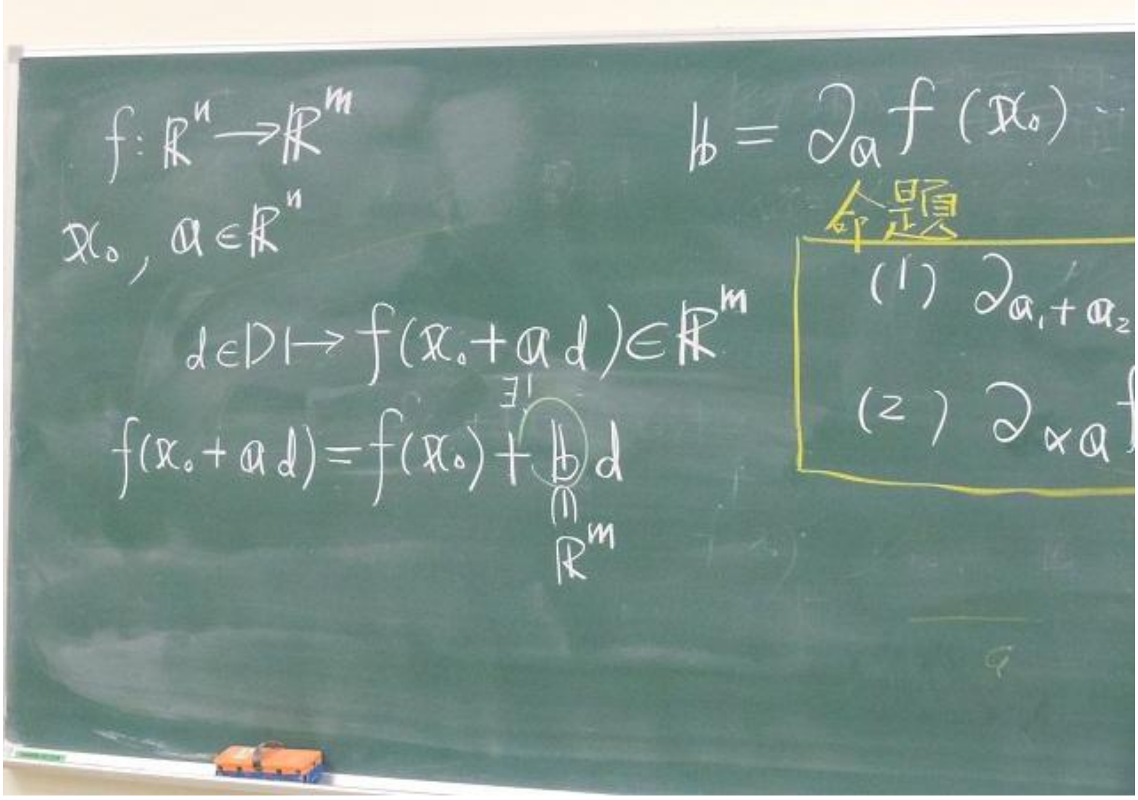
$$f_i(d) = f_i(0) + a_i d$$

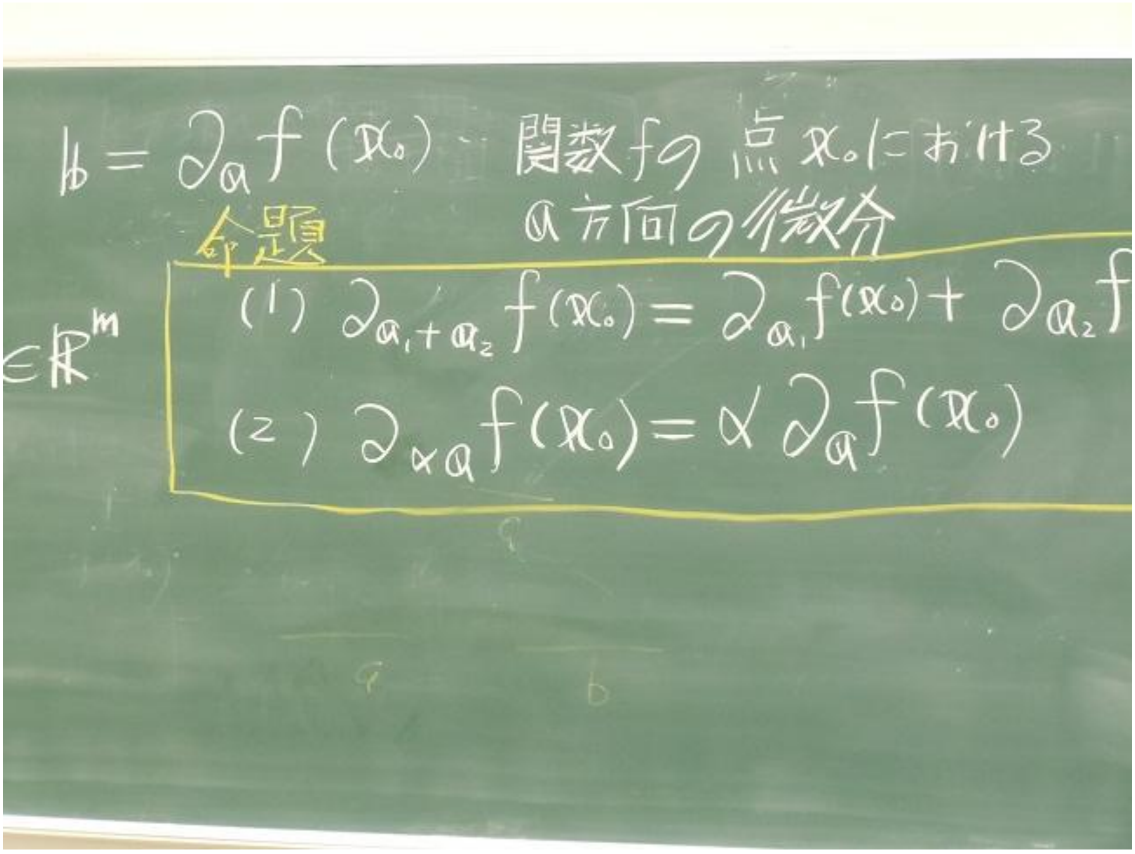
$$f(d) = \begin{pmatrix} f_1(d) \\ \vdots \\ f_n(d) \end{pmatrix} = \begin{pmatrix} f_1(0) + a_1 d \\ \vdots \\ f_n(0) + a_n d \end{pmatrix}$$

$$= \begin{pmatrix} f_1(0) \\ \vdots \\ f_n(0) \end{pmatrix} + d \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{= \mathbf{a}}$$

$$f : D \to \mathbb{R}^n$$

$$(\exists!\, \mathbf{a} \in \mathbb{R}^n)(\forall d \in D)(f(d) = f(0) + \mathbf{a} d)$$


$f: \mathbb{R}^n \to \mathbb{R}^m$
$x_0, a \in \mathbb{R}^n$
$d \in D \mapsto f(x_0 + a d) \in \mathbb{R}^m$
$\exists!$
$f(x_0 + a d) = f(x_0) + b\, d$
$b \in \mathbb{R}^m$
$b = \partial_a f(x_0)$
命題
(1) $\partial_{a_1 + a_2}$
(2) $\partial_{\alpha a}$


$b = \partial_a f(x_0)$ 関数fの点x_0における
a方向の微分
命題
(1) $\partial_{a_1 + a_2} f(x_0) = \partial_{a_1} f(x_0) + \partial_{a_2} f$
(2) $\partial_{\alpha a} f(x_0) = \alpha\, \partial_a f(x_0)$
$\in \mathbb{R}^m$

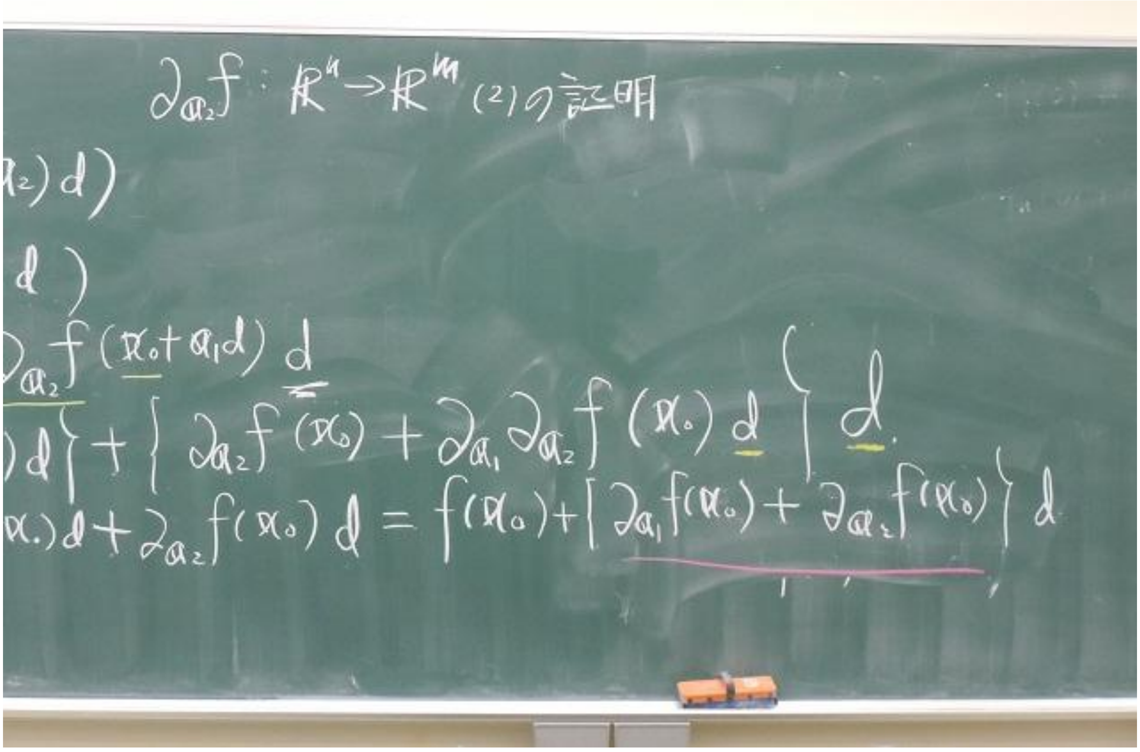
関数fの 点x_0における
a方向の微分

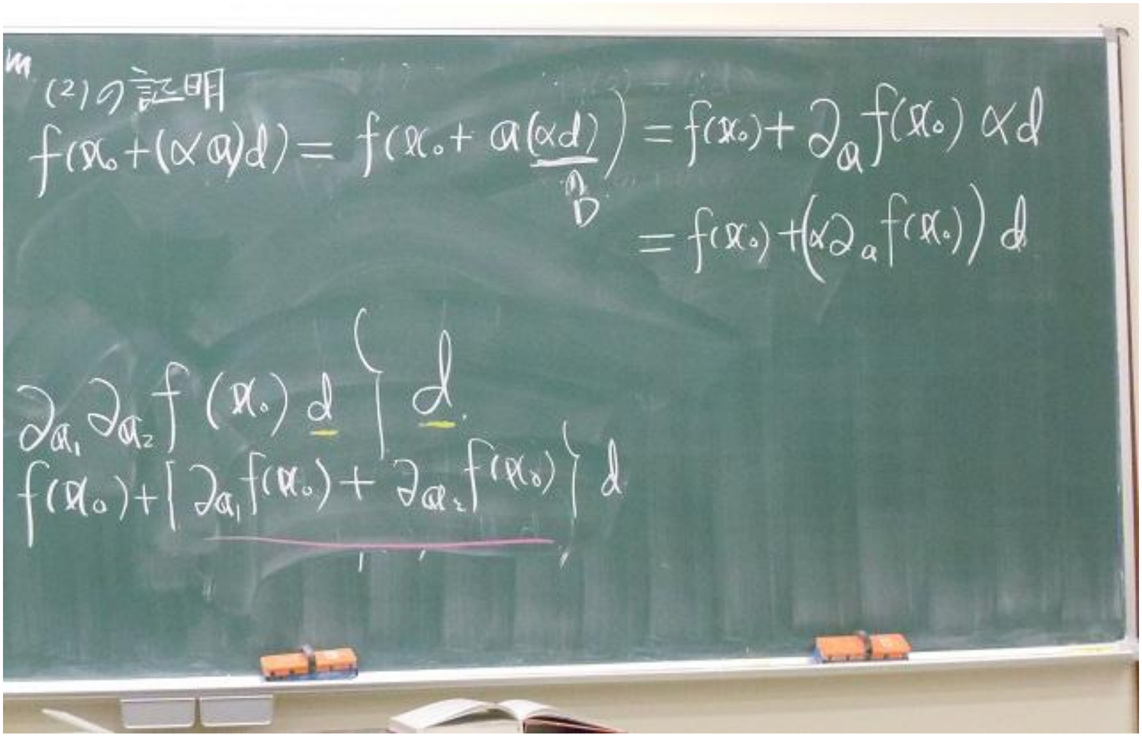
$$f(x_0) = \partial_{a_1} f(x_0) + \partial_{a_2} f(x_0) \qquad a_1, a_2 \in \mathbb{R}^n$$

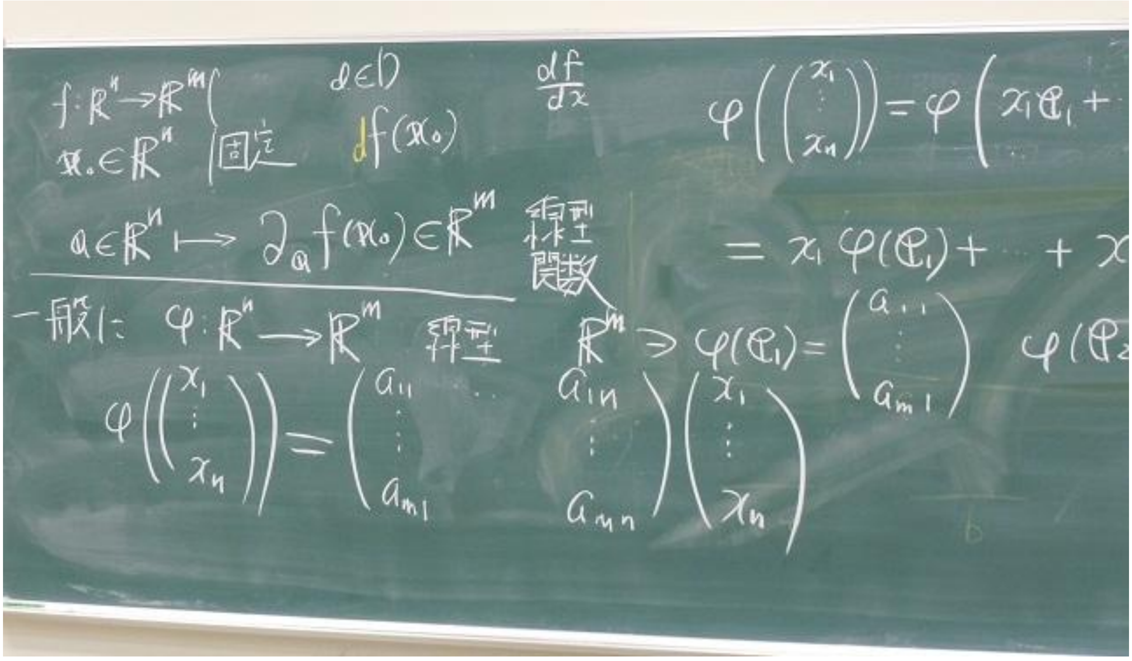
$$f(x_0) = \alpha\, \partial_a f(x_0) \qquad \alpha \in \mathbb{R}$$

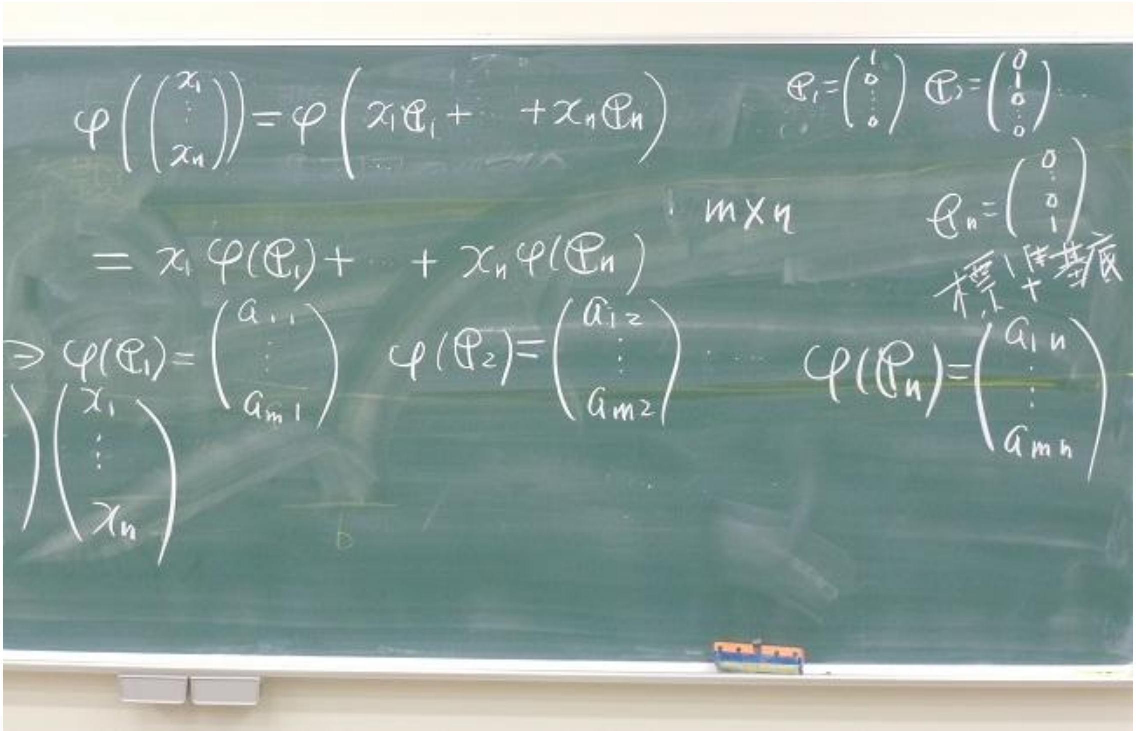
(1)の証明 $\partial_{a_2} f : \mathbb{R}^n \to \mathbb{R}^m$ (2)の証明

$$f(x_0 + (a_1 + a_2)d)$$
$$= f(x_0 + a_1 d + a_2 d)$$
$$= f(x_0 + a_1 d) + \partial_{a_2} f(x_0 + a_1 d)\, d$$
$$= \{f(x_0) + \partial_{a_1} f(x_0)\, d\} + \{\partial_{a_2} f(x_0) + \partial_{a_1}\partial_{a_2} f(x_0$$
$$= f(x_0) + \partial_{a_1} f(x_0)\, d + \partial_{a_2} f(x_0)\, d = f(x_0) + [\partial_{a_1} f($$


$\partial_{a_2} f : \mathbb{R}^n \to \mathbb{R}^m$ (2)の証明
$a_2) d)$
$d)$
$\partial_{a_2} f(x_0 + a_1 d)\, d$
$) d \} + \{ \partial_{a_2} f(x_0) + \partial_{a_1} \partial_{a_2} f(x_0)\, d \}\, d$
$x_0) d + \partial_{a_2} f(x_0)\, d = f(x_0) + \{ \partial_{a_1} f(x_0) + \partial_{a_2} f(x_0) \}\, d$


m (2)の証明
$f(x_0 + (\alpha a) d) = f(x_0 + a \underbrace{(\alpha d)}_{D}) = f(x_0) + \partial_a f(x_0)\, \alpha d$
$= f(x_0) + (\alpha \partial_a f(x_0))\, d$
$\partial_{a_1} \partial_{a_2} f(x_0)\, d \}\, d$
$f(x_0) + \{ \partial_{a_1} f(x_0) + \partial_{a_2} f(x_0) \}\, d$


$f: \mathbb{R}^n \to \mathbb{R}^m$
$x_0 \in \mathbb{R}^n$ 固定
$\frac{df}{dx}$
$df(x_0)$
$\varphi\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = \varphi(x_1 e_1 + \cdots$
$a \in \mathbb{R}^n \longmapsto \partial_a f(x_0) \in \mathbb{R}^m$ 線型関数
$= x_1 \varphi(e_1) + \cdots + x$
一般に $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ 線型
$\mathbb{R}^m \ni \varphi(e_1) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}$ $\varphi(e_2$
$\varphi\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$


$\varphi\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = \varphi(x_1 e_1 + \cdots + x_n e_n)$
$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\cdots$
$m \times n$
$e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$
標準基底
$= x_1 \varphi(e_1) + \cdots + x_n \varphi(e_n)$
$\ni \varphi(e_1) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}$ $\varphi(e_2) = \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix}$ $\cdots$ $\varphi(e_n) = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$
$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$\varphi(e_1)$

線形写像 $df(x_0)$ を $m \times n$ の行列で表わす

微分

$\partial_{e_1} f(x_0)$ $\qquad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\qquad x_0 = \begin{pmatrix} x_0^1 \\ \vdots \\ x_0^n \end{pmatrix}$

$$f(x_0 + e_1 d) = f\left(x_0 + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} d\right)$$

$$= f(x_0) + \frac{\partial f}{\partial x_1}(x_0)\, d$$

偏微分

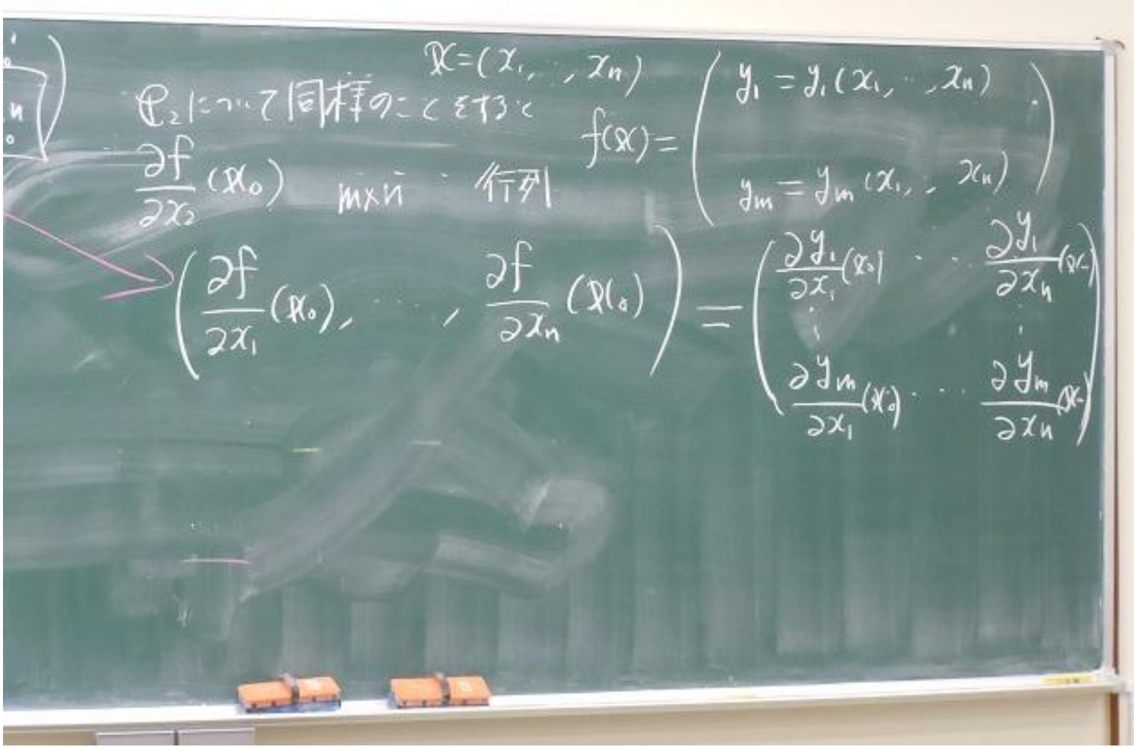
$x_0 = \begin{pmatrix} x_0^1 \\ \vdots \\ x_0^n \end{pmatrix}$ $\qquad x = (x_1, \cdots, x_n)$

e_2について同様のことをすると

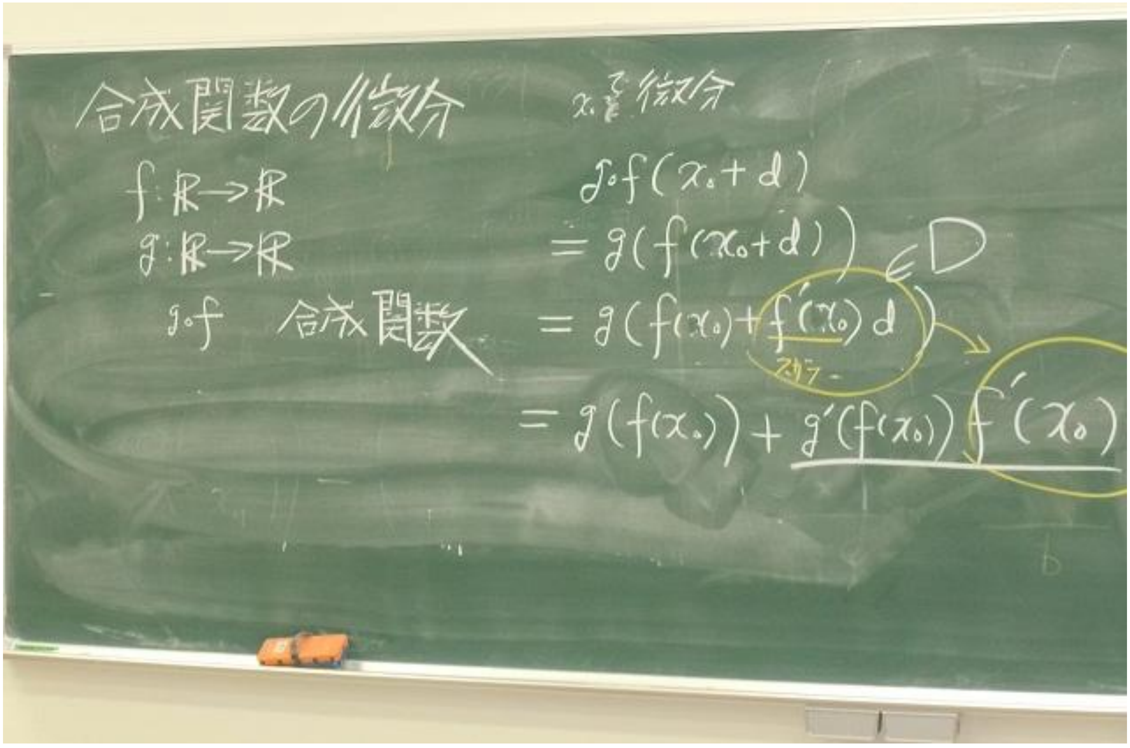
$\frac{\partial f}{\partial x_2}(x_0)$ $\qquad m \times n$ 行列

$$f(x) = \begin{pmatrix} y_1 = y_1(x_1, \cdots, x_n) \\ \vdots \\ y_m = y_m(x_1, \cdots, x_n) \end{pmatrix}$$

$$\left(\frac{\partial f}{\partial x_1}(x_0), \cdots, \frac{\partial f}{\partial x_n}(x_0)\right) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1}(x_0) & \cdots \\ \vdots & \\ \frac{\partial y_m}{\partial x_1}(x_0) & \cdots \end{pmatrix}$$



$x = (x_1, \dots, x_n)$
e_2について同様のことをすると
$\frac{\partial f}{\partial x_2}(x_0)$
$m \times n$ 行列
$f(x) = \begin{pmatrix} y_1 = y_1(x_1, \dots, x_n) \\ \vdots \\ y_m = y_m(x_1, \dots, x_n) \end{pmatrix}$
$\left(\frac{\partial f}{\partial x_1}(x_0), \dots, \frac{\partial f}{\partial x_n}(x_0)\right) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1}(x_0) & \cdots & \frac{\partial y_1}{\partial x_n}(x_0) \\ \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1}(x_0) & \cdots & \frac{\partial y_m}{\partial x_n}(x_0) \end{pmatrix}$



合成関数の微分
x_0で微分
$f: \mathbb{R} \to \mathbb{R}$
$g: \mathbb{R} \to \mathbb{R}$
$g \circ f$ 合成関数
$g \circ f(x_0 + d)$
$= g(f(x_0 + d))$
$\in D$
$= g(f(x_0) + f'(x_0)d)$
スカラー
$= g(f(x_0)) + g'(f(x_0))f'(x_0)$

$$(g\circ f)' = g'(\underbrace{f(x_0)}_{y_0})\, f'(x_0)$$

$$f'(x_0)\, d$$

多変数にくると
どうなるか

$$f : \underset{(y_1,\dots,y_m)}{\mathbb{R}^n} \longrightarrow \underset{(z_1,\dots,z_l)}{\mathbb{R}^m}$$

$$g : \mathbb{R}^m \longrightarrow \mathbb{R}^l$$

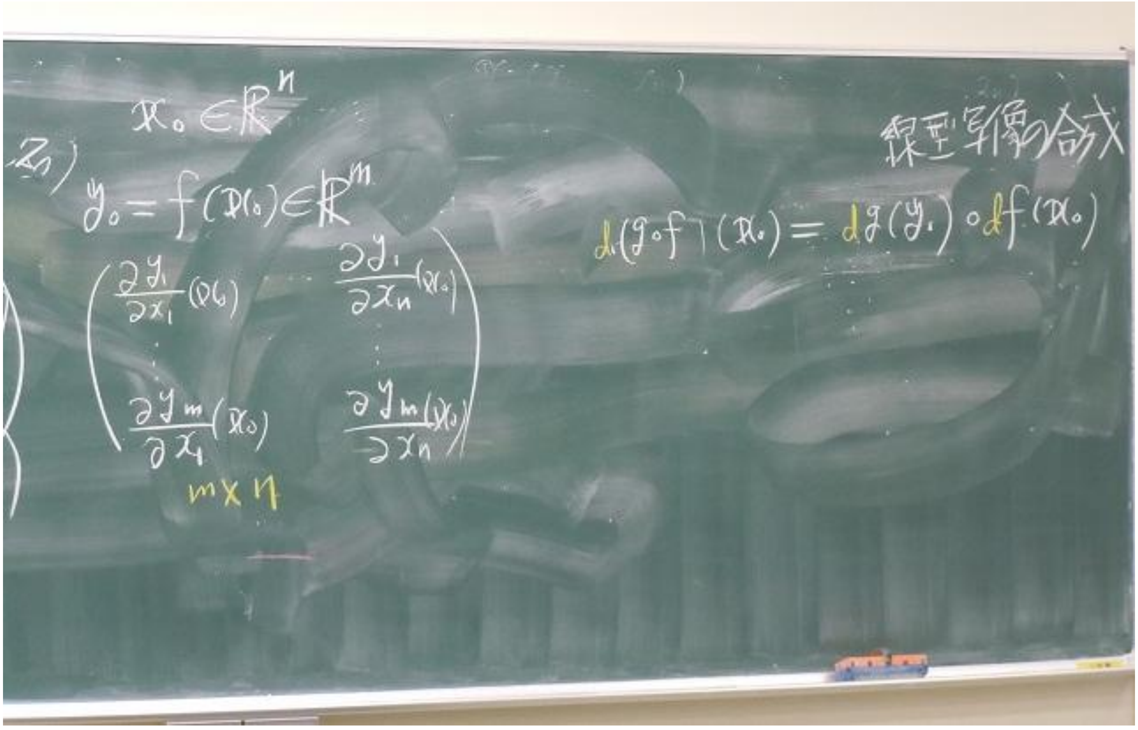
$$x_0 \in \mathbb{R}^n$$

$$y_0 = f(x_0) \in \mathbb{R}^m$$

$$[g'(y_0)][f'(x_0)]$$

$$\underset{l\times n}{} \qquad \underset{l \times m}{\begin{pmatrix} \frac{\partial z_1}{\partial y_1}(y_0) & \cdots & \frac{\partial z_1}{\partial y_m}(y_0) \\ \vdots & & \vdots \\ \frac{\partial z_l}{\partial y_1}(y_0) & \cdots & \frac{\partial z_l}{\partial y_m}(y_0) \end{pmatrix}} \underset{m\times n}{\begin{pmatrix} \frac{\partial y_1}{\partial x_1}(x_0) & \cdots & \frac{\partial y_1}{\partial x_n}(x_0) \\ \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1}(x_0) & \cdots & \frac{\partial y_m}{\partial x_n}(x_0) \end{pmatrix}}$$

$l \times n$ $\quad$ $l \times m$ $\quad$ $m \times n$



線型写像の合成

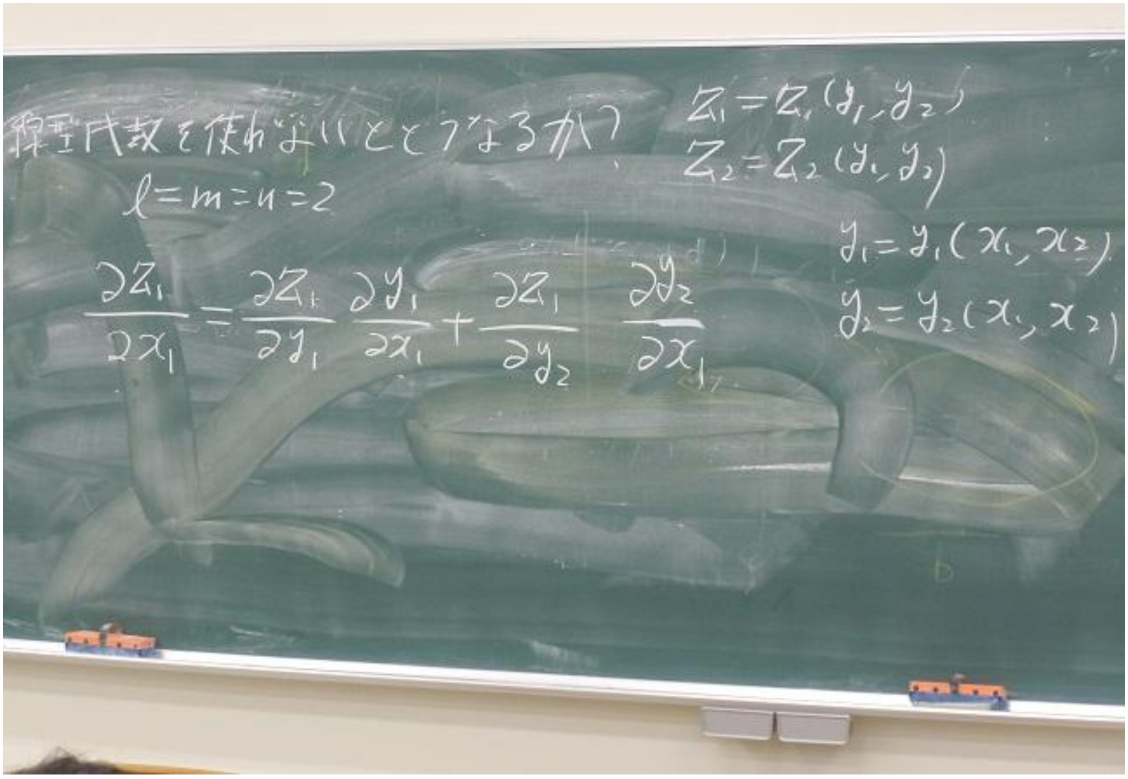
$$d(g\circ f)(x_0) = dg(y_0)\circ df(x_0)$$

例) $x_0 \in \mathbb{R}^n$

$y_0 = f(x_0) \in \mathbb{R}^m$

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_1}(x_0) & \cdots & \frac{\partial y_1}{\partial x_n}(x_0) \\ \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1}(x_0) & \cdots & \frac{\partial y_m}{\partial x_n}(x_0) \end{pmatrix}$$

$m \times n$



線型代数を使わないとどうなるか?

$l = m = n = 2$

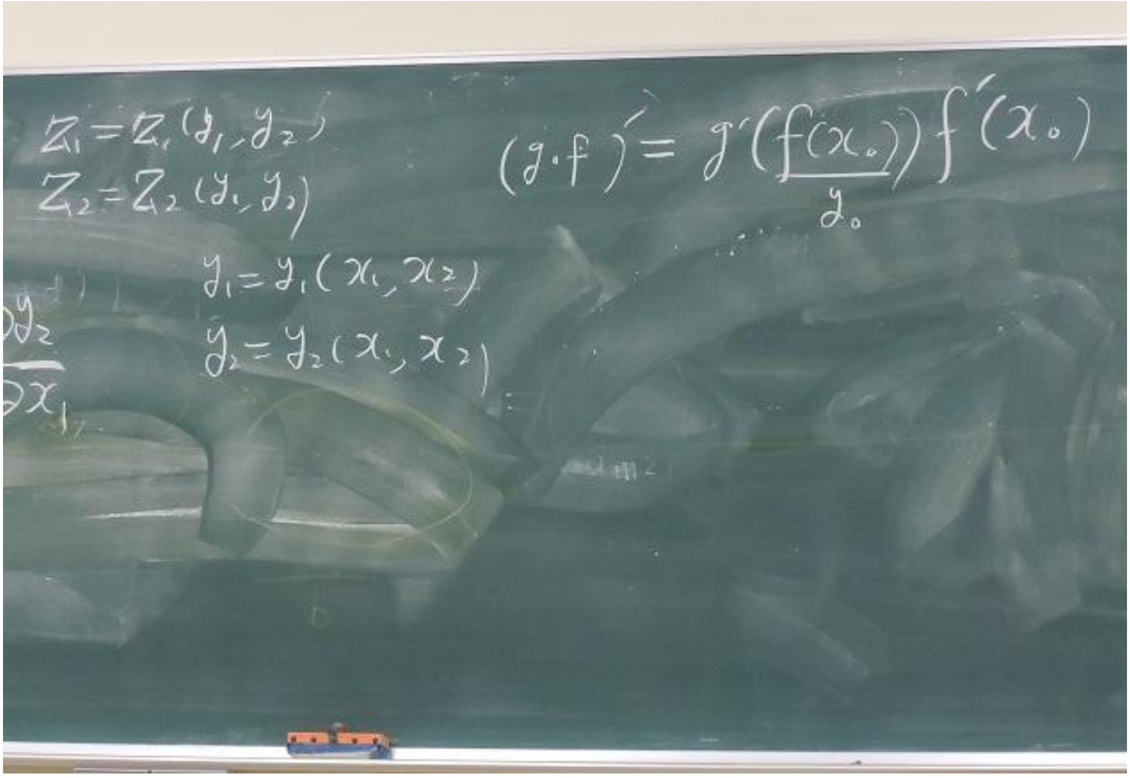
$z_1 = z_1(y_1, y_2)$

$z_2 = z_2(y_1, y_2)$

$y_1 = y_1(x_1, x_2)$

$y_2 = y_2(x_1, x_2)$

$$\frac{\partial z_1}{\partial x_1} = \frac{\partial z_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial z_1}{\partial y_2}\frac{\partial y_2}{\partial x_1}$$


$z_1 = z_1(y_1, y_2)$
$z_2 = z_2(y_1, y_2)$
$(g \circ f)' = g'(\underbrace{f(x_0)}_{y_0})f'(x_0)$
$y_1 = y_1(x_1, x_2)$
$y_2 = y_2(x_1, x_2)$