

THE OPTIMALITY OF THE GROUPED JACKKNIFE ESTIMATOR OF RATIO IN SOME REGRESSION MODEL

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In some regression model, the minimum (asymptotic) variance estimator of a ratio is discussed for some class of linear combinations of ratio estimators, and the jackknife procedure is considered. It is seen that the grouped jackknife estimator is optimal in the sense that it has asymptotically the minimum variance in the class. Higher order bias reduction of the estimators is discussed, and some examples are given.

1. Introduction

In a regression model $Y = \alpha + \beta X + U$, the problem of estimating a ratio $\rho = E(Y)/E(X)$ has been studied by many authors. The problem has been discussed from the viewpoint of the jackknife method, which Quenouille (1956) considered by dividing a sample of size n at random into blocks. In particular, the problem of the optimum choice of the number of blocks has been investigated. The jackknife estimator has also been compared with other ratio estimators in the same model. (See, e.g., Durbin (1959), Rao (1965), Rao and Webster (1966), Gray and Schucany (1972), and Rao (1988)).

In this paper we consider a class of linear combinations of ratio estimators of ρ , taking into consideration a grouped jackknife estimator. Sufficient conditions under which an estimator in this class has a bias reduction of order n^{-1} are obtained. Further the minimum (asymptotic) variance estimation of the ratio ρ is discussed for a class restricted to estimators with such a property. Consequently, we will see that the grouped jackknife estimator is optimal in the sense that it has asymptotically the minimum variance in the class. Higher order bias reduction of the estimators is discussed, and examples are given for the normal and gamma cases which were treated by Durbin (1959), Rao (1965), and Rao and Webster (1966).

2. Minimum variance in a ratio estimation

Suppose that $(X_1, Y_1), \dots, (X_n, Y_n)$ are a random sample of size n . We consider the problem of estimating the ratio $\rho = E(Y_i)/E(X_i)$. Let $Y_i = \alpha + \beta X_i + U_i$, where $E(X_i) = k_0 \neq 0$, $E(U_i|X_1, \dots, X_n) = 0$, and $V(U_i|X_1, \dots, X_n) = n\delta$ and $E(U_i U_j|X_1, \dots, X_n) = 0$ ($i \neq j$) with a constant δ of $O(n^{-1})$. Here $E(\cdot|\cdot)$ and $V(\cdot|\cdot)$ denote the conditional mean and variance, respectively. Let $\bar{X} = \sum_{i=1}^n X_i/n$, $\bar{Y} = \sum_{i=1}^n Y_i/n$, and $\bar{U} = \sum_{i=1}^n U_i/n$. Assume that $h = O(n^{-1})$. Then, a ratio estimator $r = \bar{Y}/\bar{X}$ is written as $r = \beta + \{(\alpha + \bar{U})/\bar{X}\}$. Partition the sample

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$(X_1, Y_1), \dots, (X_n, Y_n)$ into g blocks, each of size m , $n=mg$, where $g \geq 2$. By (\bar{X}'_j, \bar{Y}'_j) we denote the sample means based on a sample of size $m(g-1)$, where the j -th block of size m is deleted. Letting $r'_j = \bar{Y}'_j / \bar{X}'_j$ ($j=1, \dots, g$), we consider a class R of linear combinations of the ratio estimators r and r'_j ($j=1, \dots, g$) defined by

$$R = \left\{ \hat{r} = w_0 r + \sum_{j=1}^g w_j r'_j \mid -\infty < w_j < \infty \quad (j=0, 1, \dots, g) \right\}.$$

LEMMA 2.1. Suppose that

$$(2.1) \quad k_0 E(\bar{X}^{-1}) - 1 = h + o(h),$$

and

$$(2.2) \quad k_0 \{E(\bar{X}'^{-1}_j) - E(\bar{X}^{-1})\} = \frac{h}{g-1} + o(h) \quad (j=1, \dots, g).$$

Then the weights for any estimator $\hat{r} = w_0 r + \sum_{j=1}^g w_j r'_j$ in R with a bias reduction up to the order $o(h)$, i.e., $E(\hat{r}) = \rho + o(h)$, are of the form

$$\sum_{j=0}^g w_j = 1 + o(h); \quad \sum_{j=1}^g w_j = -(g-1) + o(g).$$

PROOF. Since

$$\rho = \frac{E(\bar{Y})}{E(\bar{X})} = \beta + \frac{\alpha}{E(\bar{X})} = \beta + \frac{\alpha}{k_0},$$

$$E(r) = \alpha E(\bar{X}^{-1}) + \beta, \quad \text{and} \quad E(r'_j) = \alpha E(\bar{X}'^{-1}_j) + \beta \quad (j=1, \dots, g),$$

it follows that for any $\hat{r} \in R$

$$E(\hat{r}) = \beta \sum_{j=0}^g w_j + \alpha \left\{ w_0 E(\bar{X}^{-1}) + \sum_{j=1}^g w_j E(\bar{X}'^{-1}_j) \right\}.$$

In order for the bias of \hat{r} to be reduced up to the order $o(h)$, it is necessary that $\sum_{j=0}^g w_j = 1 + o(h)$. Since $w_0 = 1 - \sum_{j=1}^g w_j + o(h)$, we have

$$\begin{aligned} (2.3) \quad E(\hat{r}) &= \beta + \frac{\alpha}{k_0} + \alpha \left\{ w_0 E(\bar{X}^{-1}) - \frac{1}{k_0} + \sum_{j=1}^g w_j E(\bar{X}'^{-1}_j) \right\} + o(h) \\ &= \rho + \frac{\alpha}{k_0} \left\{ k_0 w_0 E(\bar{X}^{-1}) - 1 + k_0 \sum_{j=1}^g w_j E(\bar{X}'^{-1}_j) \right\} + o(h) \\ &= \rho + \frac{\alpha}{k_0} \left\{ k_0 \left(1 - \sum_{j=1}^g w_j \right) E(\bar{X}^{-1}) - 1 + k_0 \sum_{j=1}^g w_j E(\bar{X}'^{-1}_j) \right\} + o(h) \\ &= \rho + \frac{\alpha}{k_0} \left[k_0 E(\bar{X}^{-1}) - 1 + \sum_{j=1}^g w_j k_0 \{E(\bar{X}'^{-1}_j) - E(\bar{X}^{-1})\} \right] + o(h). \end{aligned}$$

By conditions (2.1) and (2.2), we obtain

$$\begin{aligned} E(\hat{r}) &= \rho + \frac{\alpha}{k_0} \left(h + \sum_{j=1}^g w_j \frac{h}{g-1} \right) + o(h) \\ &= \rho + \frac{\alpha}{k_0} \left(1 + \frac{1}{g-1} \sum_{j=1}^g w_j \right) h + o(h). \end{aligned}$$

In order to reduce the bias of \hat{r} up to the order $o(h)$, it is necessary that

$$1 + \frac{1}{g-1} \sum_{j=1}^g w_j = o(1), \text{ i.e., } \sum_{j=1}^g w_j = -(g-1) + o(g).$$

This completes the proof.

Now we consider a subclass R_1 of R defined by

$$R_1 = \left\{ \tilde{r} = gr + \sum_{j=1}^g w_j r'_j \mid \sum_{j=1}^g w_j = -(g-1) \right\},$$

which is the simplest form of R with a bias reduction up to the order $o(h)$. Note from Lemma 2.1 that any estimator of R_1 has a bias reduction of the order n^{-1} .

The following theorem shows that any estimator of R_1 has the same bias up to the order $O(gh^4)$.

THEOREM 2.1. *Suppose that*

$$k_0 E(\bar{X}^{-1}) - 1 = h + c_1 h^2 + c_2 h^3 + O(h^4)$$

and

$$k_0 \{E(\bar{X}'^{-1}) - E(\bar{X}^{-1})\} = \frac{h}{g-1} + c_3 h^2 + c_4 h^3 + O(h^4) \quad (j=1, \dots, g),$$

where c_1 and c_2 are constants, and c_3 and c_4 are quantities of order $O(1)$, depending on g . Then for any $\tilde{r} \in R_1$,

$$(2.4) \quad E(\tilde{r}) = \rho + \frac{\alpha}{k_0} [\{c_1 - c_3(g-1)\}h^2 + \{c_2 - c_4(g-1)\}h^3] + O(gh^4).$$

PROOF. By the same argument as in the proof of Lemma 2.1, we have

$$\begin{aligned} (2.5) \quad E(\tilde{r}) &= \rho + \frac{\alpha}{k_0} \left[k_0 E(\bar{X}^{-1}) - 1 + \sum_{j=1}^g w_j k_0 \{E(\bar{X}'^{-1}) - E(\bar{X}^{-1})\} \right] \\ &= \rho + \frac{\alpha}{k_0} \left\{ h + c_1 h^2 + c_2 h^3 - (g-1) \left(\frac{h}{g-1} + c_3 h^2 + c_4 h^3 \right) + O(gh^4) \right\} \\ &= \rho + \frac{\alpha}{k_0} [\{c_1 - c_3(g-1)\}h^2 + \{c_2 - c_4(g-1)\}h^3] + O(gh^4), \end{aligned}$$

which completes the proof.

REMARK 2.1. As is easily seen from the above, any estimator \tilde{r} of R_1 has the same bias up to the order $O(gh^4)$, since the bias of \tilde{r} , i.e., the second term of the right-hand side of (2.4), is independent of the estimator \tilde{r} .

The following theorem is used to obtain the asymptotic variance of the estimator $\tilde{r} \in R_1$.

THEOREM 2.2. *Suppose that*

$$(2.6) \quad E(\bar{X}^{-2}) = k_1 + O(h^4) \text{ and } E(\bar{X}'^{-2}) = k_2 + O(h^4), \text{ with } k_2 > 0 \quad (j=1, \dots, g);$$

$$(2.7) \quad E(\bar{X}^{-1} \bar{X}'^{-1}) = k_3 + O(h^4) \quad (j=1, \dots, g) \text{ and } E(\bar{X}_i^{-1} \bar{X}_j^{-1}) = k_4 + O(h^4) \\ (i \neq j; i, j=1, \dots, g).$$

Then for any $\tilde{r} \in R_1$,

$$\begin{aligned}
 (2.8) \quad E[(\bar{r}-\beta)^2] &= \left[\alpha^2(k_2-k_4) + \delta \left\{ \frac{k_2g}{g-1} - \frac{k_4g(g-2)}{(g-1)^2} \right\} \right] \left(\sum_{j=1}^g w_j^2 \right) \\
 &\quad + \alpha^2 \{ k_1g^2 - 2k_3g(g-1) + k_4(g-1)^2 \} + \delta \{ k_1g^2 - 2k_3g(g-1) + k_4g(g-2) \} \\
 &\quad + O(g^2h^4).
 \end{aligned}$$

PROOF. For any $\bar{r} \in R_1$ we have

$$\begin{aligned}
 \bar{r} &= g\bar{r} + \sum_{j=1}^g w_j \bar{r}'_j \\
 &= g \left(\beta + \frac{\alpha + \bar{U}}{\bar{X}} \right) + \sum_{j=1}^g w_j \left(\beta + \frac{\alpha + \bar{U}'_j}{\bar{X}'_j} \right) \\
 &= \beta + \alpha \left(\frac{g}{\bar{X}} + \sum_{j=1}^g \frac{w_j}{\bar{X}'_j} \right) + \frac{g\bar{U}}{\bar{X}} + \sum_{j=1}^g \frac{w_j \bar{U}'_j}{\bar{X}'_j}.
 \end{aligned}$$

Since $E(U_i|X_1, \dots, X_n) = 0$ and $E(U_i^2|X_1, \dots, X_n) = n\delta$ for $i=1, \dots, n$, and $E(U_i U_j|X_1, \dots, X_n) = 0$ ($i \neq j$; $i, j=1, \dots, n$) it follows from conditions (2.6) and (2.7) that

$$\begin{aligned}
 E[(\bar{r}-\beta)^2] &= \alpha^2 E \left[\left(\frac{g}{\bar{X}} + \sum_{j=1}^g \frac{w_j}{\bar{X}'_j} \right)^2 \right] + E \left[\left(\frac{g\bar{U}}{\bar{X}} + \sum_{j=1}^g \frac{w_j \bar{U}'_j}{\bar{X}'_j} \right)^2 \right] \\
 &= \alpha^2 \left\{ g^2 E(\bar{X}^{-2}) + 2g \sum_{j=1}^g w_j E(\bar{X}^{-1} \bar{X}'_j^{-1}) \right. \\
 &\quad \left. + \sum_{j=1}^g w_j^2 E(\bar{X}'_j^{-2}) + \sum_{\substack{i,j=1 \\ i \neq j}}^g w_i w_j E(\bar{X}'_i^{-1} \bar{X}'_j^{-1}) \right\} \\
 &\quad + g^2 E(\bar{U}^2 \bar{X}^{-2}) + 2g \sum_{j=1}^g w_j E(\bar{U} \bar{U}'_j \bar{X}^{-1} \bar{X}'_j^{-1}) \\
 &\quad + \sum_{j=1}^g w_j^2 E(\bar{U}'_j^2 \bar{X}'_j^{-2}) + \sum_{\substack{i,j=1 \\ i \neq j}}^g w_i w_j E(\bar{U}'_i \bar{U}'_j \bar{X}'_i^{-1} \bar{X}'_j^{-1}) \\
 &\quad + O(g^2h^4) \\
 &= \alpha^2 \left[k_1g^2 - 2k_3g(g-1) + k_2 \sum_{j=1}^g w_j^2 + k_4 \left\{ \left(\sum_{j=1}^g w_j \right)^2 - \sum_{j=1}^g w_j^2 \right\} \right] \\
 &\quad + \delta \left[k_1g^2 - 2k_3g(g-1) + \frac{k_2g}{g-1} \sum_{j=1}^g w_j^2 + \frac{k_4g(g-2)}{(g-1)^2} \left\{ \left(\sum_{j=1}^g w_j \right)^2 - \sum_{j=1}^g w_j^2 \right\} \right] \\
 &\quad + O(g^2h^4) \\
 &= \left(\sum_{j=1}^g w_j^2 \right) \left[\alpha^2(k_2-k_4) + \delta \left\{ \frac{k_2g}{g-1} - \frac{k_4g(g-2)}{(g-1)^2} \right\} \right] \\
 &\quad + \alpha^2 \{ k_1g^2 - 2k_3g(g-1) + k_4(g-1)^2 \} \\
 &\quad + \delta \{ k_1g^2 - 2k_3g(g-1) + k_4g(g-2) \} \\
 &\quad + O(g^2h^4).
 \end{aligned}$$

This completes the proof.

REMARK 2.2. By Schwarz's inequality, it is easily seen that $k_2 - k_4 \geq 0$. We also have

$$\begin{aligned}
 \frac{k_2g}{g-1} - \frac{k_4g(g-2)}{(g-1)^2} &= \frac{1}{(g-1)^2} \{ k_2g(g-1) - k_4g(g-2) \} \\
 &= \frac{1}{(g-1)^2} \{ g(g-2)(k_2-k_4) + k_2 \} \\
 &\geq \frac{k_2}{(g-1)^2} > 0.
 \end{aligned}$$

From this we see that the coefficient of the term $\sum_{j=1}^g w_j^2$ on the right-hand side of (2.8) is positive.

From Theorems 2.1 and 2.2, we have the following theorem.

THEOREM 2.3. *Under the same conditions as in Theorems 2.1 and 2.2, the estimator*

$$\tilde{r}^* = gr - \frac{g-1}{g} \sum_{j=1}^g r'_j$$

has the minimum asymptotic variance in the class R_1 up to the order $O(g^2 h^4)$.

PROOF. From Theorems 2.1 and 2.2, it follows that the asymptotic variance of $\tilde{r} \in R_1$ is given by

$$\begin{aligned} V(\tilde{r}) &= V(\tilde{r} - \beta) \\ &= E[(\tilde{r} - \beta)^2] - \{E(\tilde{r} - \beta)\}^2 \\ &= \left(\sum_{j=1}^g w_j^2 \right) \left[\alpha^2(k_2 - k_4) + \delta \left\{ \frac{k_2 g}{g-1} - \frac{k_4 g(g-2)}{(g-1)^2} \right\} \right] \\ &\quad + \alpha^2[k_1 g^2 - 2k_3 g(g-1) + k_4(g-1)^2] \\ &\quad - \{1 + (c_1 - c_3(g-1))h^2 + (c_2 - c_4(g-1))h^3\}^2 / k_0^2 \\ &\quad + \delta\{k_1 g^2 - 2k_3 g(g-1) + k_4 g(g-2)\} \\ &\quad + O(g^2 h^4). \end{aligned}$$

By Remark 2.2, in order to find an estimator \tilde{r}^* with minimum asymptotic variance in R_1 up to the order $O(g^2 h^4)$, it is enough to choose w_j ($j=1, \dots, g$), which minimizes $\sum_{j=1}^g w_j^2$ under the condition $\sum_{j=1}^g w_j = -(g-1)$. By Lagrangean method, we see that such w_j 's are given by

$$w_j = -\frac{g-1}{g} \quad (j=1, \dots, g).$$

Hence the estimator

$$\tilde{r}^* = gr - \frac{g-1}{g} \sum_{j=1}^g r'_j$$

has the minimum asymptotic variance up to the order $O(g^2 h^4)$. This completes the proof.

REMARK 2.3. The estimator \tilde{r}^* in the above theorem is known as a grouped jackknife estimator (e.g., see Rao, 1988).

3. Higher order bias reduction of the estimators

In this section we will discuss the problem of a bias reduction of estimators up to the higher order under the condition $g=O(1)$. Define subclasses R_2 and R_3 of R by

$$R_2 = \left\{ \tilde{r} = w_0 r + \sum_{j=1}^g w_j r'_j \mid \sum_{j=0}^g w_j = 1; \sum_{j=1}^g w_j = -(g-1) + a_1 h + O(h^2), \right. \\ \left. \text{with } -\infty < a_1 < \infty \right\},$$

and

$$R_3 = \left\{ \tilde{r} = w_0 r + \sum_{j=1}^g w_j r'_j \left| \sum_{j=0}^g w_j = 1; \quad \sum_{j=1}^g w_j = -(g-1) + a_1 h + a_2 h^2 + O(h^3), \right. \right. \\ \left. \left. \text{with } -\infty < a_1, a_2 < \infty \right\}.$$

It is easy to see that $R_1 \subset R_3 \subset R_2$. Throughout the subsequent discussion in this section, we assume that the same conditions hold as in Theorem 2.1. We also set $a_1^* = -c_1(g-1) + c_3(g-1)^2$ and $a_2^* = -c_2(g-1) + (c_1 c_3 + c_4)(g-1)^2 - c_3^2(g-1)^3$. It is easily seen that the condition $g=O(1)$ implies $-\infty < a_1^*, a_2^* < \infty$. The following theorem asserts that the bias of any estimator of R_i is given, up to the order $O(h^i)$ ($i=3, 4$).

THEOREM 3.1. Suppose that $g=O(1)$. Then, (i) for any $\tilde{r} \in R_2$

$$(3.1) \quad E(\tilde{r}) = \rho + \frac{\alpha}{(g-1)k_0} (a_1 - a_1^*) h^2 + O(h^3);$$

and (ii) for any $\tilde{r} \in R_3$

$$(3.2) \quad E(\tilde{r}) = \rho + \frac{\alpha}{(g-1)k_0} [(a_1 - a_1^*) h^2 + \{a_2 + c_2(g-1) - c_4(g-1)^2 \\ + a_1 c_3(g-1)\} h^3] + O(h^4).$$

PROOF. Since, in the case of (i), $\tilde{r} \in R_2$, it follows that $\tilde{r} = w_0 r + \sum_{j=1}^g w_j r'_j$, with $\sum_{j=0}^g w_j = 1$ and $\sum_{j=1}^g w_j = -(g-1) + a_1 h + O(h^2)$. Then from (2.5), (3.1) holds. In a similar way to the above, we see that (3.2) holds in case (ii).

Consider the classes R_2^* and R_3^* defined as follows:

$$R_2^* = \left\{ \tilde{r} = w_0 r + \sum_{j=1}^g w_j r'_j \left| \sum_{j=0}^g w_j = 1; \quad \sum_{j=1}^g w_j = -(g-1) + a_1^* h + O(h^2), \right. \right. \\ \left. \left. \text{with } a_1^* = -c_1(g-1) + c_3(g-1)^2 \right\}, \right.$$

and

$$R_3^* = \left\{ \tilde{r} = w_0 r + \sum_{j=1}^g w_j r'_j \left| \sum_{j=0}^g w_j = 1; \quad \sum_{j=1}^g w_j = -(g-1) + a_1^* h + a_2^* h^2 + O(h^3), \right. \right\},$$

where c_1, c_2, c_3 , and c_4 are given in Theorem 2.1. Note that $R_3^* \subset R_2^*$, but R_1 is not generally included in R_2^* and R_3^* . In the following corollary it is shown that any estimator of R_i^* has a bias reduction up to the order $O(h^{i+1})$ ($i=2, 3$).

COROLLARY 3.1. Suppose that $g=O(1)$. Then (i) $E(\tilde{r}) = \rho + O(h^3)$ for all $\tilde{r} \in R_2^*$, and (ii) $E(\tilde{r}) = \rho + O(h^4)$ for all $\tilde{r} \in R_3^*$.

The assertion easily follows from Theorem 3.1.

4. Examples

Using the previous framework of the regression model, we now give examples in the normal and gamma cases, which were treated by Durbin (1959), Rao (1965), and Rao and Webster (1966).

EXAMPLE 4.1 (Normal case). Let X_1, \dots, X_n be independently, identically, and nor-

mally distributed random variables with mean 1 and variance nh , where $h=O(n^{-1})$. Then we have,

$$E(\bar{X})=k_0=1,$$

and for sufficiently large n , the asymptotic means

$$\begin{aligned} E(\bar{X}^{-1}) &= 1+h+3h^2+15h^3+O(h^4), \\ E(\bar{X}_j'^{-1}) &= 1+\frac{g}{g-1}h+\frac{3g^2}{(g-1)^2}h^2+\frac{15g^3}{(g-1)^3}h^3+O(h^4), \\ E(\bar{X}^{-2}) &= 1+3h+15h^2+105h^3+O(h^4)=k_1+O(h^4), \\ (4.1) \quad E(\bar{X}_j'^{-2}) &= 1+\frac{3g}{g-1}h+\frac{15g^2}{(g-1)^2}h^2+\frac{105g^3}{(g-1)^3}h^3+O(h^4)=k_2+O(h^4), \\ E(\bar{X}^{-1}\bar{X}_j'^{-1}) &= 1+\frac{3g-2}{g-1}h+\frac{15g^2-20g+8}{(g-1)^2}h^2+\frac{3(35g^3-70g^2+56g-16)}{(g-1)^3}h^3+O(h^4) \\ &= k_3+O(h^4), \end{aligned} \quad (j=1, \dots, g)$$

and

$$\begin{aligned} E(\bar{X}_i'^{-1}\bar{X}_j'^{-1}) &= 1+\frac{g(3g-4)}{(g-1)^2}h+\frac{g^2(15g^2-40g+27)}{(g-1)^4}h^2 \\ &\quad +\frac{g^3(105g^3-420g^2+567g-258)}{(g-1)^6}h^3+O(h^4) \\ &= k_4+O(h^4) \quad (i \neq j; i, j=1, \dots, g) \end{aligned}$$

(see Rao, 1965 for details). Thus it is clear that the conditions of Theorem 2.3 are satisfied. Hence the grouped jackknife estimator

$$\tilde{r}^* = gr - \frac{g-1}{g} \sum_{j=1}^g r_j'$$

has the minimum asymptotic variance up to the order $O(g^2h^4)$ in the class R_1 .

In the case where $g=2$, Gray and Schucany (1972, Chapter 2) treated the following as an estimator of the ratio:

$$\hat{\rho}_3 = \frac{1}{1-W}r - \frac{W}{2(1-W)}(r_1' + r_2'),$$

where

$$W = \frac{1+3h+15h^2}{2(1+6h+60h^2)}.$$

Applying the above, we obtain $-W/(1-W) = -1+6h+36h^2+O(h^3)$. On the other hand, we have $c_1=3$, $c_2=15$, $c_3=9$, and $c_4=105$ as c_i 's in Theorem 2.1 in this case. Hence it follows that $\hat{\rho}_3$ belongs to the class R_3^* , with $a_1^*=6$ and $a_2^*=36$. By Corollary 3.1, $\hat{\rho}_3$ has a bias reduction up to the order $O(h^4)$.

REMARK 4.1. In Example 4.1, the asymptotic means are given by equations (4.1). However, they do not really exist, since the underlying distribution is normal. So, let $\{\epsilon_n\}$ be a sequence of positive numbers satisfying $\epsilon_n = O(e^{-\sqrt{n}})$. Define \bar{X}_n^* as

$$\bar{X}_n^* = \begin{cases} \bar{X} & \text{for } |\bar{X}| > \epsilon_n, \\ \epsilon_n & \text{for } |\bar{X}| \leq \epsilon_n. \end{cases}$$

Since $E(\bar{X})=1$, it follows that \bar{X}_n^* and \bar{X} are asymptotically equivalent up to any fixed order, and, in addition, the asymptotic mean $E(\bar{X}_n^{*-1})$ exists. The difficulty can be avoided by using \bar{X}_n^* instead of \bar{X} . Similar discussions of the other asymptotic means in (4.1) hold.

EXAMPLE 4.2 (Gamma case). Suppose that $X_1/n, \dots, X_n/n$ are independent and identically distributed random variables according to the gamma distribution with parameter h , the density of which is given by $x^{h-1}e^{-x}/\Gamma(h)$ for $x>0$. It is well-known that $\bar{X}=\sum_{i=1}^n X_i/n$ has a gamma distribution with a parameter $s=nh$. Here we assume that $s=O(1)$ and $s>4$. Under the previous setup, we have

$$\begin{aligned} E(\bar{X}) &= k_0 = s, \\ E(\bar{X}^{-1}) &= \frac{1}{s-1}, \\ E(\bar{X}_j'^{-1}) &= \frac{g-1}{s(g-1)-g} \quad (j=1, \dots, g), \\ E(\bar{X}^{-2}) &= \frac{1}{(s-1)(s-2)}, \\ E(\bar{X}_j'^{-2}) &= \frac{(g-1)^2}{\{s(g-1)-g\}\{s(g-1)-2g\}} \quad (j=1, \dots, g), \end{aligned}$$

and

$$E(\bar{X}^{-1}\bar{X}_j'^{-1}) = \frac{g-1}{(s-2)\{g(s-1)-s\}} \quad (j=1, \dots, g)$$

(see Durbin (1959) and Rao and Webster (1966) for details). By Schwarz's inequality we also see that there exactly exists $E(\bar{X}_i'^{-1}\bar{X}_j'^{-1})$ for $i \neq j$ ($i, j=1, \dots, g$). Note that the above means are given exactly and are thus different from the asymptotic ones in the normal case. With the aid of the above expression, we get

$$k_0 E(\bar{X}^{-1}) - 1 = \frac{1}{s-1}$$

and

$$k_0 \{E(\bar{X}_j'^{-1}) - E(\bar{X}^{-1})\} = \frac{1}{(g-1)(s-1)} + \frac{g}{(g-1)(s-1)\{s(g-1)-g\}}.$$

If we consider a bias reduction of the term $1/(s-1)$, a result similar to Lemma 2.1 holds without the remainder terms. We also see that the conclusion of Theorem 2.1 leads to

$$E(\tilde{r}) = \rho - \frac{\alpha g}{s(s-1)\{s(g-1)-g\}} \quad \text{for any } \tilde{r} \in R_1.$$

Since Theorems 2.2 and 2.3 hold with exact mean and variance, the grouped jackknife estimator

$$\tilde{r}^* = gr - \frac{g-1}{g} \sum_{j=1}^g r'_j$$

has the minimum variance in the class R_1 .

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