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AN INFORMATION INEQUALITY FOR THE BAYES RISK IN A FAMILY OF UNIFORM DISTRIBUTION

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Abstract : For a family of uniform distribution on the interval $[\theta-(1/2), \theta-(1/2)]$, the information inequality for the bayes risk of any estimator of θ is given under the quadratic loss and the uniform prior distribution on an interval $[-c, c]$. The lower bound for the Bayes risk is shown to be sharp. And also the lower bound for the limit inferior of Bayes risk as $c \to \infty$ is seen to be attained by the mid-range estimator.

Key words: Cramér-Rao inequality; Bayes Estimator; lower bound; mid-range

1. Introduction : In the paper, Vincze (1979) obtained Cramer-Rao type inequality in the non-regular case, and for the uniform distribution on the interval $[\theta-(1/2), \theta-(1/2)]$ got the lower bound for the variance of unbiased estimator with the right order of magnitude, but it was not sharp. Following ideas of Vincze (1979), Khatri (1980) gave a simple general approach to the non-regular Cramer-Rao bound. In the relation to Vincze (1979), Móri (1983) also obtained the lower bound for the limit inferior of the expected quadratic risk of unbiased estimators of θ under the uniform distribution on the interval $[-c, c]$ as $c \to \infty$ and showed that it was sharp. In this paper, for a family of uniform distributions on $[\theta-(1/2), \theta-(1/2)]$, we obtain the information inequality for the Bayes risk of any estimator of θ under the quadratic loss and the uniform prior distribution on an interval $[-c, c]$ by a somewhat different way of Mori (1983). We also show that the lower bound for the Bayes risk of any estimator of θ is sharp, and that the lower bound for the limit inferior of Bayes risk of any estimator of θ as $c \to \infty$ is attained by the mid-range, which involves the result for unbiased estimators of θ by Mori (1983). The related results to the above are found in Akahira and Takeuchi (1995).

2. An information inequality for the Bayes risk of any estimator : Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables according to the uniform distribution with a density $p(x, \theta)$ on the interval $[\theta-(1/2), \theta-(1/2)]$, where $-\infty < \theta < \infty$. Let n be fixed, and let $\widehat{\theta} = \widehat{\theta}(X)$ be an estimator of θ based on the sample $\mathbf{X} = (X_1, X_2, \ldots, X_n)$. Then we consider the Bayes risk $r_c\left(\widehat{\theta}\right)$ of any estimator $\widehat{\theta}$ of θ under the quadratic loss and the uniform prior distribution on an interval $[-c, c]$, where $-\infty < c < \infty$, i.e.

$$r_c\left(\widehat{\theta}\right) := \frac{1}{2c}\int_{-c}^{c} E_\theta\left[\left(\widehat{\theta}-\theta\right)^2\right] d\theta.$$

Let $f(x, \theta) := \prod_{i=1}^{n} p(x_i, \theta)$ with $\mathrm{x} = (x_1, x_2, \ldots, x_n)$. In order to get the Bayes estimator, i.e. to minimize $r_c\left(\widehat{\theta}\right)$, it is enough to obtain the estimator minimizing

$$\int_{-c}^{c}\left\{\widehat{\theta}(\mathrm{x})-\theta\right\}^2 f(\mathrm{x}, \theta)\, d\theta$$

for almost all x. Such an estimator is easily given by

$$\widehat{\theta}_c^*(\mathrm{X}) = \int_{-c}^{c} \theta f(\mathrm{X},\theta)\,d\theta \Big/ \int_{-c}^{c} f(\mathrm{X},\theta)\,d\theta. \tag{1}$$

Here, we have

$$f(\mathrm{x},\theta) = \begin{cases} 1 & \text{for } x_{(n)} - (1/2) \le \theta \le x_{(1)} + (1/2) \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where $x_{(1)} := \min_{1\le i\le n} x_i$ and $x_{(n)} := \max_{1\le i\le n} x_i$. Let $\underline{\theta} := X_{(n)} - (1/2)$, $\overline{\theta} := X_{(1)} + (1/2)$. From (1) and (2) we have

$$\widehat{\theta}_c^*(\mathrm{X}) = \begin{cases} \frac{1}{2}(\underline{\theta} + c) & \text{for } -c < \underline{\theta},\ \underline{\theta} \le c \le \overline{\theta}, \\ \frac{1}{2}(\underline{\theta} + \overline{\theta}) & \text{for } -c < \underline{\theta},\ \overline{\theta} < c, \\ \frac{1}{2}(\overline{\theta} - c) & \text{for } \underline{\theta} \le -c \le \overline{\theta},\ \overline{\theta} < c \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

$$= \widehat{\theta}_c^*(\underline{\theta},\overline{\theta}) \ \text{(say)},$$

where $0/0 = 0$ and $c > 1/2$. Then we have following.

THEOREM 1. *The information inequality for the Bayes risk of any estimator $\widehat{\theta}$ of θ is given by*

$$\begin{aligned} r_c\left(\widehat{\theta}\right) &= \frac{1}{2c}\int_{-c}^{c} E_\theta\left[\left(\widehat{\theta} - \theta\right)^2\right] d\theta \\ &\ge \frac{1}{2(n+1)(n+2)} - \frac{1}{2c(n+1)(n+2)(n+3)} = A_0(c)\ \textit{(say)}, \end{aligned} \tag{4}$$

where $c > 1/2$, and the lower bound is sharp, that is, $\widehat{\theta}_c^*$ attains the bound.

Proof 1. *The joint density function $f_{\underline{\theta},\overline{\theta}}$ of $(\underline{\theta},\overline{\theta})$ is given by*

$$f(\mathrm{x},\theta) = \begin{cases} n(n-1)(y-z+1)^{n-2} & \textit{for } y \le \theta \le z,\ 0 \le z - y \le 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} r^*. &= \textstyle\int_{-c}^{c} E_\theta\left[\left\{\widehat{\theta}_c^* - \theta\right\}^2\right] d\theta \\ &= \textstyle\int\int_{y\le\theta\le z,\ 0\le z-y\le 1} \int_{-c}^{c} \left\{\widehat{\theta}_c^*(y,z) - \theta\right\}^2 f_{\underline{\theta},\overline{\theta}}^{\theta}(y,z)\,d\theta dy dz. \end{aligned}$$

Since

$$\begin{aligned} &\textstyle\int_{|\theta|\le c,\ y\le\theta\le z} \left(\widehat{\theta}_c^* - \theta\right)^2 f_{\underline{\theta},\overline{\theta}}^{\theta}(y,z)\,d\theta \\ &= \widehat{\theta}_c^{*2} \textstyle\int_{|\theta|\le c,\ y\le\theta\le z} f_{\underline{\theta},\overline{\theta}}^{\theta}(y,z)\,d\theta - 2\widehat{\theta}_c^* \int_{|\theta|\le c,\ y\le\theta\le z} \theta f_{\underline{\theta},\overline{\theta}}^{\theta}(y,z)\,d\theta \\ &\quad + \textstyle\int_{|\theta|\le c,\ y\le\theta\le z} \theta^2 f_{\underline{\theta},\overline{\theta}}^{\theta}(y,z)\,d\theta \\ &= n(n-1)(y-z+1)^{n-2}\left(\widehat{\theta}_c^{*2} \textstyle\int_{|\theta|\le c,\ y\le\theta\le z} d\theta - 2\widehat{\theta}_c^* \int_{|\theta|\le c,\ y\le\theta\le z} \theta d\theta \right. \\ &\quad \left. + \textstyle\int_{|\theta|\le c,\ y\le\theta\le z} \theta^2 d\theta\right) \\ &= n(n-1)(y-z+1)^{n-2}\left(I_1\widehat{\theta}_c^{*2} - 2I_2\widehat{\theta}_c^* + I_3\right) \\ &= G_n(y,z)\ \textit{(say)}, \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \min\{c,z\} - \max\{-c,y\}, \\ I_2 &:= \tfrac{1}{2}\left[(\min\{c,z\})^2 - (\max\{-c,y\})^2\right], \\ I_3 &:= \tfrac{1}{3}\left[(\min\{c,z\})^3 - (\max\{-c,y\})^3\right]. \end{aligned}$$

Next we obtain

$$\begin{aligned} r^* &= \left(\int_{-c}^{1-c}\int_{z-1}^{-c} + \int_{-c}^{1-c}\int_{-c}^{z} + \int_{1-c}^{c}\int_{z-1}^{z} + \int_{c}^{1+c}\int_{z-1}^{-c}\right) G_n(y,z)\,dydz \\ &= \left(\int\int_{J_1} + \int\int_{J_2} + \int\int_{J_3} + \int\int_{J_4}\right) G_n(y,z)\,dydz \ (say). \end{aligned} \tag{5}$$

Repeating integration by parts we have

$$\begin{aligned} J_1 &= \int_{-c}^{1-c}\int_{z-1}^{-c} n(n-1)(y-z+1)^{n-2}\left\{\frac{1}{4}(z-c)^2(z+c) - \frac{1}{2}(z-c)\left(z^2-c^2\right) + \frac{1}{3}\left(z^3+c^3\right)\right\} dydz \\ &= \frac{1}{2(n+1)(n+2)(n+3)}, \end{aligned} \tag{6}$$

$$\begin{aligned} J_2 &= \int_{-c}^{1-c}\int_{-c}^{z} n(n-1)(y-z+1)^{n-2}\left\{\frac{1}{4}(y+z)^2(z-y) - \frac{1}{2}(y+z)\left(z^2-c^2\right) + \frac{1}{3}\left(z^3+c^3\right)\right\} dydz \\ &= \frac{1}{2(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}, \end{aligned} \tag{7}$$

$$\begin{aligned} J_3 &= \int_{1-c}^{c}\int_{z-1}^{z} n(n-1)(y-z+1)^{n-2}\left\{\frac{1}{4}(y+z)^2(z-y) - \frac{1}{2}(y+z)\left(z^2-y^2\right) + \frac{1}{3}\left(z^3+c^3\right)\right\} dydz \\ &= \frac{1}{2(n+1)(n+2)}, \end{aligned} \tag{8}$$

$$\begin{aligned} J_4 &= \int_{c}^{1+c}\int_{z-1}^{c} n(n-1)(y-z+1)^{n-2}\left\{\frac{1}{4}(c+y)^2(c-y) - \frac{1}{2}(c+y)\left(c^2-y^2\right) + \frac{1}{3}\left(c^3-y^3\right)\right\} dydz \\ &= \frac{1}{2(n+1)(n+2)(n+3)}. \end{aligned} \tag{9}$$

From (5) to (9) we have

$$r^* = \frac{c}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}. \tag{10}$$

Since $\widehat{\theta}_c^*$ minimize the Bayes risk $r_c\left(\widehat{\theta}\right)$, it follows from (10) that for any estimator $\widehat{\theta}$ of θ

$$\begin{aligned} r_c\left(\widehat{\theta}\right) &= \frac{1}{2c}\int_{-c}^{c} E_\theta\left[\left(\widehat{\theta}-\theta\right)^2\right] d\theta \\ &\geq \frac{1}{2c}r^* = \frac{1}{2(n+1)(n+2)} - \frac{1}{2c(n+1)(n+2)(n+3)}. \end{aligned}$$

Thus we complete the proof.

COROLLARY 1. *For any estimator $\widehat{\theta}$ of θ*

$$\underline{\lim}_{c\to\infty} r_c\left(\widehat{\theta}\right) \geq \frac{1}{2(n+1)(n+2)} \tag{11}$$

The proof of Corollary is straightforward from the Theorem. The lower bound () is easily seen to be attained by mid-range $\widehat{\theta}_0 := \left(X_{(1)} + X_{(2)}\right)/2$.

REMARK 1. The inequality of the Corollary is same as one for any unbiased estimator given by Móri (1983).

3. Comparison of the lower bounds : In this section we compare the lower bound $A_0(c)$ with Móri's one. Let

$$\mathcal{I}_c := -A_0(c) + \frac{c^2}{3}. \tag{12}$$

In the proof of the Theorem in the paper, Móri (1983) showed that for any unbiased estimator $\widehat{\theta}$ of θ

$$r_c\left(\widehat{\theta}\right) = \frac{1}{2c}\int_{-c}^{c} V_\theta\left(\widehat{\theta}\right) d\theta \geq \frac{c^4}{9\mathcal{I}_c} - \frac{c^2}{3} = M(c) \text{ (say)}, \tag{13}$$

where $c > 1/2$. But the lower bound $M(c)$ is not sharp, as is mentioned in the paper. From (12) and (13) it seen that

$$M(c) > A_0(c) \text{ for } c > 1/2.$$

here, note that $A_0(c)$ is the lower bound for the Bayes risk for any estimator and $M(c)$ is one for any unbiased estimator. And also we have

$$M(c) = A_0(c) + \frac{3}{4c^2(n+1)^2(n+2)^2} + O\left(\frac{1}{c^3}\right) \quad c \to \infty,$$

hence

$$\lim_{c\to\infty} M(c) = \lim_{c\to\infty} A_0(c) = \frac{1}{2(n+1)(n+2)}.$$

For a family of uniform distribution on $[\theta - (\tau/2), \theta + (\tau/2)]$ with a scale τ as a nuisance parameter, we also have a similar information inequality to (4) as follows. For any estimator $\widehat{\theta}$ of θ

$$\begin{aligned} R_c\left(\widehat{\theta}\right) &= \int_{-c}^{c} E_\theta\left[\left(\frac{\widehat{\theta}-\theta}{\tau}\right)^2\right] d\theta \\ &\geq \frac{1}{2(n+1)(n+2)} - \frac{\tau}{2c(n+1)(n+2)(n+3)}, \end{aligned} \tag{14}$$

and

$$\underline{\lim}_{c\to\infty} R_c\left(\widehat{\theta}\right) \geq \frac{1}{2(n+1)(n+2)}. \tag{15}$$

In particular, letting $\tau = 1$, we have the inequality (4) from (14). When c tends to infinity, from (15) we have the same lower bound as (11).

4. Comments : In the previous section we obtain the lower bound for the Bayes risk of estimators under the quadratic loss and the uniform prior distribution on an interval $[-c, c]$, where $c > 1/2$, and show that the bound is sharp. Recently Akahira and Takeuchi (2001) shows that for small $c > 0$ the Bayes risk of any estimator in the interval of θ values of length $2c$ and centered at θ_0 can not be smaller than that of $\widehat{\theta}_0 = \left(X_{(1)} + X_{(n)}\right)/2$. More precisely they prove that for any estimator $\widehat{\theta} = \widehat{\theta}(X)$ based on the sample X of size n

$$\lim_{c\to 0}\lim_{n\to\infty} \frac{n^2}{2c}\int_{\theta_0-c}^{\theta_0+c} E_\theta\left[\left(\widehat{\theta}-\theta\right)^2\right] d\theta \geq \frac{1}{2}$$

and the lower bound is attained by $\widehat{\theta}_0$. This means that in a sense asymptotically the estimator $\widehat{\theta}_0$ can be regarded as uniformly best one.

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