THE CONSTRUCTION OF COMBINED BAYESIAN-FREQUENTIST CONFIDENCE INTERVALS FOR A POSITIVE PARAMETER

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1. INTRODUCTION

In the case when a parameter is assumed to be nonnegative or positive-valued, we consider an interval estimation problem on an unknown parameter based on the observations including errors. If the magnitude of errors is almost same size as that of the value of the parameter, the ordinary confidence interval in which the parameter is not considered to be nonnegative involves a set of negative values, and the interval restricted to nonnegative ones can be degenerate. On the above problem, various methods to construct such confidence intervals are proposed by many physicists and others (see, e.g. Feldman and Cousins (1998), Mandelkern (2002), Mandelkern and Schultz (2000a, 2000b) and Roe and Woodroofe (1999, 2001)). But, from the viewpoint of statistics, the problem is very simple and clear. Indeed, there are basically two ways to construct Bayesian and frequentist confidence intervals.

In the Bayesian case, it is enough to confine a prior distribution to the set of existence of a parameter, and in such a case, consideration whether the prior one is appropriate or not is necessary, but it is essentially same as the ordinary Bayesian case.

From the viewpoint of frequentist, we consider a set of testing hypothesis and take a corresponding acceptance region. Indeed, let $\theta_0 > 0$, and we consider the problem of testing the hypothesis H: $\theta = \theta_0$ against the alternative K: $\theta \neq \theta_0$, $\theta > 0$ with level α. Let $A(\theta_0)$ be an acceptance region of the test, and for a random sample X

$$S(X) := \{\theta_0 \mid X \in A(\theta_0)\}.$$

Since, for $\theta>0$,

$$P_\theta\{\theta \in S(X)\} = P_\theta\{X \in A(\theta)\} = 1-\alpha,$$

it follows that a confidence region $S(X)$ for θ of confidence coefficient $1-\alpha$ is obtained (see, e.g. Lehmann (1986), Bickel and Doksum (2001)). In particular, if $S(X)$ becomes an interval, then it is called a confidence interval.

Now, the problem is a better choice of the test procedure in such a testing problem. Noting that a method of unbiased tests can not be applied to the above testing problem, we propose the following procedure in this paper. Assume that a real random variable X has a probability density function (p.d.f.) $f(x,\theta)$ (w.r.t. the Lebesgue measure), where $\theta > 0$, Then we consider the problem of testing the hypothesis H: $\theta = \theta_0$ (> 0) against the alternative K: $\theta \neq \theta_0$, $\theta > 0$. Let Π be a prior distributions defined on the interval $(0,\infty)$, and take the alternative

$$K_\pi : f_\pi(x) := \int_0^\infty f(x,\theta) d\Pi(\theta).$$

Here, f_π might be improper. And we consider the problem of testing the hypothesis H_0: $f(x,\theta_0)$ against the alternative $K_\pi : f_\pi(x)$. Then the acceptance region of the most powerful test is of the form

$$A(\theta_0) := \{x \mid f_\pi(x)/f(x,\theta_0) \leq \lambda\},$$

where λ is determined from the condition $P_{\theta_0}\{X \in A(\theta_0)\} = 1-\alpha$ for given α ($0<\alpha<1$). For various prior distributions, we can get many confidence intervals. They are all admissible.

2. ORDINARY CONFIDENCE INTERVALS

Suppose that $X_1,\ldots,X_n$ are independent and identically distributed (i.i.d.) random variables with the normal distribution $N(\mu, \sigma_0^2)$, where $\mu > 0$ and σ_0 is known. Since the pivotal quantity

$$T(\overline{X},\mu) := \sqrt{n}(\overline{X}-\mu)/\sigma_0$$

is distributed according to $N(0,1)$, it follows that the interval

$$I(\overline{X}) := \left[\max\left\{0, \overline{X} - u_{\alpha/2}\frac{\sigma_0}{\sqrt{n}}\right\},\ \max\left\{0, \overline{X} + u_{\alpha/2}\frac{\sigma_0}{\sqrt{n}}\right\}\right]$$

is the confidence interval (c.i.) for μ of confidence coefficient (c.c.) $1-\alpha$, where $u_{\alpha/2}$ is the upper $100(\alpha/2)$ percentile (see figure 1). Indeed, while $I(\overline{X})$ can become sometimes degenerate, it is always true that

$$P_\mu\{\mu \in I(\overline{X})\} = 1-\alpha,$$

since the event "$\mu \in I(\overline{X})$" is equivalent to the event "$\sqrt{n}\,|\overline{X}-\mu|/\sigma_0 \leq u_{\alpha/2}$", which always has the probability $1-\alpha$. But, for $\overline{X} \leq -1$, the c.i. $I(\overline{X})$ is degenerate, hence there is still room for improvement.

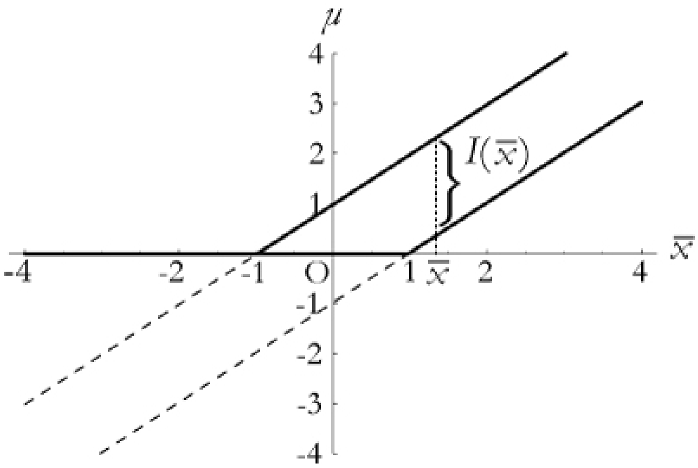

Figura 1 – 68.27% confidence limits for μ from the pivotal quantity.

In order to improve the above, Feldman and Cousins (1998) consider the c.i. based on the acceptance region of the likelihood ratio (LR) test as follows. Suppose that $X_1,...,X_n$ are i.i.d. random variables with the normal distribution $N(\mu,1)$, where $\mu>0$. Since the likelihood function L of μ, given $\overline{X} := (1/n)\sum_{i=1}^{n} X_i = (1/n)\sum_{i=1}^{n} x_i =: \overline{x}$,

$$L(\mu \mid \overline{x}) = (2\pi)^{-n/2} \exp\left[-\frac{1}{2}\left\{ \sum_{i=1}^{n} (x_i - \overline{x})^2 + n(\overline{x} - \mu)^2 \right\} \right]$$

for $\mu > 0$, it follows that the maximum likelihood estimator (MLE) is given by

$$\hat{\mu}_{ML} := \max\{\overline{X}, 0\}.$$

Then

$$L(\hat{\mu}_{ML} \mid \overline{x}) = \begin{cases} (2\pi)^{-n/2} \exp\left\{ -\frac{1}{2}\sum_{i=1}^{n} (x_i - \overline{x})^2 \right\} & \text{for } \overline{x} \geq 0, \\ (2\pi)^{-n/2} \exp\left\{ -\frac{1}{2}\sum_{i=1}^{n} x_i^{\,2} \right\} & \text{for } \overline{x} < 0, \end{cases}$$

hence the LR

$$R_\mu(\overline{x}) := \frac{L(\mu \mid \overline{x})}{L(\hat{\mu}_{ML} \mid \overline{x})} = \begin{cases} \exp\{-(n/2)(\overline{x} - \mu)^2\} & \text{for } \overline{x} \geq 0, \\ \exp\left\{ n\left(\mu\overline{x} - \frac{\mu^2}{2} \right) \right\} & \text{for } \overline{x} < 0. \end{cases}$$

If we can obtain $a(\mu)$ and $b(\mu)$ such that

$$\mathrm{R}(a(\mu)) = \mathrm{R}(b(\mu))$$

and

$$1-\alpha = \Phi(\sqrt{n}(b(\mu)-\mu)) - \Phi(\sqrt{n}(a(\mu)-\mu)) \ ,$$

then the interval $[a(\mu), b(\mu)]$ is an acceptance interval, where Φ is the cumulative distribution function (c.d.f.) of $N(0,1)$. If $S(\overline{X}) := \{\mu \mid \overline{X} \in [a(\mu), b(\mu)]\}$ is an interval, then it is c.i. for μ of c.c. $1-\alpha$ (see figure 2).

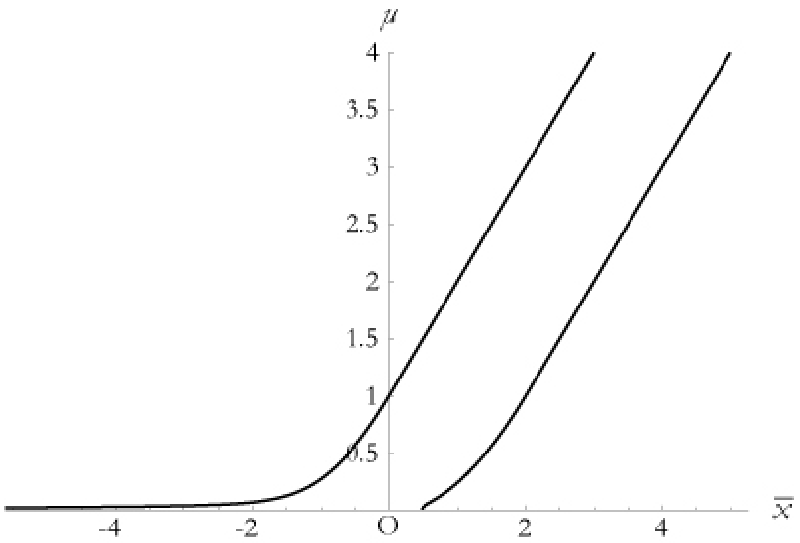

Figure 2 – 68.27% confidence limits for μ derived from the LR test.

3. BAYESIAN CONFIDENCE INTERVALS

Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables with the normal distribution $N(\mu,1)$, where $\mu > 0$. Let $\pi(\mu)$ be an improper prior distribution, i.e.

$$\pi(\mu) = \begin{cases} 1 & \text{for } \ \mu > 0, \\ 0 & \text{for } \ \mu \le 0. \end{cases}$$

Since $\overline{X}$ is normally distributed as $N(\mu, 1/n)$, the posterior density of μ given $\overline{X} = \overline{x}$ is

$$f(\mu \mid \overline{x}) := \frac{\sqrt{n}}{\sqrt{2\pi}\Phi(\sqrt{n}\,\overline{x})} \exp\left\{-\frac{n}{2}(\mu-\overline{x})^2\right\},$$

where Φ is the c.d.f. of $N(0, 1)$. If there exist $\underline{\mu}(\overline{x})$ and $\overline{\mu}(\overline{x})$ such that

$$[\underline{\mu}(\overline{x}), \overline{\mu}(\overline{x})] = \{\mu \mid f(\mu \mid \overline{x}) \geq c\} \ ,$$

$$P_\mu\{\underline{\mu}(\overline{X}) \leq \mu \leq \overline{\mu}(\overline{X})\} = 1 - \alpha,$$

then the interval

$$[\max\{\overline{X} - d, 0\}, \ \ \overline{X} + d]$$

is the c.i. for μ of c.c. $1-\alpha$, where

$$d = \begin{cases} \dfrac{1}{\sqrt{n}} \Phi^{-1}(1 - \alpha\Phi(\sqrt{n}\overline{X})) & \text{for } \ \overline{X} \leq x_0, \\ \dfrac{1}{\sqrt{n}} \Phi^{-1}\left(\dfrac{1}{2} + \dfrac{1}{2}(1-\alpha)\Phi(\sqrt{n}\overline{X})\right) & \text{for } \ \overline{X} > x_0 \end{cases}$$

with

$$x_0 := \frac{1}{\sqrt{n}} \Phi^{-1}\left(\frac{1}{1+\alpha}\right)$$

(see Mandelkern (2002) and figure 3).

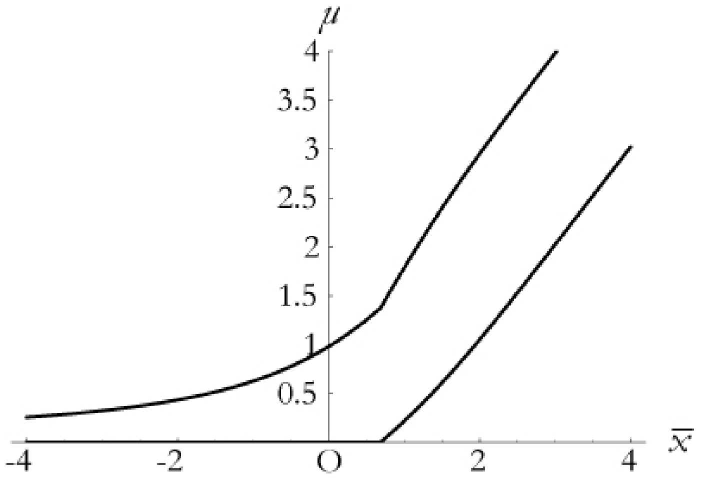

Figure 3 – 68.27% confidence limits for μ based on the Bayesian approach.

4. COMBINED BAYESIAN-FREQUENTIST CONFIDENCE INTERVALS FOR THE NORMAL MEAN IN CASE OF KNOWN VARIANCE

Under the same setup as section 3, we construct confidence intervals for μ by the combined Bayesian-frequentist approach. First we consider the problem of testing the hypothesis H: $\mu = \mu_0$ (> 0) against K: $\mu \sim \pi(\mu)$, i.e. μ is distributed as the same improper prior as section 3. Then the acceptance region of the most powerful (MP) test is of the form

$$T(\overline{X},\mu_0):=\frac{\int_0^\infty \exp\left\{-\frac{n}{2}(\overline{X}-\mu)^2\right\}d\mu}{\exp\left\{-\frac{n}{2}(\overline{X}-\mu_0)^2\right\}}=\frac{\Phi(\sqrt{n}\overline{X})}{\sqrt{n}\phi(\sqrt{n}(\overline{X}-\mu_0))}\leq\lambda,$$

where, for given α (0<α<1), λ is determined by

$$\alpha=\int_{\left\{y\left|\Phi\left(\sqrt{n}(y+\mu_0)\right)\middle/\left(\sqrt{n}\phi\left(\sqrt{n}y\right)\right)>\lambda\right.\right\}}\sqrt{n}\phi(\sqrt{n}u)du.$$

Here, $T(\overline{X},\mu_0)$ is the integrated LR or the Bayes factor (see, e.g. Robert (2001) and O'Hagan and Forster (1994)). Next, we shall show that there exist at most two solutions $\underline{y}(\mu_0)$ and $\overline{y}(\mu_0)$ of the equation

$$\frac{\Phi(\sqrt{n}(y+\mu_0))}{\sqrt{n}\phi(\sqrt{n}y)}=\lambda, \tag{1}$$

if they does. Indeed, we consider the equation $\Phi(t+c)/\phi(t)=\lambda$. Putting

$$G(t):=\Phi(t+c)-\lambda\phi(t), \tag{2}$$

we have

$$G'(t):=\phi(t+c)+\lambda t\phi(t)=\phi(t)(e^{-ct-(c^2/2)}+\lambda t)\ .$$

Here, note that $\lim_{t\to-\infty}G(t)=0$, $\lim_{t\to\infty}G(t)=1$ and $\lambda>0$. If $c<0$, then there is the only one solution of the equation $G'(t)=0$, hence that of (2) is unique. If $c>0$, then there are at most two solutions of the equation $G'(t)=0$, hence the solutions of (2) are also so. Therefore there are at most two solutions of (1), if they exist. Since $\underline{y}$ and $\overline{y}$ are solution of (1), if follows that

$$\frac{\Phi(\sqrt{n}(\overline{y}+\mu_0))}{\phi(\sqrt{n}\overline{y})}=\frac{\Phi(\sqrt{n}(\underline{y}+\mu_0))}{\phi(\sqrt{n}\underline{y})}.$$

Putting $\overline{z}=\sqrt{n}\overline{y}$, $\underline{z}=\sqrt{n}\underline{y}$ and $m:=\sqrt{n}\mu_0$, we have

$$\frac{\Phi(\overline{z}+m)}{\phi(\overline{z})}=\frac{\Phi(\underline{z}+m)}{\phi(\underline{z})}.$$

Let Y be a random variable with the normal distribution $N(0, 1/n)$. Then

$$1-\alpha = P\{\underline{z} \le \sqrt{n}Y \le \bar{z}\} = \Phi(\bar{z}) - \Phi(\underline{z}).$$

Since

$$1-\alpha = P\{\underline{z} \le \sqrt{n}Y \le \bar{z}\} = P\left\{ \frac{\underline{z}+m}{\sqrt{n}} \le \overline{X} \le \frac{\bar{z}+m}{\sqrt{n}} \right\},$$

we have the acceptance region

$$\left[\frac{1}{\sqrt{n}}(\underline{z}+m)\ ,\ \frac{1}{\sqrt{n}}(\bar{z}+m)\right],$$

hence we can construct a confidence interval (see figure 4).

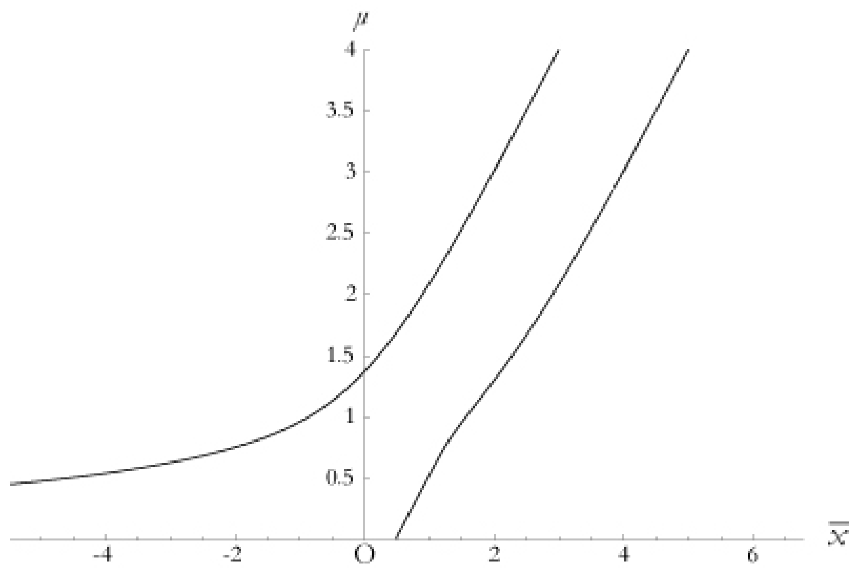

Figure 4 – 68.27% confidence limits for μ based on the combined Bayesian-frequentist approach.

Next, as a proper prior distribution, we consider an exponential distribution

$$\pi_\theta(\mu) = \begin{cases} \dfrac{1}{\theta}e^{-\mu/\theta} & \text{for } \mu > 0, \\ 0 & \text{for } \mu \le 0, \end{cases}$$

where $\theta > 0$. Here, note that the proper prior $\pi_\theta(\mu)$ converges to the improper uniform prior of type $\pi(\mu)$ as $\theta \to \infty$ in the sense that $\theta\pi_\theta(\mu) \to 1$ as $\theta \to \infty$. In a similar way to the above, it is shown that the acceptance region of the MP test is of the form

$$T(\overline{X},\mu_0) := \frac{\int_0^\infty \sqrt{\frac{n}{2\pi}}\left[\exp\left\{-\frac{n}{2}(\overline{X}-\mu)^2\right\}\right]\frac{1}{\theta}e^{-\mu/\theta}d\mu}{\sqrt{\frac{n}{2\pi}}\exp\left\{-\frac{n}{2}(\overline{X}-\mu_0)^2\right\}}$$

$$= \frac{\frac{1}{\theta}\Phi\left(\sqrt{n}\overline{X}-\frac{1}{\sqrt{n}\theta}\right)\exp\left\{-\frac{n}{2}\left(\frac{2\overline{X}}{n\theta}-\frac{1}{n^2\theta^2}\right)\right\}}{\sqrt{n}\phi(\sqrt{n}(\overline{X}-\mu_0))} \le \lambda.$$

Here, for given α ($0<\alpha<1$), λ is determined by

$$\alpha = \int_{\{\overline{x}|T(\overline{x},\mu_0)>\lambda\}} \sqrt{\frac{n}{2\pi}}\exp\left\{-\frac{n}{2}(\overline{x}-\mu_0)^2\right\}d\overline{x}.$$

Now, there are at most two solutions $t_1(\mu_0)$ and $t_2(\mu_0)$ ($t_1(\mu_0) < t_2(\mu_0)$) of the equation $T(\overline{x},\mu_0)=\lambda$ if they exist. Then we have

$$1-\alpha = P_{\mu_0}\{T(\overline{X},\mu_0)\le\lambda\} = P_{\mu_0}\{t_1(\mu_0)\le\overline{X}\le t_2(\mu_0)\},$$

and if the above equality is reduced to

$$P_{\mu_0}\{a(\overline{X})\le\mu_0\le b(\overline{X})\} = 1-\alpha,$$

the interval $[a(\overline{X}), b(\overline{X})]$ is c.i. for μ of c.c. $1-\alpha$ (see figure 5).

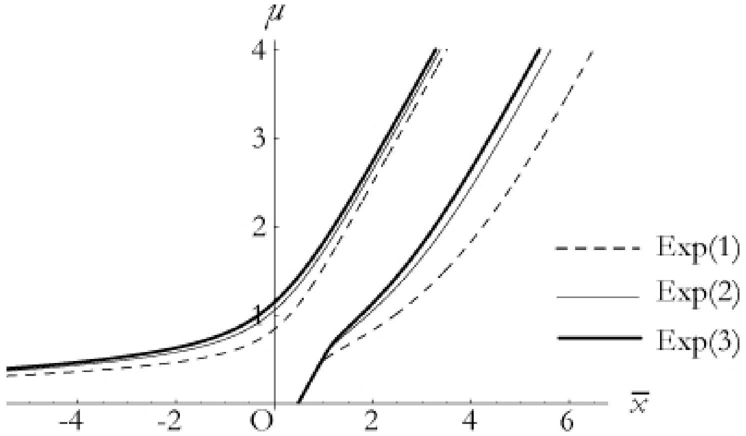

Figure 5 – Bayesian-frequentist 68.27% confidence limits for μ when the prior distribution is exponential one Exp(θ).

Next we numerically compare ordinary, LR, Bayesian, combined Bayesian-frequentist confidence limits (see figure 6 and table 1). As is seen in figure 6 and table 1, if the value of $\overline{X}$ is less than -1, the LR c.i. is comparatively better than others, and in a neighborhood of $\overline{X} = 0$ the combined Bayesian-frequentist c.i. is better than others in the sense that the length of c.i. is shorter. However, the combined one seems to be good in the sense that its width is monotone increasing with some appropriate extent for $\overline{X} < 0$.

In purely non-Bayesian sense, our formulation can be interpreted as follows. We have the problem of testing H: $\mu = \mu_0$ against K: $\mu \neq \mu_0$ when $\mu > 0$. Now denote by $\beta_\varphi(\mu)$ the power of a test φ under μ. We want to maximize $\beta_\varphi(\mu)$. Since there exists no uniformly most powerful test, we can not maximize $\beta_\varphi(\mu)$ simultaneously for all $\mu > 0$. We may instead maximize the average power of $\beta_\varphi(\mu)$ over $\mu > 0$ with some weight function $\pi(\mu)$, that is,

$$\tilde{\beta}_\varphi(\pi) = \int_0^\infty \beta_\varphi(\mu) d\pi(\mu),$$

where π is a probability measure over the interval $(0,\infty)$. It is easily shown that maximizing $\tilde{\beta}_\varphi(\pi)$ is given by the most powerful test of the hypothesis against the simple alternative hypothesis : $\mu \sim \pi(\mu)$.

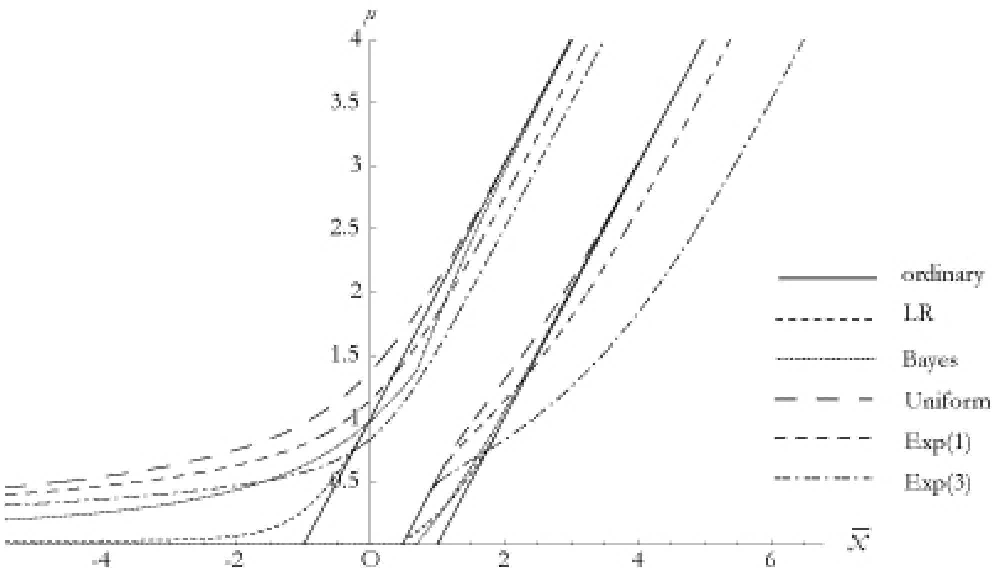

Figure 6 – 68.27% confidence limits for μ.

TABLE 1

Ordinary, Bayesian, and LR 68.27% confidence intervals of μ for n=1

$\overline{x}$	Ordinary	LR	Bayes
-4	[0, 0]	[0, 0.028]	[0, 0.264]
-3	[0, 0]	[0, 0.038]	[0, 0.334]
-2	[0, 0]	[0, 0.069]	[0, 0.446]
-1	[0, 0]	[0, 0.272]	[0, 0.642]
0	[0, 1.000]	[0, 1.000]	[0, 1.000]
1	[0, 2.000]	[0.241, 2.000]	[0.203, 1.797]
2	[1.000, 3.000]	[1.000, 3.000]	[1.032, 2.968]
3	[2.000, 4.000]	[2.000, 4.000]	[2.001, 3.998]
4	[3.000, 5.000]	[3.000, 5.000]	[3.000, 5.000]

Bayesian-frequentist 68.27% confidence intervals of μ for $n=1$

$\bar{x}$	Uniform	Exp(1)	Exp(2)	Exp(3)
-4	[0, 0.540]	[0, 0.361]	[0, 0.435]	[0, 0.466]
-3	[0, 0.626]	[0, 0.406]	[0, 0.495]	[0, 0.534]
-2	[0, 0.752]	[0, 0.467]	[0, 0.581]	[0, 0.630]
-1	[0, 0.961]	[0, 0.574]	[0, 0.728]	[0, 0.794]
0	[0, 1.372]	[0, 0.841]	[0, 1.054]	[0, 1.146]
1	[0.525, 2.089]	[0.488, 1.540]	[0.523, 1.727]	[0.524, 1.826]
2	[1.299, 3.012]	[0.830, 2.500]	[1.053, 2.642]	[1.135, 2.724]
3	[2.082, 4.000]	[1.248, 3.494]	[1.651, 3.628]	[1.798, 3.724]
4	[3.011, 5.000]	[1.832, 4.493]	[2.443, 4.627]	[2.648, 4.723]

5. COMBINED BAYESIAN-FREQUENTIST APPROACH IN CASE OF UNKNOWN VARIANCE

In the previous sections, we assume that the variance is known. In this section we construct a Bayesian-frequentist type confidence intervals of μ when the variance σ^2 is unknown. Suppose that $X_1,\dots,X_n$ are i.i.d. random variables according to the normal distribution $N(\mu, \sigma^2)$ with a p.d.f. $f(x\,;\,\mu,\sigma^2)$, where $n \geq 2$. Now we consider the problem of testing $H:\mu=\mu_0(>0)$, $\sigma \sim \pi(\sigma)=1/\sigma$ for $\sigma>0$ against $K:(\mu,\sigma)\sim\pi(\mu,\sigma)=1/\sigma$ for $\mu>0$, $\sigma>0$. Then we have

$$\int_0^\infty \prod_{i=1}^n f(x_i;\mu,\sigma^2)\frac{d\sigma}{\sigma}$$

$$=\int_0^\infty \frac{1}{(\sqrt{2\pi})^n\sigma^{n+1}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2\right\}d\sigma$$

$$=\int_0^\infty \frac{1}{(\sqrt{2\pi})^n}(2\tau)^{\frac{n}{2}-1}\exp\left\{-\tau\sum_{i=1}^n (x_i-\mu)^2\right\}d\tau$$

$$=\frac{2^{(n/2)-1}}{(\sqrt{2\pi})^n}\left\{\sum_{i=1}^n (x_i-\mu)^2\right\}^{-n/2}\int_0^\infty y^{(n/2)-1}e^{-y}dy$$

$$\left(\text{after a tranformation } y=\tau\sum\nolimits_{i=1}^n (x_i-\mu)^2\right)$$

$$=C'\left\{\sum_{i=1}^n (x_i-\mu)^2\right\}^{-n/2}$$

$$=C'\{(n-1)s_0^2+n(\bar{x}-\mu)^2\}^{-n/2}$$

$$=C''s_0^{-n}\left\{1+\frac{n(\bar{x}-\mu)^2}{(n-1)s_0^2}\right\}^{-n/2}$$

$$=Cs_0^{-n}\frac{\Gamma(n/2)}{\sqrt{\pi(n-1)}\Gamma((n-1)/2)}\left\{1+\frac{n(\bar{x}-\mu)^2}{(n-1)s_0^2}\right\}^{-n/2}$$

$$=Cs_0^{-n}f_{n-1}(\sqrt{n}(\bar{x}-\mu)/s_0) \qquad \text{(say)}, \tag{3}$$

where

$$C' := \frac{2^{(n/2)-1}\Gamma(n/2)}{(\sqrt{2\pi})^n}, \qquad C'' := C'(n-1)^{-n/2},$$

$$C := C''\frac{\sqrt{\pi(n-1)}\Gamma((n-1)/2)}{\Gamma(n/2)}, \quad s_0 := \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2}.$$

Note that f_{n-1} is a p.d.f. of the t-distribution with degrees $n-1$ of freedom. From (3) we have under the hypothesis H

$$\int_0^\infty \prod_{i=1}^{n} f(x_i;\mu_0,\sigma^2)\frac{d\sigma}{\sigma} = Cs_0^{-n} f_{n-1}\left(\frac{\sqrt{n}(\bar{x}-\mu_0)}{s_0}\right),$$

and under the alternative

$$\int_0^\infty \int_0^\infty \prod_{i=1}^{n} f(x_i;\mu,\sigma^2)\frac{1}{\sigma}d\sigma d\mu$$

$$= \int_0^\infty Cs_0^{-n} f_{n-1}\left(\frac{\sqrt{n}(\bar{x}-\mu)}{s_0}\right)d\mu$$

$$= \frac{Cs_0^{-n+1}}{\sqrt{n}}\int_{-\infty}^{\sqrt{n}\bar{x}/s_0}\frac{\Gamma(n/2)}{\sqrt{\pi(n-1)}\Gamma((n-1)/2)}\left(1+\frac{t^2}{n-1}\right)^{-n/2}dt$$

(after a transformation $t = \sqrt{n}(\bar{x}-\mu)/s_0$)

$$= \frac{Cs_0^{-n+1}}{\sqrt{n}}F_{n-1}(\sqrt{n}\bar{x}/s_0),$$

where F_{n-1} is the c.d.f. of f_{n-1}.

In the above testing problem, the acceptance region of the MP test is of the form

$$T(\overline{X},S_0,\mu_0) := \frac{(S_0/\sqrt{n})\,F_{n-1}(\sqrt{n}\overline{X}/S_0)}{f_{n-1}(\sqrt{n}(\overline{X}-\mu_0)/S_0)} \le \lambda. \tag{4}$$

By the approximation of the c.d.f. F_{n-1} to the normal distribution, we have

$$\begin{aligned} F_{n-1}(t) &= \Phi(t) - \frac{1}{4(n-1)}t(t^2+1)\phi(t) + O\left(\frac{1}{n^2}\right) \\ &=: \Phi_{n-1}(t) + O\left(\frac{1}{n^2}\right) \qquad \text{(say)}, \end{aligned} \tag{5}$$

where Φ and ϕ are the c.d.f. and p.d.f. of the standard normal distribution N(0,1), respectively. Then

$$\begin{aligned} f_{n-1}(t) = F'_{n-1}(t) &= \phi(t) + \frac{1}{4(n-1)} t^2(t^2+1)\phi(t) + O\left(\frac{1}{n^2}\right) \\ &=: \phi_{n-1}(t) + O\left(\frac{1}{n^2}\right) \qquad \text{(say).} \end{aligned} \tag{6}$$

So, instead of T in (4) we use

$$\tilde{T}(\overline{X}, S_0, \mu_0) := \frac{(S_0/\sqrt{n})\ \Phi_{n-1}(\sqrt{n}\overline{X}/S_0)}{\phi_{n-1}(\sqrt{n}(\overline{X}-\mu_0)/S_0)}$$

as the approximation of T by (5) and (6). Putting $t = \sqrt{n}(\bar{x}-\mu)/s_0$, we have at most two solutions $\underline{t}$ and $\bar{t}$ of t of the equation

$$\frac{(s_0/\sqrt{n})\ \Phi_{n-1}\left(t + \dfrac{\sqrt{n}\mu_0}{s_0}\right)}{\phi_{n-1}(t)} = \lambda, \tag{7}$$

if they exist. But $\underline{t}$ and $\bar{t}$ can not be represented as functions of μ_0/s_0 since s and t are not independent. So, instead of s_0 in (7) we use σ as a known standard deviation for the present, and the solutions $\underline{t}_0$ and $\bar{t}_0$ become functions of μ_0/σ such that

$$P\left\{\underline{t}_0\left(\frac{\mu_0}{\sigma}\right) \le \frac{\sqrt{n}(\overline{X}-\mu_0)}{\sigma} \le \bar{t}_0\left(\frac{\mu_0}{\sigma}\right)\right\} = 1-\alpha.$$

Since

$$P\left\{\mu_0 + \frac{\sigma}{\sqrt{n}}\underline{t}_0\left(\frac{\mu_0}{\sigma}\right) \le \overline{X} \le \mu_0 + \frac{\sigma}{\sqrt{n}}\bar{t}_0\left(\frac{\mu_0}{\sigma}\right)\right\} = 1-\alpha,$$

it follows that the acceptance region is

$$\left[\mu_0 + \frac{\sigma}{\sqrt{n}}\underline{t}_0\left(\frac{\mu_0}{\sigma}\right),\ \ \mu_0 + \frac{\sigma}{\sqrt{n}}\bar{t}_0\left(\frac{\mu_0}{\sigma}\right)\right]. \tag{8}$$

Since σ is generally unknown, we propose the acceptance region

$$\left[\mu_0 + \frac{S_0}{\sqrt{n}}\underline{t}_0\left(\frac{\mu_0}{S_0}\right),\quad \mu_0 + \frac{S_0}{\sqrt{n}}\bar{t}_0\left(\frac{\mu_0}{S_0}\right)\right], \tag{9}$$

where S_0 is substituted for σ in (8). From (9) we numerically construct a c.i. for μ (see figure 7 and table 2).

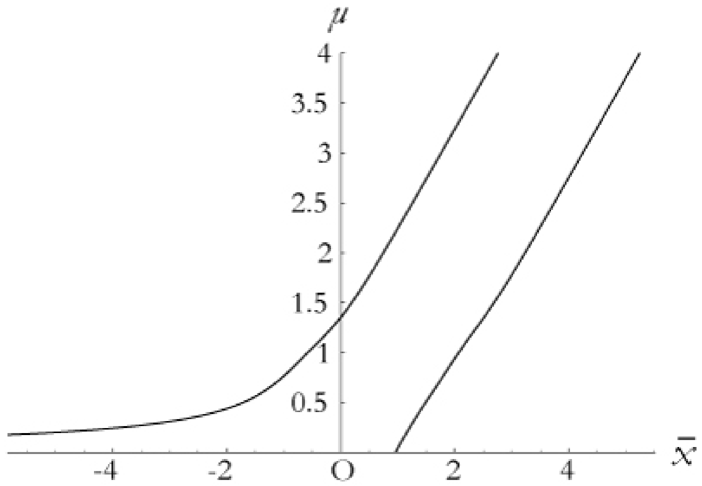

Figure 7 – Bayesian-frequentist 95% confidence limits for μ in the case of unknown variance σ.

TABLE 2

The coverage probability (%) of the Bayesian-frequentist type 95% confidence interval for μ in the case when σ is unknown

(i) $n = 5$

σ \ μ	0.1	0.3	0.5	1.0	1.3
0.5	94.94	94.97	95.00	94.89	95.10
1.0	94.94	95.27	94.88	95.13	95.07
1.5	94.97	95.10	95.01	95.07	95.01
2.0	95.14	94.91	95.06	94.84	95.03

(ii) $n = 10$

σ \ μ	0.1	0.3	0.5	1.0	1.3
0.5	95.05	95.18	95.04	95.10	95.06
1.0	94.90	95.10	95.17	95.01	94.93
1.5	94.92	95.02	94.88	94.96	95.00
2.0	95.07	95.10	95.09	94.77	94.90

As is seen in table 2, the value of coverage probability is very close to that of confidence coefficient.

6. CONCLUDING REMARKS

In this paper, we propose the combined Bayesian-frequentist c.i. for μ in both of cases when σ is known and unknown. We give a numerical comparison of the combined one with ordinary, LR and Bayesian c.i.'s. As a result, the combined one is comparatively better than others in the sense that its width is monotone increasing with some appropriate extent for $\overline{X} < 0$. Note that using the integrated LR is closely connected with the Bayes factor of H against K (see, e.g. Robert (2001) and O'Hagan and Forster (1994)). In the case when σ is unknown as is seen in table 2, the Bayesian-frequentist type c.i. can be recommended. For σ unknown case, one may choose as the prior $d\mu d\sigma/\sigma^2$ instead of $d\mu d\sigma/\sigma$. Now, in the formula (3) we have $n + 1$ instead of n, and the results will not make much difference. However, it seems to be more natural to the degree $n - 1$ of freedom instead of n. Hence our choice is $d\mu d\sigma/\sigma$. On the choice of the prior, see, e.g. Berger and Pericchi (2001).

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RIASSUNTO

La costruzine di intervalli di confidenza combinati bayesiani-frequentisti per un parametro positivo

Negli esperimenti della fisica moderna, si verificano numerosi casi in cui il parametro può assumere solo valori non negativi o positivi. Per tali casi nel contributo si adotta un approccio combinato bayesiano-frequentista per costruire gli intervalli di confidenza. In seguito gli intervalli vengono confrontati con quelli ottenuti dai due approcci bayesiano e classico per i casi normali.

SUMMARY

The construction of combined Bayesian-frequentist confidence intervals for a positive parameter

In current physics experiments, there are many cases when the value of a parameter is theoretically assumed to be nonnegative or positive. In such cases, a combined Bayesian-frequentist approach to confidence intervals for a positive parameter is adopted in this paper, and the confidence intervals are constructed. Comparisons of the confidence intervals with ordinary and Bayesian ones are done in the normal cases.