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THE COMPARISON OF ESTIMATORS OF RATIO FOR A REGRESSION MODEL

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In some regression model, the mean square errors of a ratio estimator, a grouped jackknife estimator, and an estimator based on the least square estimators (LSEs) are obtained and compared up to the order $O(n^{-3})$, where n is the size of the sample. The bias-adjusted ratio estimator and the jackknife estimator are also compared up to the order $O(n^{-3})$. Then it is concluded that the estimator based on the LSEs is an asymptotically better estimator of ratio up to the order $O(n^{-3})$. Some examples are given.

Key words and phrases: Regression model, mean square error, ratio estimator, least square estimator, grouped jackknife estimator

1. Introduction

In a regression model $Y=\alpha+\beta X+U$, we consider the estimation problem of a ratio $\rho=E(Y)/E(X)$. In regards to this problem, the jackknife method by Quenouille (1956), based on splitting the sample at random into groups, has been studied by many authors, and the optimum number of groups has been discussed. (See, e.g., Durbin (1959), Rao (1965) and Rao and Webster (1966)). A comparison of ratio-type estimators has also been done by Gray and Schucany (1972), Cochran (1977), Rao (1969, 1988) and others. Ina previous paper by Akahira and Kawai (1990), it was shown that the grouped jackknife estimator asymptotically had the minimum variance for a class of linear combinations of ratio estimators.

In this paper we consider a ratio estimator, a grouped jackknife estimator, and an estimator based on the least square estimators (LSEs), and obtain their mean square errors (MSEs). We also compare the estimators using the MSE up to the order $O(n^{-3})$, where n is the sample size. Then it is revealed that the estimator based on the LSEs is an asymptotically better estimator of ratio. Further, a bias reduction of the ratio estimator is carried out, and the MSE of the bias-adjusted ratio estimator is obtained. This is then compared with the MSE of the jackknife estimator up to the order $O(n^{-3})$. Following methods similar to those of Akahira and Kawai (1990), it is shown that the jackknife estimator has the minimum variance in a class of estimators of ratio. Finally some examples on the above are given.

2. Estimators of the Ratio in Some Regression Model

Suppose that $(X_1, Y_1), \cdots, (X_n, Y_n)$ are a random sample of size n. We consider the problem of estimating the ratio $\rho=E(Y_i)/E(X_i)$. Let $Y_i=\alpha+\beta X_i+U_i$, where $E(X_i)=k_0\neq 0$, $E(U_i|X_1, \cdots, X_n)=0$ and $V(U_i|X_1, \cdots, X_n)=\delta$ and $E(U_iU_j|X_1, \cdots, X_n)=$

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$0\ (i\neq j)$ with a constant δ of the order $O(1)$. Here $E(\cdot|\cdot)$ and $V(\cdot|\cdot)$ denote the conditional mean and variance, respectively. Then it is noted that, under the regression model, $\rho=\beta+(\alpha/k_0)$. Let $\bar{X}=\sum_{i=1}^n X_i/n$, $\bar{Y}=\sum_{i=1}^n Y_i/n$ and $\bar{U}=\sum_{i=1}^n U_i/n$. As estimators of ρ, a ratio estimator and a grouped jackknife estimator are known. The ratio estimator is given by

$$r=\frac{\bar{Y}}{\bar{X}}=\beta+\frac{\alpha+\bar{U}}{\bar{X}}\ .$$

On the other hand, we split the sample $(X_1, Y_1), \cdots, (X_n, Y_n)$ of size n into g groups of size m, i.e., $n=mg$ for $g\geq 2$. For each $j=1, \cdots, g$, $\bar{X}'_j$ and $\bar{Y}'_j$ are the sample means after omitting the j-th group. When we set

$$r'_j=\frac{\bar{Y}'_j}{\bar{X}'_j} \qquad (j=1, \cdots, g)\ ,$$

the jackknife estimator $r_{JK,g}$ is given by

$$r_{JK,g}=gr-\frac{g-1}{g}\sum_{j=1}^{g} r'_j\ .$$

Let $\hat{\alpha}_{LS}$ and $\hat{\beta}_{LS}$ be the least square estimators (LSEs) of α and β, that is,

$$\hat{\beta}_{LS}=\frac{\sum_{i=1}^n (X_i-\bar{X})(Y_i-\bar{Y})}{\sum_{i=1}^n (X_i-\bar{X})^2} \qquad \text{and} \qquad \hat{\alpha}_{LS}=\bar{Y}-\hat{\beta}_{LS}\bar{X}\ .$$

Then we also consider the following estimator $\hat{\rho}$ based on the LSEs

$$\hat{\rho}=\hat{\beta}_{LS}+\frac{\hat{\alpha}_{LS}}{k_0}\ ,$$

where k_0 is known. Since $E(\hat{\alpha}_{LS})=\alpha$ and $E(\hat{\beta}_{LS})=\beta$, it follows that $\hat{\rho}$ is the unbiased estimator of ρ. In the subsequent discussion we shall compare the above estimators using their MSEs.

3. Comparison of the Estimators

Suppose that $X_1, \cdots, X_n$ are independent and identically distributed (i.i.d.) random variables with mean k_0, variance σ^2, and, for each $l=3, 4, \cdots$, we denote the l-th order moment by $\mu_l=E[(X_1-k_0)^l]$. In this section and section 4 we assume that k_0 is known. In the regression model, without loss of generality we may assume $k_0=1$. Then we compare the mean square errors of $\hat{\rho}$, r and $r_{JK,g}$ up to $O(n^{-3})$. First, in order to get the mean square error of $\hat{\rho}$, we need the following lemma.

LEMMA 3.1. *Assume that $X_1, \cdots, X_n$ are i.i.d. ramdom variables with a continuous density function with mean 1, variance σ^2 and finite μ_6, where $0<\sigma^2<\infty$. Then*

$$E\left[\frac{\sum_{i=1}^n (X_i-1)^2}{\sum_{i=1}^n (X_i-\bar{X})^2}\right]=1+\frac{1}{n}+O\left(\frac{1}{n^2}\right),$$

where $\bar{X}=\sum_{i=1}^n X_i/n$.

PROOF. Putting $W_n=\sum_{i=1}^n (X_i-1)^2/\sum_{i=1}^n (X_i-\bar{X})^2$, we have

$$W_n=1+\frac{n(\bar{X}-1)^2}{\sum_{i=1}^n (X_i-\bar{X})^2}.$$

Let $Z=\sqrt{n}(\bar{X}-1)$ and $U=\sqrt{n}(S^2-\sigma^2)$ with $S^2=\sum_{i=1}^n (X_i-\bar{X})^2/n$. Then

$$\text{(3.1)} \qquad W_n=1+\frac{Z^2}{nS^2}$$

$$=1+\frac{1}{\sigma^2 n}Z^2-\frac{1}{\sigma^4 n\sqrt{n}}UZ^2+\frac{1}{\sigma^6 n^2}U^2Z^2+o_p\left(\frac{1}{n^2}\right).$$

It is clear that $E(Z^2)=\sigma^2$. Then we have

$$\text{(3.2)} \qquad E(UZ^2)=\frac{1}{\sqrt{n}}(\mu_4-4\sigma^4)-\frac{1}{n\sqrt{n}}(\mu_4-3\sigma^4)=O\left(\frac{1}{\sqrt{n}}\right).$$

Since

$$E[S^2(\bar{X}-1)^2]=\left(\frac{1}{n^2}-\frac{1}{n^3}\right)\mu_4+(n-1)\left(\frac{1}{n^2}-\frac{3}{n^3}\right)\sigma^4,$$

we obtain

$$\text{(3.3)} \qquad E(U^2Z^2)=\sigma^2\mu_4+\sigma^6+2\mu_3^2+O\left(\frac{1}{n}\right)=O(1).$$

From (3.1), (3.2) and (3.3), we have

$$E(W_n)=1+\frac{1}{n}+O\left(\frac{1}{n^2}\right).$$

This completes the proof.

From Lemma 3.1, the mean square error of $\hat{\rho}$, $MSE(\hat{\rho})$, is given by

$$\begin{aligned}
MSE(\hat{\rho})&=V(\hat{\rho})\\
&=E[V(\hat{\rho}\mid X_1,\cdots,X_n)]+V[E(\hat{\rho}\mid X_1,\cdots,X_n)]\\
&=E[V(\hat{\rho}\mid X_1,\cdots,X_n)]\\
&=E[V(\hat{\beta}_{LS}\mid X_1,\cdots,X_n)+V(\hat{\alpha}_{LS}\mid X_1,\cdots,X_n)\\
&\qquad +2Cov(\hat{\beta}_{LS},\hat{\alpha}_{LS}\mid X_1,\cdots,X_n)]\\
&=\frac{\delta}{n}E\left[\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n (X_i-\bar{X})^2}+\frac{n}{\sum_{i=1}^n (X_i-\bar{X})^2}-\frac{2\sum_{i=1}^n X_i}{\sum_{i=1}^n (X_i-\bar{X})^2}\right]\\
&=\frac{\delta}{n}E\left[\frac{\sum_{i=1}^n (X_i-1)^2}{\sum_{i=1}^n (X_i-\bar{X})^2}\right]\\
&=\frac{\delta}{n}+\frac{\delta}{n^2}+O(n^{-3}).
\end{aligned}$$

Next we shall get the mean square error of the ratio estimator r. Letting $Z=\sqrt{n}(\bar{X}-1)$, we have

$$\text{(3.4)} \qquad E\left(\frac{1}{\bar{X}}\right)=E\left[\frac{1}{1+(Z/\sqrt{n})}\right]$$

$$=1-\frac{E(Z)}{\sqrt{n}}+\frac{E(Z^2)}{n}-\frac{E(Z^3)}{n\sqrt{n}}+\frac{E(Z^4)}{n^2}+O\left(\frac{1}{n^3}\right)$$

$$=1+\frac{\sigma^2}{n}+\frac{1}{n^2}(-\mu_3+3\sigma^4)+O\left(\frac{1}{n^3}\right),$$

hence

(3.5)
$$E(r-\beta)=E\left(\frac{\alpha+\bar{U}}{\bar{X}}\right)=\alpha E\left(\frac{1}{\bar{X}}\right)$$
$$=\alpha\left\{1+\frac{\sigma^2}{n}+\frac{1}{n^2}(-\mu_3+3\sigma^4)\right\}+O\left(\frac{1}{n^3}\right).$$

We also obtain

(3.6)
$$E\left(\frac{1}{\bar{X}^2}\right)=E\left[\left\{\frac{1}{1+(Z/\sqrt{n})}\right\}^2\right]$$
$$=1-\frac{2E(Z)}{\sqrt{n}}+\frac{3E(Z^2)}{n}-\frac{4E(Z^3)}{n\sqrt{n}}+\frac{5E(Z^4)}{n^2}+O\left(\frac{1}{n^3}\right)$$
$$=1+\frac{3\sigma^2}{n}+\frac{1}{n^2}(-4\mu_3+15\sigma^4)+O\left(\frac{1}{n^3}\right),$$

which yields

(3.7)
$$E[(r-\beta)^2]=E\left[\left(\frac{\alpha+\bar{U}}{\bar{X}}\right)^2\right]=\left(\alpha^2+\frac{\delta}{n}\right)E\left(\frac{1}{\bar{X}^2}\right)$$
$$=\left(\alpha^2+\frac{\delta}{n}\right)\left\{1+\frac{3\sigma^2}{n}+\frac{1}{n^2}(-4\mu_3+15\sigma^4)\right\}+O\left(\frac{1}{n^3}\right).$$

If follows from (3.5) and (3.7) that the mean square error of r, $MSE(r)$, is given by
$$MSE(r)=E[\{(r-\beta)-\alpha\}^2]$$
$$=\alpha^2\left\{\frac{\sigma^2}{n}+\frac{1}{n^2}(-2\mu_3+9\sigma^4)\right\}+\frac{\delta}{n}\left(1+\frac{3}{n}\sigma^2\right)+O\left(\frac{1}{n^3}\right).$$

Further, we obtain the mean square error of the jackknife estimator $r_{JK,g}$. Letting $Z'_j=\sqrt{n-m}(\bar{X}'_j-1)$ $(j=1, \cdots, n)$, we have
$$E(r_{JK,g}-\beta)=\alpha\left\{gE\left(\frac{1}{\bar{X}}\right)-\frac{g-1}{g}\sum_{j=1}^{g}E\left(\frac{1}{\bar{X}'_j}\right)\right\}$$
$$=\alpha\left\{gE\left[\frac{1}{1+(Z/\sqrt{n})}\right]-(g-1)E\left[\frac{1}{1+(Z'_j/\sqrt{n-m})}\right]\right\}$$
$$=\alpha\left[g\left\{1+\frac{\sigma^2}{n}+\frac{1}{n^2}(-\mu_3+3\sigma^4)\right\}\right.$$
$$\left.-(g-1)\left\{1+\frac{1}{n}\frac{g}{g-1}\sigma^2+\frac{1}{n^2}\left(\frac{g}{g-1}\right)^2(-\mu_3+3\sigma^4)\right\}\right]+O\left(\frac{1}{n^3}\right)$$
$$=\alpha\left\{1+\frac{1}{n^2}\frac{g}{g-1}(\mu_3-3\sigma^4)\right\}+O\left(\frac{1}{n^3}\right),$$

which yields
$$E[(r_{JK,g}-\beta)^2]=\alpha^2E\left[\left(\frac{g}{\bar{X}}-\frac{g-1}{g}\sum_{j=1}^{g}\frac{1}{\bar{X}'_j}\right)^2\right]+E\left[\left(\frac{g\bar{U}}{\bar{X}}-\frac{g-1}{g}\sum_{j=1}^{g}\frac{\bar{U}'_j}{\bar{X}'_j}\right)^2\right]$$
$$=\alpha^2\left\{1+\frac{\sigma^2}{n}+\frac{2}{n^2}\frac{g}{g-1}(\mu_3-2\sigma^4)\right\}+\frac{\delta}{n}\left(1+\frac{1}{n}\frac{g}{g-1}\sigma^2\right)+O\left(\frac{1}{n^3}\right).$$

Hence the mean square error of $r_{JK,g}$, $MSE(r_{JK,g})$, is given by
$$MSE(r_{JK,g})=E[\{(r_{JK,g}-\beta)-\alpha\}^2]$$
$$=\alpha^2\left(\frac{\sigma^2}{n}+\frac{2}{n^2}\frac{g}{g-1}\sigma^4\right)+\frac{\delta}{n}\left(1+\frac{1}{n}\frac{g}{g-1}\sigma^2\right)+O\left(\frac{1}{n^3}\right).$$

From the above, we get the following theorem in the setup of section 2.

THEOREM 3.1. *Assume that $X_1, \cdots, X_n$ are i.i.d. random variables with a continuous density function with mean 1, variance σ^2 and finite μ_6, where $0<\sigma^2<\infty$. Suppose that $E(\bar{X}^{-1})$ and $E(\bar{X}^{-2})$ asymptotically exist up to the order $O(n^{-3})$ in the sence of (3.4) and (3.6), respectively, and also $E(\bar{X}_j'^{-2})$, $E(\bar{X}^{-1}\bar{X}_j'^{-1})$ $(j=1, \cdots, g)$ and $E(\bar{X}_i'^{-1}\bar{X}_j'^{-1})$ $(i\neq j;\ i, j=1, \cdots, g)$ asymptotically exist up to the order $O(n^{-3})$ in a similar sence to the above. Then the MSEs of $\hat{\rho}$, r and $r_{JK,g}$ are given up to the order $O(n^{-3})$ as follows.*

$$MSE(\hat{\rho})=V(\hat{\rho})=\frac{1}{n}\left(\delta+\frac{\delta}{n}\right)+O(n^{-3}),$$

$$MSE(r)=\frac{1}{n}(\delta+\alpha^2\sigma^2)+\frac{1}{n^2}\{\alpha^2(-2\mu_3+9\sigma^4)+3\delta\sigma^2\}+O(n^{-3}),$$

$$MSE(r_{JK,g})=\frac{1}{n}(\delta+\alpha^2\sigma^2)+\frac{1}{n^2}\left\{\frac{2g}{g-1}\alpha^2\sigma^4+\frac{g}{g-1}\delta\sigma^2\right\}+O(n^{-3}),$$

and moreover, if $g=n$, then

$$MSE(r_{JK,n})=\frac{1}{n}(\delta+\alpha^2\sigma^2)+\frac{1}{n^2}(2\alpha^2\sigma^4+\delta\sigma^2)+O(n^{-3}).$$

Further, the comparison of the estimators $\hat{\rho}$, r and $r_{JK,n}$ with respect to the MSE is given up to the order $O(n^{-3})$ as follows.

$MSE(\hat{\rho})\leq MSE(r_{JK,n})\leq MSE(r)$ for $\alpha\neq 0$ and $\mu_3\leq(7/2)\sigma^4+\delta\sigma^2/\alpha^2$,

$MSE(\hat{\rho})\leq MSE(r)\leq MSE(r_{JK,n})$ for $\alpha\neq 0$ and $\mu_3>(7/2)\sigma^4+\delta\sigma^2/\alpha^2$,

$MSE(r_{JK,n})\leq MSE(r)\leq MSE(\hat{\rho})$ for $\alpha=0$ and $0<\sigma^2\leq 1/3$,

$MSE(r_{JK,n})\leq MSE(\hat{\rho})\leq MSE(r)$ for $\alpha=0$ and $1/3<\sigma^2\leq 1$,

$MSE(\hat{\rho})\leq MSE(r_{JK,n})\leq MSE(r)$ for $\alpha=0$ and $\sigma^2>1$.

The proof of the first half of Theorem 3.1 is given above. That of the latter half is omitted since it is straightforward from the first one. It is easily seen that $MSE(r_{JK,g})\geq MSE(r_{JK,n})+O(n^{-3})$ for $2\leq g\leq n$. Hence $r_{JK,n}$ will often be used in the class of the grouped jackknife estimators.

REMARK 3.1. In the problem of estimating ρ, it is seen from Theorem 3.1 that $\hat{\rho}$ is asymptotically better than the other estimators up to the order $O(n^{-3})$ for $\alpha\neq 0$ or for $\alpha=0$ and $\sigma^2>1$. Since $\hat{\rho}$ is also unbiased for ρ and further seems to be simpler than $r_{JK,n}$ in the construction, $\hat{\rho}$ is recommended as an asymptotically better estimator up to the $O(n^{-3})$, though α and σ^2 are unknown.

4. Bias Adjustment of the Estimators

Although $\hat{\rho}$ is an unbiased estimator of ρ, the ratio estimator r and the jackknife estimator $r_{JK,n}$ have a bias up to the order $O(n^{-3})$,

$$E(r)=\rho+\frac{\alpha\sigma^2}{n}-\frac{\alpha}{n^2}(\mu_3-3\sigma^4)+O(n^{-3})$$

$$E(r_{JK,n})=\rho+\frac{\alpha}{n^2}(\mu_8-3\sigma^4)+O(n^{-3}).$$

So it is desirable to compare r with $r_{JK,n}$ after adjusting the bias of the order $O(n^{-1})$ of r. Let

$$r^*=r-\frac{1}{n}\hat{\alpha}_{LS}\hat{\sigma}^2,$$

where $\hat{\sigma}^2=\{1/(n-1)\}\sum_{i=1}^n(X_i-\bar{X})^2$. Since

$$\begin{aligned}E[\hat{\alpha}_{LS}\hat{\sigma}^2]&=\frac{1}{n-1}E\left[\bar{Y}\sum_{i=1}^n(X_i-\bar{X})^2-\bar{X}\sum_{i=1}^n(X_i-\bar{X})(Y_i-\bar{Y})\right]\\&=\frac{1}{n(n-1)}E\left[\sum_{i=1}^n\sum_{j=1}^n X_i^2Y_j-\sum_{i=1}^n\sum_{j=1}^n X_iX_jY_j\right]\\&=E(X_i^2Y_j)-E(X_iX_jY_j)\\&=\alpha\sigma^2,\end{aligned}$$

it follows that $\hat{\alpha}_{LS}\hat{\sigma}^2$ is an unbiased estimator of $\alpha\sigma^2$, hence

$$E(r^*)=\rho-\frac{\alpha}{n^2}(\mu_3-3\sigma^4)+O(n^{-3}),$$

which means that r^* is a bias-adjusted ratio estimator from r up to the order $O(n^{-3})$. It is also noted that the estimator $\bar{r}=(r^*+r_{JK,n})/2$ is asymptotically unbiased up to the order $O(n^{-3})$, i.e. $E(\bar{r})=\rho+O(n^{-3})$. From the above, in the setup of section 2, we have the following.

THEOREM 4.1. *Assume that $X_1,\cdots,X_n$ are i.i.d. random variables with mean 1, variance σ^2 and finite μ_6, where $0<\sigma^2<\infty$. Suppose that $E[\bar{X}^{-1}(X_1-\bar{X})^2]$ asymptotically exists up to the order $O(n^{-2})$ in a similar sense of (3.4). Then the mean square error, $MSE(r^*)$, of the bias-adjusted ratio estimator r^* is given by*

$$MSE(r^*)=MSE(r)-\frac{1}{n^2}\{\alpha^2(\mu_3+\sigma^4)+2\delta\sigma^2\}+O(n^{-3}), \tag{4.1}$$

PROOF. We have

$$\begin{aligned}MSE(r^*)&=E[(r^*-\rho)^2]=E\left[(r-\alpha-\beta-\frac{1}{n}\hat{\alpha}_{LS}\hat{\sigma}^2)^2\right]\\&=MSE(r)-\frac{2}{n}E[r\hat{\alpha}_{LS}\hat{\sigma}^2]+\frac{2}{n}\alpha(\alpha+\beta)\sigma^2+\frac{1}{n^2}E[\hat{\alpha}_{LS}^2\sigma^4].\end{aligned}\tag{4.2}$$

We also obtain

$$\begin{aligned}E[r\hat{\alpha}_{LS}\hat{\sigma}^2]&=\frac{1}{n-1}E\left[\left(\beta+\frac{\alpha+\bar{U}}{\bar{X}}\right)\left\{\bar{Y}\sum_{i=1}^n(X_i-\bar{X})^2-\bar{X}\sum_{i=1}^n(X_i-\bar{X})(Y_i-\bar{Y})\right\}\right]\\&=\alpha\beta\sigma^2+\frac{1}{n-1}E\left[(\alpha+\bar{U})\frac{\bar{Y}}{\bar{X}}\sum_{i=1}^n(X_i-\bar{X})^2-(\alpha+\bar{U})\sum_{i=1}^n(X_i-\bar{X})(Y_i-\bar{Y})\right]\\&=\alpha\beta\sigma^2+\frac{1}{n-1}E\left[(\alpha+\bar{U})\left(\beta+\frac{\alpha+\bar{U}}{\bar{X}}\right)\sum_{i=1}^n(X_i-\bar{X})^2\right]\\&\quad-\frac{\alpha}{n-1}E\left[\sum_{i=1}^n(X_i-\bar{X})(Y_i-\bar{Y})\right]\end{aligned}\tag{4.3}$$

$$=2\alpha\beta\sigma^2+\frac{\alpha^2}{n-1}E\left[\frac{1}{\bar{X}}\sum_{i=1}^n(X_i-\bar{X})^2\right]+\frac{\delta}{n(n-1)}E\left[\frac{1}{\bar{X}}\sum_{i=1}^n(X_i-\bar{X})^2\right]$$

$$-\frac{\alpha}{n-1}E\left[\sum_{i=1}^n(X_i-\bar{X})\{\beta(X_i-\bar{X})+(U_i-\bar{U})\}\right]$$

$$=\alpha\beta\sigma^2+\frac{1}{n-1}\left(\alpha^2+\frac{\delta}{n}\right)E\left[\frac{1}{\bar{X}}\sum_{i=1}^n(X_i-\bar{X})^2\right].$$

Since

$$E\left[\frac{1}{\bar{X}}\sum_{i=1}^n(X_i-\bar{X})^2\right]=(n-1)\sigma^2\left\{1+\frac{1}{n}\left(\sigma^2+\frac{\mu_3}{\sigma^2}\right)+O\left(\frac{1}{n^2}\right)\right\}$$

it follows from (4.3) that

$$E[r\hat{\alpha}_{LS}\hat{\sigma}^2]=\alpha(\alpha+\beta)\sigma^2+\frac{1}{n}\delta\sigma^2+\frac{1}{n}\alpha^2(\mu_3+\sigma^4)+O\left(\frac{1}{n^2}\right). \tag{4.4}$$

Since

$$E[\hat{\alpha}_{LS}^2\hat{\sigma}^4]=\frac{1}{n^2(n-1)^2}E\left[\left(\sum_{i=1}^n\sum_{j=1}^n X_i^2Y_j-\sum_{i=1}^n\sum_{j=1}^n X_iX_jY_j\right)^2\right]$$

$$=\alpha^2\sigma^4+O\left(\frac{1}{n}\right),$$

it follows from (4.2) and (4.4) that

$$MSE(r^*)=MSE(r)-\frac{1}{n^2}\{\alpha^2(\mu_3+\sigma^4)+2\delta\sigma^2\}+O(n^{-3}).$$

This completes the proof.

It is noted from (4.1) that, for $\mu_3\geq 0$,

$$MSE(r^*)\leq MSE(r)+O(n^{-3}).$$

If we compare r^* with $r_{JK,n}$, then

$$MSE(r^*)\leq MSE(r_{JK,n})+O(n^{-3}) \qquad \text{for } \mu_3\geq 2\sigma^4 \tag{4.5}$$

$$MSE(r^*)\geq MSE(r_{JK,n})+O(n^{-3}) \qquad \text{for } \mu_3\leq 2\sigma^4 \tag{4.6}$$

If the distribution of X_1 is normal, then r^* is asymptotically better than $r_{JK,n}$ in the sense that (4.5) holds. The comparison of the other estimators with r^* will be given in Example 6.1.

5. Optimum Property of the Grouped Jackknife Estimator in Some Class

As stated in the previous section, the jackknife estimator is not always asymptotically better than the others. However, if we restrict the estimators to some class of linear combinations of ratio estimators, then the grouped jackknife estimator has asymptotically the minimum variance (see Akahira and Kawai (1990)). In this section, it is shown that the grouped jackknife estimator has the minimum variance in some class.

We consider a class of linear combinations of the ratio estimators r and r_j' $(j=1, \cdots, g)$ defined by

$$\mathcal{R}_1=\left\{\hat{r}=gr+\sum_{j=1}^{g} w_j r'_j \,\middle|\, \sum_{j=1}^{g} w_j=-(g-1)\right\}.$$

It is easily seen that the grouped jackknife estimator $r_{JK,g}$ belongs to $\mathcal{R}_1$. Then, in the setup of section 2, we have the following.

THEOREM 5.1. *Assume that* $k_0E(1/\bar{X})=1+h$, $k_0\{E(1/\bar{X}'_j)-E(1/\bar{X})\}=h/(g-1)$ $(j=1, \cdots, g)$, *and there exist* $E(1/\bar{X}^2)$, $E(1/\bar{X}'^2_j)$ $(j=1, \cdots, g)$, $E[1/(\bar{X}\bar{X}'_j)]$ $(j=1, \cdots, g)$ *and* $E[1/(\bar{X}'_i\bar{X}'_j)]$ $(i\neq j;\ i, j=1, \cdots, g)$, *where* k_0 is *unknown and* h *is a constant of order* $O(n^{-1})$. *Then any estimator* $\hat{r}$ *of the class* $\mathcal{R}_1$ *is unbiased for* ρ, *and the jackknife estimator* $r_{JK,g}$ *has the minimum variance in the class* $\mathcal{R}_1$.

The proof is omitted since it is similar to those of Lemmas 2.1 and 2.3 of Akahira and Kawai (1990). An example for Theorem 5.1 will be given later.

6. Examples

In this section we give an example of the comparison of the estimators $\hat{\rho}$, r, r^* and $r_{JK,n}$ with respect to the MSE up to the order $O(n^{-3})$ in the normal case according to Theorems 4.1 and 4.2. An example is also given on the optimality of the jackknife estimator $r_{JK,g}$ in the class $\mathcal{R}_1$ in the inverse Gaussian case according to Theorem 5.1. The examples on the latter property are given in the normal and gamma cases in Akahira and Kawai (1990).

EXAMPLE 6.1 (Normal Case). Assume that $X_1, \cdots, X_n$ are independently, identically and normally distributed random variables with mean 1 and variance σ^2. According to Theorems 4.1 and 4.2, it is seen that the MSEs of $\hat{\rho}$, r, r^* and $r_{JK,n}$ are given up to the order $O(n^{-3})$ as follows.

$$MSE(\hat{\rho})=\frac{\delta}{n}+\frac{\delta}{n^2}+O(n^{-3}),$$

$$MSE(r)=\frac{1}{n}(\delta+\alpha^2\sigma^2)+\frac{\sigma^2}{n^2}(9\alpha^2\sigma^2+3\delta)+O(n^{-3}),$$

$$MSE(r^*)=\frac{1}{n}(\delta+\alpha^2\sigma^2)+\frac{\sigma^2}{n^2}(8\alpha^2\sigma^2+\delta)+O(n^{-3}),$$

$$MSE(r_{JK,n})=\frac{1}{n}(\delta+\alpha^2\sigma^2)+\frac{\sigma^2}{n^2}(2\alpha^2\sigma^2+\delta)+O(n^{-3}).$$

It is noted that the MSEs of r and $r_{JK,n}$ coincide with those of Durbin (1959) and Rao (1965) for $\mu=1$. Their comparisons are given up to the order $O(n^{-3})$ as follows.

$$MSE(\hat{\rho})\leq MSE(r_{JK,n})\leq MSE(r^*)\leq MSE(r) \quad \text{for } \alpha\neq 0,$$

$$MSE(r_{JK,n})=MSE(r^*)\leq MSE(r)\leq MSE(\hat{\rho}) \quad \text{for } \alpha=0 \text{ and } 0<\sigma^2\leq 1/3,$$

$$MSE(r_{JK,n})=MSE(r^*)\leq MSE(\hat{\rho})\leq MSE(r) \quad \text{for } \alpha=0 \text{ and } 1/3<\sigma^2\leq 1,$$

$$MSE(\hat{\rho})\leq MSE(r_{JK,n})=MSE(r^*)\leq MSE(r) \quad \text{for } \alpha=0 \text{ and } \sigma^2>1.$$

Note that the asymptotic values of $E(\bar{X}^{-1})$, $E(\bar{X}^{-2})$, etc. are given in Rao (1965) (see also Remark 4.1 of Akahira and Kawai (1990)).

EXAMPLE 6.2 (Inverse Gaussian case). Assume that $X_1, \cdots, X_n$ are i.i.d. random variables with the inverse Gaussian distribution with parameter (μ, λ) $(\mu>0, \lambda>0)$, which is denoted as $IG(\mu, \lambda)$, with the density function given by $\sqrt{\lambda/(2\pi x^3)}\exp\{-\lambda(x-\mu)^2/(2\mu^2 x)\}$ for $x>0$ and zero otherwise. Since $\bar{X}$ has the distribution $IG(\mu, n\lambda)$, the following holds.

$$k_0=E(\bar{X})=\mu\,,$$

$$E\left(\frac{1}{\bar{X}}\right)=\frac{1}{\mu}+\frac{1}{n\lambda}\,,$$

$$E\left(\frac{1}{\bar{X}'_j}\right)=\frac{1}{\mu}+\frac{g}{n(g-1)\lambda} \qquad (j=1, \cdots, g)\,,$$

$$E\left(\frac{1}{\bar{X}^2}\right)=\frac{1}{\mu^2}+3\frac{1}{\mu}\frac{1}{n\lambda}+3\frac{1}{n^2\lambda^2}\,,$$

$$E\left(\frac{1}{\bar{X}'^2_j}\right)=\frac{1}{\mu^2}+3\frac{1}{\mu}\frac{g}{n(g-1)\lambda}+3\frac{g^2}{n^2(g-1)^2\lambda^2} \qquad (j=1, \cdots, g)\,.$$

By Schwarz's inequality, we see that $E[1/(\bar{X}\bar{X}'_j)]$ $(j=1, \cdots, g)$ and $E[1/(\bar{X}'_i\bar{X}'_j)]$ $(i\neq j;\ i, j=1, \cdots, g)$ exist. We also have

$$k_0E\left(\frac{1}{\bar{X}}\right)-1=\frac{\mu}{n\lambda}\,,$$

$$k_0\left\{E\left(\frac{1}{\bar{X}'_j}\right)-E\left(\frac{1}{\bar{X}}\right)\right\}=\frac{\mu}{n\lambda}\,\frac{1}{g-1}\,.$$

From the above it is easily seen that the conditions of Theorem 5.1 hold. A straightforward calculation also leads to the fact that any estimator $\hat{r}$ of $\mathcal{R}_1$ is unbiased for ρ, i.e.,

$$E(\hat{r})=\beta+\frac{\alpha}{\mu}=\rho\,.$$

From Theorem 5.1 we see that the jackknife estimator $r_{JK,g}$ is unbiased for ρ and has the minimum variance in the class $\mathcal{R}_1$.

Though the bias-adjusted ratio estimator r^* does not belong to the class $\mathcal{R}_1$, when comparing r^* with $r_{JK,n}$, we have from (4.5) and (4.6)

$$MSE(r^*)\leq MSE(r_{JK,n})+O(n^{-3}) \qquad \text{for } \mu=1\,.$$

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