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Axiomatic Differential Geometry III-3 -Its Landscape- Chapter 3: The Old Kingdom of Differential Geometers

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Abstract

The principal objective in this paer is to study the relationship between the old kingdom of differential geometry (the category of smooth manifolds) and its new kingdom (the category of functors on the category of Weil algebras to some smooth category). It is shown that the canonical embedding of the old kingdom into the new kingdom preserves Weil functors.

1 Introduction

Roughly speaking, the path to axiomatic differential geometry is composed of five acts. **Act One** was Weil's algebraic treatment of nilpotent infinitesimals in [31], namely, the introduction of so-called *Weil algebras*. It showed that nilpotent infinitesimals could be grasped algebraically. While nilpotent infinitesimals are imaginary entities, Weil algebras are real ones. **Act Two** began almost at the same time with Steenrod's introduction of *convenient categories* of topological spaces (cf. [30]), consisting of a string of proposals of convenient categories of smooth spaces. Its principal slogan was that the category of differential geometry should be (locally) cartesian closed. The string was panoramized by [29] as well as [1]. **Act Three** was so-called *synthetic differential geometry*, in which synthetic methods as well as nilpotent infinitesimals play a predominant role. It demonstrated amply that differential geometry could be made *axiomatic* in the same sense that Euclidean geometry is so, though it should resort to reincarnation of nilpotent infinitesimals. In any case, synthetic differential geometers were forced to fabricate their own world, called *well-adapted* *models*, where they could indulge in their favorite nilpotent infinitesimals incessantly. Their unblushing use of moribund nilpotent infinitesimals alienated most of orthodox mathematicians, because nilpotent infinitesimals were almost eradicated as genuine hassle and replaced by so-called $\varepsilon - \delta$ arguments in the 19th century. The reader is referred to [10] and [13] for good treatises on synthetic differential geometry. **Act Four** was the introduction of *Weil functors* and their thorough study by what was called the Czech school of differential geometers in the 1980's, for which the reader is referred to Chapter VIII of [11] and §31 of [12]. Weil functors, which are a direct generalization of the tangent bundle functor, opens a truly realistic path of axiomatizing differential geometry without nilpotent infinitesimals. Then **Act Five** is our *axiomatic differential geometry*, which is tremendously indebted to all previous four acts. For axiomatic differential geometry, the reader is referred to [17], [18], [19], [20], [21] and [22].

In our previous two papers [21] and [22], we have developed model theory for axiomatic differential geometry, in which the category $\mathcal{K}_{\mathbf{Smooth}}$ of functors on the category $\mathbf{Weil}_{\mathbf{R}}$ of Weil algebras to the smooth category **Smooth** (by which we mean any proposed or possible convenient category of smooth spaces) and their natural transformations play a crucial role. We will study the relationship between the category **Mf** of smooth manifolds and smooth mappings and our new kingdom $\mathcal{K}_{\mathbf{Smooth}}$ as well as that between **Smooth** and $\mathcal{K}_{\mathbf{Smooth}}$ in this paper.

2 Convenient Categories of Smooth Spaces

The category of topological spaces and continuous mappins is by no means cartesian closed. In 1967 Steenrod [30] popularized the idea of *convenient category* by announcing that the category of compactly generated spaces and continuous mappings renders a good setting for algebraic topology. The proposed category is cartesian closed, complete and cocomplete, and contains all CW complexes.

At about the same time, an attempt to give a convenient category of smooth spaces began, and we have a few candidates at present. For a thorough study upon the relationship among these already proposed candidates, the reader is referred to [29], in which he or she will find, by way of example, that the category of Frölicher spaces is a full subcategory of that of Souriau spaces, and the category of Souriau spaces is in turn a full subcategory of that of Chen spaces. We have no intention to discuss which is the best convenient category of smooth spaces here, but we note in passing that both the category of Souriau spaces and that of Chen spaces are locally cartesian closed, while that of Frölicher spaces is not. At present we content ourselves with denoting some of such convenient categories of smooth spaces by **Smooth**, which is required to be complete and cartesian closed at least, containing the category **Mf** of smooth manifolds as a full subcategory. Obviously the category **Mf** contains the set $\mathbf{R}$ of real numbers.

3 Weil Functors

Weil algebras were introduced by Weil himself [31]. For a thorough treatment of Weil algebras as smooth algebras, the reader is referred to III.5 in [10].

Notation 1 *We denote by* $\mathbf{Weil}_{\mathbf{R}}$ *the category of Weil algebras over* $\mathbf{R}$.

Let us endow the category **Smooth** with Weil functors.

Proposition 2 *Let* W *be an object in the category* $\mathbf{Weil}_{\mathbf{R}}$ *with its finite presentation*

$$W = C^{\infty}\left(\mathbf{R}^{n}\right)/I$$

as a smooth algebra in the sense of III.5 of [10]. Let $X, Y \in \mathbf{Smooth}$, $f, g \in \mathbf{Smooth}\left(\mathbf{R}^{n}, X\right)$, *and* $h \in \mathbf{Smooth}\left(X, Y\right)$. *If*

$$f \sim_{W} g,$$

then

$$h \circ f \sim_{W} h \circ g$$

Proof. Given $\varsigma \in \mathbf{Smooth}\left(Y, \mathbf{R}\right)$, we have

$$\begin{aligned} &\varsigma \circ (h \circ f) - \varsigma \circ (h \circ g) \\ &= (\varsigma \circ h) \circ f - (\varsigma \circ h) \circ g \in I \end{aligned}$$

so that we have the desired result. ■

Corollary 3 *We can naturally make* $\underline{\mathbf{T}}^{W}_{\mathbf{Smooth}}$ *a functor*

$$\underline{\mathbf{T}}^{W}_{\mathbf{Smooth}} : \mathbf{Smooth} \to \mathbf{Smooth}$$

Proposition 4 *Let* W_1 *and* W_2 *be objects in the category* $\mathbf{Weil}_{\mathbf{R}}$ *with their finite presentations*

$$\begin{aligned} W_1 &= C^{\infty}\left(\mathbf{R}^{n}\right)/I \\ W_2 &= C^{\infty}\left(\mathbf{R}^{m}\right)/J \end{aligned}$$

as smooth algebras. Let

$$\varphi : W_1 \to W_2$$

be a morphism in the category $\mathbf{Weil}_{\mathbf{R}}$, *so that there exists a morphism*

$$\overleftarrow{\varphi} : \mathbf{R}^{m} \to \mathbf{R}^{n}$$

in the category **Smooth** *such that the composition with* $\overleftarrow{\varphi}$ *renders a mapping*

$$C^{\infty}\left(\mathbf{R}^{n}\right) \to C^{\infty}\left(\mathbf{R}^{m}\right)$$

inducing φ. *Let* $X \in \mathbf{Smooth}$ *and* $f, g \in \mathbf{Smooth}\left(\mathbf{R}^{n}, X\right)$. *If*

$$f \sim_{W_1} g$$

then

$$f \circ \overleftarrow{\varphi} \sim_{\mathbf{W}_2} g \circ \overleftarrow{\varphi}$$

Proof. Given any $\varsigma \in \mathbf{Smooth}(Y, \mathbf{R})$, we have

$$\begin{aligned} & \varsigma \circ (f \circ \overleftarrow{\varphi}) - \varsigma \circ (g \circ \overleftarrow{\varphi}) \\ & = (\varsigma \circ f) \circ \overleftarrow{\varphi} - (\varsigma \circ g) \circ \overleftarrow{\varphi} \\ & = (\varsigma \circ f - \varsigma \circ g) \circ \overleftarrow{\varphi} \in J \end{aligned}$$

since $\varsigma \circ f - \varsigma \circ g \in I$, and the composition with $\overleftarrow{\varphi} : \mathbf{R}^n \to \mathbf{R}^m$ maps I into J. ■

Corollary 5 *The above procedure automatically induces a natural transformation*

$$\underline{\alpha}_{\varphi}^{\mathbf{Smooth}} : \underline{\mathbf{T}}_{\mathbf{Smooth}}^{W_1} \Rightarrow \underline{\mathbf{T}}_{\mathbf{Smooth}}^{W_2}$$

Notation 6 *Given an object W in the category $\mathbf{Weil}_{\mathbf{R}}$, the restriction of the functor $\underline{\mathbf{T}}_{\mathbf{Smooth}}^{W}$ to the category $\mathbf{Mf}$ is denoted by $\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}$. Given a morphism $\varphi : W_1 \to W_2$ in the category $\mathbf{Weil}_{\mathbf{R}}$, the corresponding restriction of $\underline{\alpha}_{\varphi}^{\mathbf{Smooth}}$ is denoted by $\underline{\alpha}_{\varphi}^{\mathbf{Mf}}$.*

Remark 7 *Weil functors*

$$\underline{\mathbf{T}}_{\mathbf{Mf}}^{W} : \mathbf{Mf} \to \mathbf{Mf}$$

are given distinct (but equivalent) definitions and studied thoroughly in Chapter VIII of [11] in the finite-dimensional case and §31 of [12] in the infinite-dimensional case.

It is well known that

Proposition 8 *We have the following:*

1. *Given an object W in the category $\mathbf{Weil}_{\mathbf{R}}$, the functor*

   $$\underline{\mathbf{T}}_{\mathbf{Mf}}^{W} : \mathbf{Mf} \to \mathbf{Mf}$$

   abides by the following conditions:

   - *$\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}$ preserves finite products.*
   - *The functor*

     $$\underline{\mathbf{T}}_{\mathbf{Mf}}^{\mathbf{R}} : \mathbf{Mf} \to \mathbf{Mf}$$

     is the identity functor.
   - *We have*

     $$\underline{\mathbf{T}}_{\mathbf{Mf}}^{W_2} \circ \underline{\mathbf{T}}_{\mathbf{Mf}}^{W_1} = \underline{\mathbf{T}}_{\mathbf{Mf}}^{W_1 \otimes_{\mathbf{R}} W_2}$$

     for any objects W_1 and W_2 in the category $\mathbf{Weil}_{\mathbf{R}}$.

2. *Given a morphism $\varphi : W_1 \to W_2$ in the category $\mathbf{Weil}_{\mathbf{R}}$, $\underline{\alpha}_{\varphi}^{\mathbf{Mf}} : \underline{\mathbf{T}}_{\mathbf{Mf}}^{W_1} \Rightarrow \underline{\mathbf{T}}_{\mathbf{Mf}}^{W_2}$ is a natural transformation subject to the following conditions:*

- *We have*
$$\underline{\alpha}_{\mathrm{id}_W}^{\mathbf{Mf}}=\mathrm{id}_{\underline{\mathbf{T}}_{\mathbf{Mf}}^W}$$
for any identity morphism $\mathrm{id}_W:W\rightarrow W$ *in the category* $\mathbf{Weil}_{\mathbf{R}}$.
- *We have*
$$\underline{\alpha}_{\psi}^{\mathbf{Mf}}\cdot\underline{\alpha}_{\varphi}^{\mathbf{Mf}}=\underline{\alpha}_{\psi\circ\varphi}^{\mathbf{Mf}}$$
for any morphisms $\varphi:W_1\rightarrow W_2$ *and* $\psi:W_2\rightarrow W_3$ *in the category* $\mathbf{Weil}_{\mathbf{R}}$.
- *Given an object* W *and a morphism* $\varphi:W_1\rightarrow W_2$ *in the category* $\mathbf{Weil}_{\mathbf{R}}$, *the diagrams*
$$\begin{array}{ccc}\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\circ\underline{\mathbf{T}}_{\mathbf{Mf}}^{W_1} & \underset{\Rightarrow}{\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\circ\underline{\alpha}_{\varphi}^{\mathbf{Mf}}} & \underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\circ\underline{\mathbf{T}}_{\mathbf{Mf}}^{W_2}\\ \| & & \|\\ \underline{\mathbf{T}}_{\mathbf{Mf}}^{W_1\otimes_{\mathbf{R}}W} & \underset{\underline{\alpha}_{\varphi\otimes_{\mathbf{R}}\mathrm{id}_W}^{\mathbf{Mf}}}{\Rightarrow} & \underline{\mathbf{T}}_{\mathbf{Mf}}^{W_2\otimes_{\mathbf{R}}W}\end{array}$$
and
$$\begin{array}{ccc}\underline{\mathbf{T}}_{\mathbf{Mf}}^{W\otimes_{\mathbf{R}}W_1} & \underset{\Rightarrow}{\underline{\alpha}_{\mathrm{id}_W\otimes_{\mathbf{R}}\varphi}^{\mathbf{Mf}}} & \underline{\mathbf{T}}_{\mathbf{Mf}}^{W\otimes_{\mathbf{R}}W_2}\\ \| & & \|\\ \underline{\mathbf{T}}_{\mathbf{Mf}}^{W_1}\circ\underline{\mathbf{T}}_{\mathbf{Mf}}^{W} & \underset{\underline{\alpha}_{\varphi}^{\mathbf{Mf}}\circ\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}}{\Rightarrow} & \underline{\mathbf{T}}_{\mathbf{Mf}}^{W_2}\circ\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\end{array}$$
are commutative.

3. *Given an object* W *in the category* $\mathbf{Weil}_{\mathbf{R}}$, *we have*
$$\underline{\mathbf{T}}^W(\mathbf{R})=W$$
4. *Given a morphism* $\varphi:W_1\rightarrow W_2$ *in the category* $\mathbf{Weil}_{\mathbf{R}}$, *we have*
$$\underline{\alpha}_{\varphi}(\mathbf{R})=\varphi$$

4 A New Kingdom for Differential Geometers

Notation 9 *We introduce the following notation:*

1. *We denote by* $\mathcal{K}_{\mathbf{Smooth}}$ *the category whose objects are functors from the category* $\mathbf{Weil}_{\mathbf{R}}$ *to the category* $\mathbf{Smooth}$ *and whose morphisms are their natural transformations.*
2. *Given an object* W *in the category* $\mathbf{Weil}_{\mathbf{R}}$, *we denote by*
$$\mathbf{T}_{\mathbf{Smooth}}^W:\mathcal{K}_{\mathbf{Smooth}}\rightarrow\mathcal{K}_{\mathbf{Smooth}}$$
the functor obtained as the composition with the functor
$$W\otimes_{\mathbf{R}}\cdot:\mathbf{Weil}_{\mathbf{R}}\rightarrow\mathbf{Weil}_{\mathbf{R}}$$

so that for any object M in the category $\mathcal{K}_{\mathbf{Smooth}}$, we have

$$\mathbf{T}_{\mathbf{Smooth}}^{W}(M) = M(W \otimes_{\mathbf{R}} \cdot)$$

3. *Given a morphism $\varphi : W_1 \to W_2$ in the category $\mathbf{Weil}_{\mathbf{R}}$, we denote by*

$$\alpha_{\varphi}^{\mathbf{Smooth}} : \mathbf{T}_{\mathbf{Smooth}}^{W_1} \Rightarrow \mathbf{T}_{\mathbf{Smooth}}^{W_2}$$

the natural transformation such that, given an object W in the category $\mathbf{Weil}_{\mathbf{R}}$, the morphism

$$\alpha_{\varphi}^{\mathbf{Smooth}}(M) : \mathbf{T}_{\mathbf{Smooth}}^{W_1}(M) \to \mathbf{T}_{\mathbf{Smooth}}^{W_2}(M)$$

is

$$M(\varphi \otimes_{\mathbf{R}} \mathrm{id}_W) : M(W_1 \otimes_{\mathbf{R}} W) \to M(W_2 \otimes_{\mathbf{R}} W)$$

4. *We denote by $\mathbb{R}_{\mathbf{Smooth}}$ the functor*

$$\mathbf{R} \otimes_{\mathbf{R}} \cdot : \mathbf{Weil}_{\mathbf{R}} \to \mathbf{Smooth}$$

We have established the following proposition in [21] and [22].

Proposition 10 *We have the following:*

1. *$\mathcal{K}_{\mathbf{Smooth}}$ is a category which is complete and cartesian closed.*
2. *Given an object W in the category $\mathbf{Weil}_{\mathbf{R}}$, the functor*

$$\mathbf{T}_{\mathbf{Smooth}}^{W} : \mathcal{K}_{\mathbf{Smooth}} \to \mathcal{K}_{\mathbf{Smooth}}$$

abides by the following conditions:

- *$\mathbf{T}_{\mathbf{Smooth}}^{W}$ preserves limits.*
- *The functor*

$$\mathbf{T}_{\mathbf{Smooth}}^{\mathbf{R}} : \mathcal{K}_{\mathbf{Smooth}} \to \mathcal{K}_{\mathbf{Smooth}}$$

is the identity functor.

- *We have*

$$\mathbf{T}_{\mathbf{Smooth}}^{W_1} \circ \mathbf{T}_{\mathbf{Smooth}}^{W_2} = \mathbf{T}_{\mathbf{Smooth}}^{W_1 \otimes_{\mathbf{R}} W_2}$$

for any objects W_1 and W_2 in the category $\mathbf{Weil}_{\mathbf{R}}$.

- *We have*

$$\mathbf{T}_{\mathbf{Smooth}}^{W}\left(M^N\right) = \mathbf{T}_{\mathbf{Smooth}}^{W}(M)^{\mathbf{T}_{\mathbf{Smooth}}^{W}(N)}$$

for any objects M and N in the category $\mathcal{K}_{\mathbf{Smooth}}$.

3. *Given a morphism $\varphi : W_1 \to W_2$ in the category $\mathbf{Weil}_{\mathbf{R}}$,*

$$\alpha_{\varphi} : \mathbf{T}_{\mathbf{Smooth}}^{W_1} \Rightarrow \mathbf{T}_{\mathbf{Smooth}}^{W_2}$$

is a natural transformation subject to the following conditions:

- *We have*
$$\alpha_{\mathrm{id}_W}^{\mathbf{Smooth}}=\mathrm{id}_{\mathbf{T}^W}$$
for any identity morphism $\mathrm{id}_W:W\rightarrow W$ *in the category* $\mathbf{Weil}_{\mathbf{R}}$*.*
- *We have*
$$\alpha_{\psi}^{\mathbf{Smooth}}\circ\alpha_{\varphi}^{\mathbf{Smooth}}=\alpha_{\psi\circ\varphi}^{\mathbf{Smooth}}$$
for any morphisms $\varphi:W_1\rightarrow W_2$ *and* $\psi:W_2\rightarrow W_3$ *in the category* $\mathbf{Weil}_{\mathbf{R}}$*.*
- *Given objects* M *and* N *in the category* $\mathcal{K}_{\mathbf{Smooth}}$*, the diagram*
$$\begin{array}{ccc}
\begin{array}{c}
\mathbf{T}_{\mathbf{Smooth}}^{W_1}(M)^{\mathbf{T}_{\mathbf{Smooth}}^{W_1}(N)}\\
\|\\
\mathbf{T}_{\mathbf{Smooth}}^{W_1}\left(M^N\right)\\
\alpha_{\varphi}^{\mathbf{Smooth}}\left(M^N\right)\ \downarrow\\
\mathbf{T}_{\mathbf{Smooth}}^{W_2}\left(M^N\right)\\
\|\\
\mathbf{T}_{\mathbf{Smooth}}^{W_2}(M)^{\mathbf{T}_{\mathbf{Smooth}}^{W_2}(N)}
\end{array}
&
\begin{array}{c}
\xrightarrow{\alpha_{\varphi}^{\mathbf{Smooth}}(M)^{\mathbf{T}_{\mathbf{Smooth}}^{W_1}(N)}}\\
\nearrow\ \mathbf{T}_{\mathbf{Smooth}}^{W_2}(M)^{\alpha_{\varphi}^{\mathbf{Smooth}}(N)}
\end{array}
&
\mathbf{T}_{\mathbf{Smooth}}^{W_2}(M)^{\mathbf{T}_{\mathbf{Smooth}}^{W_1}(N)}
\end{array}$$
is commutative.
- *Given an object* W *and a morphism* $\varphi:W_1\rightarrow W_2$ *in the category* $\mathbf{Weil}_{\mathbf{R}}$*, the diagrams*
$$\begin{array}{ccc}
\mathbf{T}_{\mathbf{Smooth}}^{W}\circ\mathbf{T}_{\mathbf{Smooth}}^{W_1} & \overset{\mathbf{T}_{\mathbf{Smooth}}^{W}\circ\alpha_{\varphi}^{\mathbf{Smooth}}}{\Rightarrow} & \mathbf{T}_{\mathbf{Smooth}}^{W}\circ\mathbf{T}_{\mathbf{Smooth}}^{W_2}\\
\| & & \|\\
\mathbf{T}_{\mathbf{Smooth}}^{W\otimes_{\mathbf{R}}W_1} & \underset{\alpha_{\mathrm{id}_W\otimes_{\mathbf{R}}\varphi}^{\mathbf{Smooth}}}{\Rightarrow} & \mathbf{T}_{\mathbf{Smooth}}^{W\otimes_{\mathbf{R}}W_2}
\end{array}$$
and
$$\begin{array}{ccc}
\mathbf{T}_{\mathbf{Smooth}}^{W_1\otimes_{\mathbf{R}}W} & \overset{\alpha_{\varphi\otimes_{\mathbf{R}}\mathrm{id}_W}^{\mathbf{Smooth}}}{\Rightarrow} & \mathbf{T}_{\mathbf{Smooth}}^{W_2\otimes_{\mathbf{R}}W}\\
\| & & \|\\
\mathbf{T}_{\mathbf{Smooth}}^{W_1}\circ\mathbf{T}_{\mathbf{Smooth}}^{W} & \underset{\alpha_{\varphi}^{\mathbf{Smooth}}\circ\mathbf{T}_{\mathbf{Smooth}}^{W}}{\Rightarrow} & \mathbf{T}_{\mathbf{Smooth}}^{W_2}\circ\mathbf{T}_{\mathbf{Smooth}}^{W}
\end{array}$$
are commutative.

4. *Given an object* W *in the category* $\mathbf{Weil}_{\mathbf{R}}$*, we have*
$$\mathbf{T}_{\mathbf{Smooth}}^{W}(\mathbb{R}_{\mathbf{Smooth}})=\mathbb{R}_{\mathbf{Smooth}}\otimes_{\mathbf{R}}W$$

5. *Given a morphism* $\varphi:W_1\rightarrow W_2$ *in the category* $\mathbf{Weil}_{\mathbf{R}}$*, we have*
$$\alpha_{\varphi}^{\mathbf{Smooth}}(\mathbb{R}_{\mathbf{Smooth}})=\mathbb{R}_{\mathbf{Smooth}}\otimes_{\mathbf{R}}\varphi$$

5 From the Old Kingdom to the New One

Notation 11 *We write*

$$i_{\mathbf{Smooth}}:\mathbf{Smooth}\rightarrow\mathcal{K}_{\mathbf{Smooth}}$$

for the functor

$$i_{\mathbf{Smooth}}\left(\underline{M}\right):W\in\mathrm{Obj}\,\mathbf{We}il_{\mathbf{R}}\mapsto\underline{\mathbf{T}}_{\mathbf{Smooth}}^{W}\underline{M}\in\mathrm{Obj}\,\mathcal{K}_{\mathbf{Smooth}}$$
$$i_{\mathbf{Smooth}}\left(\underline{M}\right):\varphi\in\mathrm{Mor}\,\mathbf{We}il_{\mathbf{R}}\mapsto\underline{\alpha}_{\varphi}^{\mathbf{Smooth}}\left(\underline{M}\right)\in\mathrm{Mor}\mathcal{K}_{\mathbf{Smooth}}$$

provided with an object object $\underline{M}$ in the category **Smooth**, *and*

$$i_{\mathbf{Smooth}}\left(f\right)\left(W\right)=\underline{\mathbf{T}}_{\mathbf{Smooth}}^{W}f:\underline{\mathbf{T}}_{\mathbf{Smooth}}^{W}\underline{M}_{1}\rightarrow\underline{\mathbf{T}}_{\mathbf{Smooth}}^{W}\underline{M}_{2}$$

provided with a morphism $f:\underline{M}_{1}\rightarrow\underline{M}_{2}$ in the category **Smooth** *and an object W in the category $\mathbf{We}il_{\mathbf{R}}$. The restriction of $i_{\mathbf{Smooth}}$ to the subcategory* **Mf** *is denoted by*

$$i_{\mathbf{Mf}}:\mathbf{Mf}\rightarrow\mathcal{K}_{\mathbf{Smooth}}$$

Theorem 12 *Given an object W in the category $\mathbf{We}il_{\mathbf{R}}$, the diagram*

$$\begin{array}{ccc}\mathbf{Mf} & \underrightarrow{i_{\mathbf{Mf}}} & \mathcal{K}_{\mathbf{Smooth}}\\ \underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\downarrow & & \downarrow\mathbf{T}_{\mathbf{Smooth}}^{W}\\ \mathbf{Mf} & \overrightarrow{i_{\mathbf{Mf}}} & \mathcal{K}_{\mathbf{Smooth}}\end{array}$$

is commutative.

Proof. Given an object $\underline{M}$ in the category **Mf**, we have

$$\begin{aligned}&\left(\mathbf{T}_{\mathbf{Smooth}}^{W}\circ i_{\mathbf{Mf}}\right)\left(\underline{M}\right)\\&=i_{\mathbf{Mf}}\left(\underline{M}\right)\circ\left(W\otimes_{\mathbf{R}}\cdot\right)\\&=\underline{\mathbf{T}}_{\mathbf{Mf}}^{W\otimes_{\mathbf{R}}\cdot}\underline{M}\\&=\underline{\mathbf{T}}_{\mathbf{Mf}}^{\cdot}\left(\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\underline{M}\right)\\&=i_{\mathbf{Mf}}\left(\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\underline{M}\right)\\&=\left(i_{\mathbf{Mf}}\circ\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\right)\left(\underline{M}\right)\end{aligned}$$

Given a morphism

$$\underline{f}:\underline{M}_{1}\rightarrow\underline{M}_{2}$$

in the category **Mf**, we have

$$
\begin{aligned}
&\left(\mathbf{T}_{\mathbf{Smooth}}^{W}\circ i_{\mathbf{Mf}}\right)\left(\underline{f}\right)\\
&=i_{\mathbf{Mf}}\left(\underline{f}\right)\circ\left(W\otimes_{\mathbf{R}}\cdot\right)\\
&=\underline{\mathbf{T}}_{\mathbf{Mf}}^{W\otimes_{\mathbf{R}}\cdot}\underline{f}\\
&=\underline{\mathbf{T}}_{\mathbf{Mf}}^{\cdot}\left(\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\underline{f}\right)\\
&=i_{\mathbf{Mf}}\left(\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\underline{f}\right)\\
&=\left(i_{\mathbf{Mf}}\circ\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\right)\left(\underline{f}\right)
\end{aligned}
$$

■

Theorem 13 *Given a morphism $\varphi : W_1 \to W_2$ in the category $\mathbf{Weil_R}$, the diagram*

$$
\begin{array}{ccc}
i_{\mathbf{Mf}}\circ\underline{\mathbf{T}}_{\mathbf{Mf}}^{W_1} & \overset{i_{\mathbf{Mf}}\circ\underline{\alpha}_{\varphi}^{\mathbf{Mf}}}{\Rightarrow} & i_{\mathbf{Mf}}\circ\underline{\mathbf{T}}_{\mathbf{Mf}}^{W_2}\\
\| & & \|\\
\mathbf{T}_{\mathbf{Smooth}}^{W_1}\circ i_{\mathbf{Mf}} & \underset{\alpha_{\varphi}^{\mathbf{Smooth}}\circ i_{\mathbf{Mf}}}{\Rightarrow} & \mathbf{T}_{\mathbf{Smooth}}^{W_2}\circ i_{\mathbf{Mf}}
\end{array}
$$

is commutative.

Proof. Given an object $\underline{M}$ in the category **Mf**, we have

$$
\begin{aligned}
&\left(i_{\mathbf{Mf}}\circ\underline{\alpha}_{\varphi}^{\mathbf{Mf}}\right)\left(\underline{M}\right)\\
&=i_{\mathbf{Mf}}\left(\underline{\alpha}_{\varphi}^{\mathbf{Mf}}\left(\underline{M}\right)\right)\\
&=\underline{\mathbf{T}}_{\mathbf{Mf}}^{\cdot}\left(\underline{\alpha}_{\varphi}^{\mathbf{Mf}}\left(\underline{M}\right)\right)\\
&=\underline{\alpha}_{\varphi}^{\mathbf{Mf}}\left(\underline{\mathbf{T}}_{\mathbf{Mf}}^{\cdot}\left(\underline{M}\right)\right)\\
&=\alpha_{\varphi}^{\mathbf{Smooth}}\left(i_{\mathbf{Mf}}\left(\underline{M}\right)\right)\\
&=\left(\alpha_{\varphi}^{\mathbf{Smooth}}\circ i_{\mathbf{Mf}}\right)\left(\underline{M}\right)
\end{aligned}
$$

■

6 Microlinearity

Definition 14 *Given a category $\mathcal{K}$ endowed with a functor $\mathbf{T}^W : \mathcal{K} \to \mathcal{K}$ for each object W in the category $\mathbf{Weil_R}$ and a natural transformation $\alpha_\varphi : \mathbf{T}^{W_1} \Rightarrow \mathbf{T}^{W_2}$ for each morphism $\varphi : W_1 \to W_2$ in the category $\mathbf{Weil_R}$, an object M in the category $\mathcal{K}$ is called* <u>*microlinear*</u> *if any limit diagram $\mathcal{D}$ in the category $\mathbf{Weil_R}$ makes the diagram $\mathbf{T}^{\mathcal{D}}M$ a limit diagram in the category $\mathcal{K}$, where the diagram $\mathbf{T}^{\mathcal{D}}M$ consists of objects*

$$\mathbf{T}^W M$$

for any object W in the diagram $\mathcal{D}$ and morphisms

$$\alpha_{\varphi}\left(M\right):\mathbf{T}^{W_1}M\rightarrow\mathbf{T}^{W_2}M$$

for any morphism $\varphi:W_1\rightarrow W_2$ in the diagram $\mathcal{D}$.

Proposition 15 *Every manifold as an object in the category* **Smooth** *is microlinear.*

Proof. This can be established in three steps.

1. The first step is to show that $\mathbf{R}^n$ is micorlinear for any natural number n, which follows easily from
$$\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\mathbf{R}^n=\underline{\mathbf{T}}_{\mathbf{Smooth}}^{W}\mathbf{R}^n=W^n$$
and
$$\underline{\alpha}_{\varphi}^{\mathbf{Mf}}\left(\mathbf{R}^n\right)=\underline{\alpha}_{\varphi}^{\mathbf{Smooth}}\left(\mathbf{R}^n\right)=\varphi^n$$
for any morphism $\varphi:W_1\rightarrow W_2$ in the category $\mathbf{Weil}_{\mathbf{R}}$.
2. The second step is to show that any open subset of $\mathbf{R}^n$ is microlinear in homage to the result in the first step.
3. The third step is to establish the desired result by remarking that a smooth manifold is no other than an overlapping family of open subsets of $\mathbf{R}^n$.

The details can safely be left to the reader. ■

Theorem 16 *The embedding*

$$i_{\mathbf{Smooth}}:\mathbf{Smooth}\rightarrow\mathcal{K}_{\mathbf{Smooth}}$$

maps smooth manifolds to microlinear objects in the category $\mathcal{K}_{\mathbf{Smooth}}$.

Proof. Let $\mathcal{D}$ be a limit diagram in the category $\mathbf{Weil}_{\mathbf{R}}$. Let $\underline{M}$ be a smooth manifold in the category **Smooth**. Given an object W in the category $\mathbf{Weil}_{\mathbf{R}}$, the diagram $\left(\mathbf{T}_{\mathbf{Smooth}}^{\mathcal{D}}\left(i_{\mathbf{Smooth}}\left(\underline{M}\right)\right)\right)(W)$, which consists of objects

$$\underline{\mathbf{T}}_{\mathbf{Mf}}^{W'\otimes_{\mathbf{R}}W}\underline{M}=\underline{\mathbf{T}}_{\mathbf{Mf}}^{W\otimes_{\mathbf{R}}W'}\underline{M}=\underline{\mathbf{T}}_{\mathbf{Mf}}^{W'}\left(\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\underline{M}\right)$$

for any object W' in the category $\mathbf{Weil}_{\mathbf{R}}$ and morphisms

$$\begin{array}{ccc}
\underline{\mathbf{T}}_{\mathbf{Mf}}^{W_1\otimes_{\mathbf{R}}W}\underline{M} & \underrightarrow{\underline{\alpha}_{\varphi\otimes_{\mathbf{R}}\mathrm{id}_W}^{\mathbf{Mf}}\left(\underline{M}\right)} & \underline{\mathbf{T}}_{\mathbf{Mf}}^{W_2\otimes_{\mathbf{R}}W}\underline{M}\\
\| & & \|\\
\underline{\mathbf{T}}_{\mathbf{Mf}}^{W\otimes_{\mathbf{R}}W_1}\underline{M} & & \underline{\mathbf{T}}_{\mathbf{Mf}}^{W\otimes_{\mathbf{R}}W_2}\underline{M}\\
\| & & \|\\
\underline{\mathbf{T}}_{\mathbf{Mf}}^{W_1}\left(\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\underline{M}\right) & \overrightarrow{\underline{\alpha}_{\varphi}^{\mathbf{Mf}}\left(\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\underline{M}\right)} & \underline{\mathbf{T}}_{\mathbf{Mf}}^{W_2}\left(\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\underline{M}\right)
\end{array}$$

for any morphism $\varphi:W_1\rightarrow W_2$ in the category $\mathbf{Weil}_{\mathbf{R}}$, is a limit diagram in the category **Smooth**, because $\underline{\mathbf{T}}_{\mathbf{Mf}}^{W}\underline{M}$ is a microlinear object in the category **Smooth** in homage to Proposition 15. Therefore the diagram $\mathbf{T}_{\mathbf{Smooth}}^{\mathcal{D}}\left(i_{\mathbf{Smooth}}\left(\underline{M}\right)\right)$ is a limit diagram in the category $\mathcal{K}_{\mathbf{Smooth}}$ thanks to Theorem 7.5.2 and Remarks 7.5.3 in [27]. ■

7 Transversal Limits

Definition 17 *A cone $\mathcal{D}$ in the category* **Smooth** *is called a* <u>*transversal limit diagram*</u> *if the diagram $\mathbf{T}_{\mathbf{Smooth}}^{W}\mathcal{D}$ is a limit diagram for any object W in the category $\mathbf{Weil}_{\mathbf{R}}$. In this case, the vertex of the cone is called a* <u>*transversal limit*</u>*.*

It is easy to see that

Proposition 18 *A transversal limit diagram is a limit diagram, so that a transversal limit is a limit.*

Proof. Since

$$\mathbf{T}_{\mathbf{Smooth}}^{\mathbf{R}}\mathcal{D} = \mathcal{D}$$

for any cone $\mathcal{D}$ in the category **Smooth**, the desired conclusion follows immediately. ■

What makes the notion of a transversal limit significant is the following theorem.

Theorem 19 *The embedding*

$$i_{\mathbf{Smooth}} : \mathbf{Smooth} \to \mathcal{K}_{\mathbf{Smooth}}$$

maps transversal limit diagrams in the category **Smooth** *to limit diagrams in the category $\mathcal{K}_{\mathbf{Smooth}}$.*

Proof. This follows directly in homage to Theorem 7.5.2 and Remarks 7.5.3 in [27]. ■

Now we are going to show that the above embedding preserves vertical Weil functors, as far as fibered manifolds are concerned. Let us recall the definition of vertical Weil functor given in [17].

Definition 20 *Let us suppose that we are given a left exact category $\mathcal{K}$ endowed with a functor $\mathbf{T}^{W} : \mathcal{K} \to \mathcal{K}$ for each object W in the category $\mathbf{Weil}_{\mathbf{R}}$ and a natural transformation $\alpha_{\varphi} : \mathbf{T}^{W_1} \Rightarrow \mathbf{T}^{W_2}$ for each morphism $\varphi : W_1 \to W_2$ in the category $\mathbf{Weil}_{\mathbf{R}}$. Given a morphism $\pi : E \to M$ in the category $\mathcal{K}$, its* <u>*vertical Weil functor*</u> *$\overrightarrow{\mathbf{T}}^{W}(\pi)$ is defined to be the equalizer of the parallel morphisms*

$$\mathbf{T}^{W}(E) \underset{\overrightarrow{\mathbf{T}^{W}(\pi)}\ \mathbf{T}^{W}(M)\ \overrightarrow{\alpha_{W\to\mathbf{R}}(M)}\ \mathbf{T}^{\mathbf{R}}(M)\ \overrightarrow{\alpha_{\mathbf{R}\to W}(M)}}{\overset{\mathbf{T}^{W}(\pi)}{\xrightarrow{\qquad\qquad\qquad\qquad\qquad\qquad}}} \mathbf{T}^{W}(M)$$

Lemma 21 *The equalizer of the above diagram in the category* **Smooth** *is transversal, as far as $\pi : E \to M$ is a fibered manifold in the sense of 2.4 in [11].*

Proof. The proof is similar to that in Proposition 15.

1. In case that $E = \mathbf{R}^{m+n}$, $M = \mathbf{R}^m$, and π is the canonical projection, the equalizer is the canonical injection

$$\mathbf{R}^m \times W^n \to W^{m+n} = \mathbf{T}^W(E)$$

and it is easy to see that it is transversal.

2. Then we prove the statement in case that $E = U \times V$, $M = U$, and π is the canonical projection, where U is an open subset of $\mathbf{R}^m$, and V is an open subset of $\mathbf{R}^n$.

3. The desired statement in full generality follows from the above case by remarking that the fiber bundle $\pi : E \to M$ is no other than an overlapping family of such special cases.

The details can safely be left to the reader. ■

Theorem 22 *Given an object W in the category* $\mathbf{Weil_R}$ *and a fibered manifold $\pi : E \to M$ in the category* **Smooth**, *we have*

$$i_{\mathbf{Smooth}}\left(\overrightarrow{\underline{\mathbf{T}}}^W_{\mathbf{Smooth}}(\pi)\right) = \overrightarrow{\mathbf{T}}^W_{\mathbf{Smooth}}\left(i_{\mathbf{Smooth}}(\pi)\right)$$

Proof. In homage to Theorems 12 and 13, the functor $i_{\mathbf{Smooth}}$ maps the diagram

$$\begin{tikzcd}
\underline{\mathbf{T}}^W_{\mathbf{Smooth}}(E) \arrow[rrr, bend left, "\underline{\mathbf{T}}^W_{\mathbf{Smooth}}(\pi)"] \arrow[r, "\underline{\mathbf{T}}^W_{\mathbf{Smooth}}(\pi)"'] & \underline{\mathbf{T}}^W_{\mathbf{Smooth}}(M) \arrow[r, "\underline{\alpha}^{\mathbf{Smooth}}_{W\to\mathbf{R}}(M)"'] & \underline{\mathbf{T}}^{\mathbf{R}}_{\mathbf{Smooth}}(M) \arrow[r, "\underline{\alpha}^{\mathbf{Smooth}}_{\mathbf{R}\to W}(M)"'] & \underline{\mathbf{T}}^W_{\mathbf{Smooth}}(M)
\end{tikzcd}$$

in the category **Smooth** into the diagram

$$\begin{tikzcd}
\mathbf{T}^W_{\mathbf{Smooth}}(i_{\mathbf{Smooth}}(E)) \arrow[rrr, bend left, "\mathbf{T}^W_{\mathbf{Smooth}}(i_{\mathbf{Smooth}}(\pi))"] \arrow[r, "\mathbf{T}^W_{\mathbf{Smooth}}(i_{\mathbf{Smooth}}(\pi))"'] & \mathbf{T}^W_{\mathbf{Smooth}}(i_{\mathbf{Smooth}}(M)) \arrow[r, "\alpha^{\mathbf{Smooth}}_{W\to\mathbf{R}}(i_{\mathbf{Smooth}}(M))"'] & \mathbf{T}^{\mathbf{R}}_{\mathbf{Smooth}}(i_{\mathbf{Smooth}}(M)) \arrow[r, "\alpha^{\mathbf{Smooth}}_{\mathbf{R}\to W}(i_{\mathbf{Smooth}}(M))"'] & \mathbf{T}^W_{\mathbf{Smooth}}(i_{\mathbf{Smooth}}(M))
\end{tikzcd}$$

in the category $\mathcal{K}_{\mathbf{Smooth}}$. Since the equalizer of the former diagram is transversal by Lemma 21, it is preserved by the functor $i_{\mathbf{Smooth}}$ by Theorem 19, so that the desired result follows. ■

Corollary 23 *Given a morphism $\varphi : W_1 \to W_2$ in the category* $\mathbf{Weil_R}$ *and a fibered manifold $\pi : E \to M$ in the category* **Smooth**, *we have*

$$\begin{aligned}
& i_{\mathbf{Smooth}}\left(\overrightarrow{\underline{\alpha}}^{\mathbf{Smooth}}_{\varphi}(\pi)\right) \\
& = \alpha^{\mathbf{Smooth}}_{\varphi}\left(\left(i_{\mathbf{Smooth}}(\pi)\right)\right)
\end{aligned}$$

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