Axiomatic Differential Geometry III-2 -Its Landscape- Chapter 2: Model Theory II

Hirokazu NISHIMURA
Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki, 305-8571, JAPAN

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Abstract

Given a complete and (locally) cartesian closed category **U**, it is shown that the category of functors from the category of Weil algebras to the category **U** is (locally, resp.) cartesian closed. The corresponding axiomatization for differential geometry based upon Weil functors is then given.

1 Introduction

Cartesian closedness is one of the desirable properties that every good category is expected to possess. Indeed, it is surely behind Steenrod's epoch-making notion of a *convenient category* of topological spaces, for which the reader is referred to [30]. Unlike many other desirable properties (e.g., completeness), cartesian closedness is not stable under slicing, and *slicing* within the realm of category theory corresponds to *fibered manifolds* within the realm of differential geometry. Therefore the importance of *locally cartesian closedness* in the arena of differential geometry could not be exaggerated. A few convenient categories of smooth spaces have been proposed (cf. [1] and [29] for their panoramic expositions), but not all of them are locally cartesian closed. By way of example, the category of Chen spaces (cf. [4]) and that of Souriau spaces (cf. [28]) are locally cartesian closed, while that of Frölicher spaces (cf. [6] and [7]) is not.

The principal objective in this paper. as a sequel to [21], is to show that, given a category **U** which is complete and (locally, resp.) cartesian closed, the category $\mathcal{K}_{\mathbf{U}}$ of functors on the category of Weil algebras to **U** is not only complete but also (locally, resp.) cartesian closed, which will be explained in §4 and §5. A corresponding axiomatization is given in §6.

2 Preliminaries

2.1 Category Theory

Given a category $\mathbf{C}$ and a morphism

$$f : A \to B$$

in $\mathbf{C}$, we write

$$A = \mathrm{dom}\, f$$
$$B = \mathrm{cod}\, f$$

2.2 Weil Algebras

Let k be a commutative ring. The category of Weil algebras over k (also called Weil k-algebras) is denoted by $\mathbf{Weil}_k$. It is well known that the category $\mathbf{Weil}_k$ is left exact. The initial and terminal object in $\mathbf{Weil}_k$ is k itself. Given two objects W_1 and W_2 in the category $\mathbf{Weil}_k$, we denote their tensor algebra by $W_1 \otimes_k W_2$. For a good treatise on Weil algebras, the reader is referred to § 1.16 of [10]. Given a left exact category $\mathcal{K}$ and a k-algebra object $\mathbb{R}$ in $\mathcal{K}$, there is a canonical functor $\mathbb{R}\underline{\otimes}_k\cdot$ (denoted by $\mathbb{R}\otimes\cdot$ in [10]) from the category $\mathbf{Weil}_k$ to the category of k-algebra objects and their homomorphisms in $\mathcal{K}$.

3 The Main Example

Let $\mathbf{U}$ be a complete and cartesian closed category with $\mathbb{R}$ being a k-algebra object in $\mathbf{U}$. We have in mind a convenient category of smooth spaces as $\mathbf{U}$.

Notation 1 *We introduce the following notation:*

1. *We denote by $\mathcal{K}_{\mathbf{U}}$ the category whose objects are functors from the category $\mathbf{Weil}_k$ to the category $\mathbf{U}$ and whose morphisms are their natural transformations.*

2. *Given an object W in the category $\mathbf{Weil}_k$, we denote by*

$$\mathbf{T}_{\mathbf{U}}^{W} : \mathcal{K}_{\mathbf{U}} \to \mathcal{K}_{\mathbf{U}}$$

the functor obtained as the composition with the functor

$$\_ \otimes_k W : \mathbf{Weil}_k \to \mathbf{Weil}_k$$

so that for any object M in the category $\mathcal{K}_{\mathbf{U}}$, we have

$$\mathbf{T}_{\mathbf{U}}^{W}(M) = M(\_ \otimes_k W)$$

3. *Given a morphism $\varphi : W_1 \to W_2$ in the category $\mathbf{Weil}_k$, we denote by*

$$\alpha_\varphi^{\mathbf{U}} : \mathbf{T}_{\mathbf{U}}^{W_1} \Rightarrow \mathbf{T}_{\mathbf{U}}^{W_2}$$

the natural transformation such that, given an object W in the category $\mathbf{Weil}_k$, the morphism

$$\alpha_\varphi^{\mathbf{U}} (M) : \mathbf{T}_{\mathbf{U}}^{W_1} (M) \to \mathbf{T}_{\mathbf{U}}^{W_2} (M)$$

is

$$M (W \otimes_k \varphi) : M (W \otimes_k W_1) \to M (W \otimes_k W_2)$$

4. *We denote by $\mathbb{R}_{\mathbf{U}}$ the functor*

$$\mathbb{R}\underline{\otimes}_k \text{-} : \mathbf{Weil}_k \to \mathbf{U}$$

4 Cartesian Closedness

Theorem 2 *The category $\mathcal{K}_{\mathbf{U}}$ is cartesian closed.*

Proof. The proof is a modification of Exercise 1.3.7 in [8]. Let M and N be objects in the category $\mathcal{K}_{\mathbf{U}}$. Given an object W in the category $\mathbf{Weil}_k$, we let $M^N (W)$ denote the intersection of all the equalizers

$$\prod_{\operatorname{dom}\varphi = W} M (\operatorname{cod}\varphi)^{N(\operatorname{cod}\varphi)} \begin{array}{l} \to M (\operatorname{cod}\varphi_1)^{N(\operatorname{cod}\varphi_1)} \xrightarrow{M(\varphi_2)^{N(\operatorname{cod}\varphi_1)}} \\ \to M (\operatorname{cod}\varphi_2 \circ \varphi_1)^{N(\operatorname{cod}\varphi_2 \circ \varphi_1)} \xrightarrow[M(\operatorname{cod}\varphi_2 \circ \varphi_1)^{N(\varphi_2)}]{} \end{array} M (\operatorname{cod}\varphi_2 \circ \varphi_1)^{N(\operatorname{cod}\varphi_1)}$$

where φ ranges over all morphisms in the category $\mathbf{Weil}_k$ with $\operatorname{dom}\varphi = W$, φ_1 and φ_2 range over all morphisms in the category $\mathbf{Weil}_k$ with $\operatorname{dom}\varphi_1 = W$ and $\operatorname{cod}\varphi_1 = \operatorname{dom}\varphi_2$, and the morphisms

$$\prod_{\operatorname{dom}\varphi = W} M (\operatorname{cod}\varphi)^{N(\operatorname{cod}\varphi)} \to M (\operatorname{cod}\varphi_1)^{N(\operatorname{cod}\varphi_1)}$$

and

$$\prod_{\operatorname{dom}\varphi = W} M (\operatorname{cod}\varphi)^{N(\operatorname{cod}\varphi)} \to M (\operatorname{cod}\varphi_2 \circ \varphi_1)^{N(\operatorname{cod}\varphi_2 \circ \varphi_1)}$$

are the canonical projections. Given a morphism $\psi : W_1 \to W_2$ in the category $\mathbf{Weil}_k$, the canonical morphism

$$\prod_{\operatorname{dom}\varphi = W_1} M (\operatorname{cod}\varphi)^{N(\operatorname{cod}\varphi)} \to$$

$$\prod_{\operatorname{dom}\varphi' = W_2} M (\operatorname{cod}\varphi' \circ \psi)^{N\left(\operatorname{cod}\varphi' \circ \psi\right)} = \prod_{\operatorname{dom}\varphi' = W_2} M (\operatorname{cod}\varphi')^{N\left(\operatorname{cod}\varphi'\right)}$$

naturally gives rise to a morphism $M^{N}(W_1) \rightarrow M^{N}(W_2)$ in the category $\mathcal{K}_{\mathbf{U}}$, which we let $M^{N}(\psi)$. It is easy to see that M^{N} becomes an object in the category $\mathcal{K}_{\mathbf{U}}$, which works as the exponentiation of M by N within the category $\mathcal{K}_{\mathbf{U}}$. ■

Corollary 3 *Given an object W in the category $\mathbf{Weil}_k$ and objects M and N in the category $\mathcal{K}_{\mathbf{U}}$, we have*

$$\mathbf{T}_{\mathbf{U}}^{W}\left(M^{N}\right) = \mathbf{T}_{\mathbf{U}}^{W}(M)^{\mathbf{T}_{\mathbf{U}}^{W}(N)}$$

Corollary 4 *Given a morphism $\varphi : W_1 \rightarrow W_2$ in the category $\mathbf{Weil}_k$ and objects M and N in the category $\mathcal{K}_{\mathbf{U}}$, the morphism*

$$\alpha_{\varphi}^{\mathbf{U}}(M)^{\mathbf{T}_{\mathbf{U}}^{W_1}(N)} : \mathbf{T}_{\mathbf{U}}^{W_1}\left(M^{N}\right) = \mathbf{T}_{\mathbf{U}}^{W_1}(M)^{\mathbf{T}_{\mathbf{U}}^{W_1}(N)} \rightarrow \mathbf{T}_{\mathbf{U}}^{W_2}(M)^{\mathbf{T}_{\mathbf{U}}^{W_1}(N)}$$

is equal to the morphism

$$\mathbf{T}_{\mathbf{U}}^{W_1}(M)^{\mathbf{T}_{\mathbf{U}}^{W_1}(N)} = \mathbf{T}_{\mathbf{U}}^{W_1}\left(M^{N}\right) \underrightarrow{\alpha_{\varphi}^{\mathbf{U}}\left(M^{N}\right)} \mathbf{T}_{\mathbf{U}}^{W_2}\left(M^{N}\right) = \mathbf{T}_{\mathbf{U}}^{W_2}(M)^{\mathbf{T}_{\mathbf{U}}^{W_2}(N)}$$
$$\underrightarrow{\mathbf{T}_{\mathbf{U}}^{W_2}(M)^{\alpha_{\varphi}^{\mathbf{U}}(N)}} \mathbf{T}_{\mathbf{U}}^{W_2}(M)^{\mathbf{T}_{\mathbf{U}}^{W_1}(N)}$$

5 Locally Cartesian Closedness

In this section we assume that the category $\mathbf{U}$ is locally cartesian closed.

Theorem 5 *The category $\mathcal{K}_{\mathbf{U}}$ is locally cartesian closed.*

Proof. Our present discussion is a localization of the discussion in the proof of Theorem 2 in a sense. Let L be an object in the category $\mathcal{K}_{\mathbf{U}}$. Let $\pi_1 : M \rightarrow L$ and $\pi_2 : N \rightarrow L$ be objects in the slice category $\mathcal{K}_{\mathbf{U}}/L$. Given an object W in the category $\mathbf{Weil}_k$, we let $\left(M^{N}\right)_L(W)$ denote the intersection of all the equalizers

$$\left(\prod_{\operatorname{dom}\varphi = W} M(\operatorname{cod}\varphi)^{N(\operatorname{cod}\varphi)}\right)_L$$
$$\rightarrow \left(M(\operatorname{cod}\varphi_1)^{N(\operatorname{cod}\varphi_1)}\right)_L \overrightarrow{\left(M(\varphi_2)^{N(\operatorname{cod}\varphi_1)}\right)_L}$$
$$\rightarrow \left(M(\operatorname{cod}\varphi_2 \circ \varphi_1)^{N(\operatorname{cod}\varphi_2 \circ \varphi_1)}\right)_L \overrightarrow{\left(M(\operatorname{cod}\varphi_2 \circ \varphi_1)^{N(\varphi_2)}\right)_L}$$
$$\left(M(\operatorname{cod}\varphi_2 \circ \varphi_1)^{N(\operatorname{cod}\varphi_1)}\right)_L$$

where φ ranges over all morphisms in the category $\mathbf{Weil}_k$ with $\operatorname{dom}\varphi = W$, φ_1 and φ_2 range over all morphisms in the category $\mathbf{Weil}_k$ with $\operatorname{dom}\varphi_1 = W$ and $\operatorname{cod}\varphi_1 = \operatorname{dom}\varphi_2$, and the morphisms

$$\left(\prod_{\operatorname{dom}\varphi = W} M(\operatorname{cod}\varphi)^{N(\operatorname{cod}\varphi)}\right)_L \rightarrow \left(M(\operatorname{cod}\varphi_1)^{N(\operatorname{cod}\varphi_1)}\right)_L$$

and

$$\left(\prod_{\operatorname{dom}\varphi=W} M\left(\operatorname{cod}\varphi\right)^{N\left(\operatorname{cod}\varphi\right)}\right)_{L}\rightarrow\left(M\left(\operatorname{cod}\varphi_{2}\circ\varphi_{1}\right)^{N\left(\operatorname{cod}\varphi_{2}\circ\varphi_{1}\right)}\right)_{L}$$

are the canonical projections, and $(-)_L$ denotes the categorical operation within the slice category $\mathcal{K}_{\mathbf{U}}/L$ so that $(M \times N)_L$ denotes the fibered product $M \times_L N$ by way of example. Given a morphism $\psi : W_1 \rightarrow W_2$ in the category $\mathbf{Weil}_k$, the canonical morphism

$$\begin{gathered}\left(\prod_{\operatorname{dom}\varphi=W_{1}} M\left(\operatorname{cod}\varphi\right)^{N\left(\operatorname{cod}\varphi\right)}\right)_{L}\rightarrow\\ \left(\prod_{\operatorname{dom}\varphi^{\prime}=W_{2}} M\left(\operatorname{cod}\varphi^{\prime}\circ\psi\right)^{N\left(\operatorname{cod}\varphi^{\prime}\circ\psi\right)}\right)_{L}=\left(\prod_{\operatorname{dom}\varphi^{\prime}=W_{2}} M\left(\operatorname{cod}\varphi^{\prime}\right)^{N\left(\operatorname{cod}\varphi^{\prime}\right)}\right)_{L}\end{gathered}$$

naturally gives rise to a morphism $\left(M^{N}\right)_{L}(W_1) \rightarrow \left(M^{N}\right)_{L}(W_2)$ in the slice category $\mathcal{K}_{\mathbf{U}}/L$, which we let $\left(M^{N}\right)_{L}(\psi)$. It is easy to see that $\left(M^{N}\right)_{L}$ becomes an object in the slice category $\mathcal{K}_{\mathbf{U}}/L$, which works as the exponentiation of M by N within the slice category $\mathcal{K}_{\mathbf{U}}/L$. ■

Corollary 6 *Given an object W in the category $\mathbf{Weil}_k$, an object L in the category $\mathcal{K}_{\mathbf{U}}$, and objects $\pi_1 : M \rightarrow L$ and $\pi_2 : N \rightarrow L$ in the slice category $\mathcal{K}_{\mathbf{U}}/L$, we have*

$$\left(\mathbf{T}_{\mathbf{U}}\right)_{L}^{W}\left(\left(M^{N}\right)_{L}\right)=\left(\left(\mathbf{T}_{\mathbf{U}}\right)_{L}^{W}(M)^{\left(\mathbf{T}_{\mathbf{U}}\right)_{L}^{W}(N)}\right)_{L}$$

where $\left(\mathbf{T}_{\mathbf{U}}\right)_{L}^{W}(M)$ denotes the equalizer of

$$\mathbf{T}_{\mathbf{U}}^{W}(M)\ \underset{\overrightarrow{\mathbf{T}_{\mathbf{U}}^{W}(\pi_1)}\ \mathbf{T}_{\mathbf{U}}^{W}(L)\ \overrightarrow{\alpha_{W\rightarrow k}^{\mathbf{U}}(L)}\ \mathbf{T}_{\mathbf{U}}^{k}(L)\ \overrightarrow{\alpha_{k\rightarrow W}^{\mathbf{U}}(L)}}{\overset{\mathbf{T}_{\mathbf{U}}^{W}(\pi_1)}{\longrightarrow}}\ \mathbf{T}_{\mathbf{U}}^{W}(L)$$

with $W \rightarrow k$ and $k \rightarrow W$ being the canonical morphisms in the category $\mathbf{Weil}_k$. We can naturally extend $\left(\mathbf{T}_{\mathbf{U}}\right)_{L}^{W}$ to a functor

$$\mathcal{K}_{\mathbf{U}}/L \rightarrow \mathcal{K}_{\mathbf{U}}/L$$

in the sense that, given any commutative diagram

$$\begin{array}{ccc} M & \underrightarrow{f} & N \\ \pi_1 \searrow & & \swarrow \pi_2 \\ & L & \end{array}$$

within the category $\mathcal{K}$, there exists a unique morphism

$$\left(\mathbf{T}_{\mathbf{U}}\right)_{L}^{W}(f) : \left(\mathbf{T}_{\mathbf{U}}\right)_{L}^{W}(M) \rightarrow \left(\mathbf{T}_{\mathbf{U}}\right)_{L}^{W}(N)$$

making the diagram

$$\begin{array}{ccc} \left(\mathbf{T}_{\mathbf{U}}\right)_L^W(M) & \underrightarrow{\left(\mathbf{T}_{\mathbf{U}}\right)_L^W(f)} & \left(\mathbf{T}_{\mathbf{U}}\right)_L^W(N) \\ \downarrow & & \downarrow \\ \mathbf{T}_{\mathbf{U}}^W(M) & \overrightarrow{\mathbf{T}_{\mathbf{U}}^W(f)} & \mathbf{T}_{\mathbf{U}}^W(N) \end{array}$$

commutative, where the two vertical arrows are the canonical injections.

Corollary 7 *Given a morphism $\varphi : W_1 \to W_2$ in the category $\mathbf{Weil}_k$, an object L in the category $\mathcal{K}_{\mathbf{U}}$, and objects $\pi_1 : M \to L$ and $\pi_2 : N \to L$ in the slice category $\mathcal{K}_{\mathbf{U}}/L$, the morphism*

$$\begin{aligned} \left(\alpha^{\mathbf{U}}\right)_\varphi^L(M)^{\left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_1}(N)} : \left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_1}\left(\left(M^N\right)_L\right) = \left(\left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_1}(M)^{\left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_1}(N)}\right)_L \\ \to \left(\left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_2}(M)^{\left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_1}(N)}\right)_L \end{aligned}$$

is equal to the morphism

$$\begin{aligned} \left(\left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_1}(M)^{\left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_1}(N)}\right)_L = \left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_1}\left(\left(M^N\right)_L\right) \underrightarrow{\left(\alpha^{\mathbf{U}}\right)_\varphi^L\left(\left(M^N\right)_L\right)} \\ \left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_2}\left(\left(M^N\right)_L\right) = \left(\left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_2}(M)^{\left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_2}(N)}\right)_L \underrightarrow{\left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_2}(M)^{\left(\alpha^{\mathbf{U}}\right)_\varphi^L(N)}} \\ \left(\left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_2}(M)^{\left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_1}(N)}\right)_L \end{aligned}$$

where the natural transformation

$$\left(\alpha^{\mathbf{U}}\right)_\varphi^L : \left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_1} \Rightarrow \left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_2}$$

is induced by the natural transformation

$$\alpha_\varphi^{\mathbf{U}} : \mathbf{T}_{\mathbf{U}}^{W_1} \Rightarrow \mathbf{T}_{\mathbf{U}}^{W_2}$$

in the sense of making the diagram

$$\begin{array}{ccc} \left(\mathbf{T}_{\mathbf{U}}\right)_L^{W_1}(M) & \underrightarrow{\left(\alpha^{\mathbf{U}}\right)_\varphi^L(\pi_1)} & \mathbf{T}_{\mathbf{U}}^{W_2}(M) \\ \downarrow & & \downarrow \\ \mathbf{T}_{\mathbf{U}}^{W_1}(M) & \overrightarrow{\alpha_\varphi^{\mathbf{U}}(M)} & \mathbf{T}_{\mathbf{U}}^{W_2}(M) \end{array}$$

commutative.

6 The Axiomatics

Definition 8 *A DG-category (DG stands for Differential Geometry) is a quadruple $(\mathcal{K}, \mathbf{T}, \alpha, \mathbb{R})$, where*

1. $\mathcal{K}$ *is a category which is complete and cartesian closed.*

2. *Given an object* W *in the category* $\mathbf{Weil}_k$, $\mathbf{T}^W : \mathcal{K} \to \mathcal{K}$ *is a functor subject to the conditions:*

   - $\mathbf{T}^W$ *preserves limits.*
   - $\mathbf{T}^k : \mathcal{K} \to \mathcal{K}$ *is the identity functor.*
   - *We have*
     $$\mathbf{T}^{W_2} \circ \mathbf{T}^{W_1} = \mathbf{T}^{W_1 \otimes_k W_2}$$
     for any objects W_1 *and* W_2 *in the category* $\mathbf{Weil}_k$.
   - *We have*
     $$\mathbf{T}^W \left(M^N \right) = \mathbf{T}^W \left(M \right)^{\mathbf{T}^W (N)}$$
     for any objects M *and* N *in the category* $\mathcal{K}$.

3. *Given a morphism* $\varphi : W_1 \to W_2$ *in the category* $\mathbf{Weil}_k$, $\alpha_\varphi : \mathbf{T}^{W_1} \Rightarrow \mathbf{T}^{W_2}$ *is a natural transformation subject to the conditions:*

   - *We have*
     $$\alpha_{\mathrm{id}_W} = \mathrm{id}_{\mathbf{T}^W}$$
     for any identity morphism $\mathrm{id}_W : W \to W$ *in the category* $\mathbf{Weil}_k$.
   - *We have*
     $$\alpha_\psi \cdot \alpha_\varphi = \alpha_{\psi \circ \varphi}$$
     for any morphisms $\varphi : W_1 \to W_2$ *and* $\psi : W_2 \to W_3$ *in the category* $\mathbf{Weil}_k$.
   - *Given objects* M *and* N *in the category* $\mathcal{K}$, *the morphism*
     $$\alpha_\varphi \left(M \right)^{\mathbf{T}^{W_1}(N)} : \mathbf{T}^{W_1} \left(M^N \right) = \mathbf{T}^{W_1} \left(M \right)^{\mathbf{T}^{W_1}(N)} \to \mathbf{T}^{W_2} \left(M \right)^{\mathbf{T}^{W_1}(N)}$$
     is equal to the morphism
     $$\mathbf{T}^{W_1} \left(M \right)^{\mathbf{T}^{W_1}(N)} = \mathbf{T}^{W_1} \left(M^N \right) \underrightarrow{\alpha_\varphi \left(M^N \right)} \mathbf{T}^{W_2} \left(M^N \right) = \mathbf{T}^{W_2} \left(M \right)^{\mathbf{T}^{W_2}(N)}$$
     $$\underrightarrow{\mathbf{T}^{W_2} \left(M \right)^{\alpha_\varphi (N)}} \mathbf{T}^{W_2} \left(M \right)^{\mathbf{T}^{W_1}(N)}$$
     within the category $\mathcal{K}$.

4. *Given an object* W *in the category* $\mathbf{Weil}_k$, *we have*
   $$\mathbf{T}^W \left(\mathbb{R} \right) = \mathbb{R} \underline{\otimes}_k W$$

5. *Given a morphism* $\varphi : W_1 \to W_2$ *in the category* $\mathbf{Weil}_k$, *we have*
   $$\alpha_\varphi \left(\mathbb{R} \right) = \mathbb{R} \underline{\otimes}_k \varphi$$

Notation 9 *Given an object W in the category $\mathbf{Weil}_k$, an object L in the category $\mathcal{K}$, and an object $\pi : M \to L$ in the slice category $\mathcal{K}/L$, we denote by $\mathbf{T}_L^W(M)$ the equalizer of*

$$\mathbf{T}^W(M) \underset{\overrightarrow{\mathbf{T}^W(\pi)}\ \mathbf{T}^W(L)\ \overrightarrow{\alpha_{W\to k}(L)}\ \mathbf{T}^k(L)\ \overrightarrow{\alpha_{k\to W}(L)}}{\overset{\mathbf{T}^W(\pi)}{\rightrightarrows}} \mathbf{T}^W(L)$$

with $W \to k$ and $k \to W$ being the canonical morphisms within the category $\mathbf{Weil}_k$. We can naturally extend $\mathbf{T}_L^W$ to a functor

$$\mathcal{K}/L \to \mathcal{K}/L$$

in the sense that, given any commutative diagram

$$\begin{array}{ccc} M & \overset{f}{\rightarrow} & N \\ \pi_1 \searrow & & \swarrow \pi_2 \\ & L & \end{array}$$

within the category $\mathcal{K}$, there exists a unique morphism

$$\mathbf{T}_L^W(f) : \mathbf{T}_L^W(M) \to \mathbf{T}_L^W(N)$$

making the diagram

$$\begin{array}{ccc} \mathbf{T}_L^W(M) & \overrightarrow{\mathbf{T}_L^W(f)} & \mathbf{T}_L^W(N) \\ \downarrow & & \downarrow \\ \mathbf{T}^W(M) & \overrightarrow{\mathbf{T}^W(f)} & \mathbf{T}^W(N) \end{array}$$

commutative, where the two vertical arrows are the canonical injections.

Notation 10 *Given a morphism $\varphi : W_1 \to W_2$ in the category $\mathbf{Weil}_k$, an object L in the category $\mathcal{K}$, and an object $\pi : M \to L$ within the slice category $\mathcal{K}/L$, we denote by α_φ^L the natural transformation*

$$\mathbf{T}_L^{W_1} \Rightarrow \mathbf{T}_L^{W_2}$$

making the diagram

$$\begin{array}{ccc} \mathbf{T}_L^{W_1}(M) & \overrightarrow{\alpha_\varphi^L(M)} & \mathbf{T}_L^{W_2}(M) \\ \downarrow & & \downarrow \\ \mathbf{T}^{W_1}(M) & \overrightarrow{\alpha_\varphi(M)} & \mathbf{T}^{W_2}(M) \end{array}$$

commutative for any object W in the category $\mathbf{Weil}_k$, where $\begin{array}{c} \mathbf{T}_L^{W_1}(M) \\ \downarrow \\ \mathbf{T}^{W_1}(M) \end{array}$ *and* $\begin{array}{c} \mathbf{T}_L^{W_2}(M) \\ \downarrow \\ \mathbf{T}^{W_2}(M) \end{array}$ *are the canonical injections.*

Definition 11 *A* <u>*local DG-category*</u> *is a DG-category* $(\mathcal{K}, \mathbf{T}, \alpha, \mathbb{R})$ *subject to the conditions:*

1. *The category* $\mathcal{K}$ *is not only cartesian closed but, what is even more, locally cartesian closed.*

2. *Given an object* W *in the category* $\mathbf{Weil}_k$*, an object* L *in the category* $\mathcal{K}$*, and objects* $\pi_1 : M \to L$ *and* $\pi_2 : N \to L$ *in the slice category* $\mathcal{K}/L$*, we have*
$$\mathbf{T}_L^W\left(\left(M^N\right)_L\right) = \left(\mathbf{T}_L^W(M)^{\mathbf{T}_L^W(N)}\right)_L$$
within the category $\mathcal{K}/L$*.*

3. *Given an object* W *in the category* $\mathbf{Weil}_k$*, an object* L *in the category* $\mathcal{K}$*, and objects* $\pi_1 : M \to L$ *and* $\pi_2 : N \to L$ *in the slice category* $\mathcal{K}/L$*, the morphism*
$$\alpha_\varphi^L(M)^{\mathbf{T}_L^{W_1}(N)} : \mathbf{T}_L^{W_1}\left(\left(M^N\right)_L\right) = \left(\mathbf{T}_L^{W_1}(M)^{\mathbf{T}_L^{W_1}(N)}\right)_L \to \left(\mathbf{T}_L^{W_2}(M)^{\mathbf{T}_L^{W_1}(N)}\right)_L$$
is equal to the morphism
$$\left(\mathbf{T}_L^{W_1}(M)^{\mathbf{T}_L^{W_1}(N)}\right)_L = \mathbf{T}_L^{W_1}\left(\left(M^N\right)_L\right) \xrightarrow{\alpha_\varphi^L\left(\left(M^N\right)_L\right)}$$
$$\mathbf{T}_L^{W_2}\left(\left(M^N\right)_L\right) = \left(\mathbf{T}_L^{W_2}(M)^{\mathbf{T}_L^{W_2}(N)}\right)_L \xrightarrow{\mathbf{T}_L^{W_2}(M)^{\alpha_\varphi^L(N)}} \left(\mathbf{T}_L^{W_2}(M)^{\mathbf{T}_L^{W_1}(N)}\right)_L$$
within the category $\mathcal{K}/L$*.*

Proposition 12 *Given a local DG-category* $(\mathcal{K}, \mathbf{T}, \alpha, \mathbb{R})$ *and an object* L *in the category* $\mathcal{K}$*, the quadruple*
$$\left(\mathcal{K}/L, \mathbf{T}_L, \alpha^L, \begin{array}{c} L \times \mathbb{R} \\ \downarrow \\ L \end{array}\right),$$
which may be considerd to be the <u>*localization of the DG-category*</u> $(\mathcal{K}, \mathbf{T}, \alpha, \mathbb{R})$ <u>*with respect to*</u> L *in a sense, is a local DG-category, where* $\begin{array}{c} L \times \mathbb{R} \\ \downarrow \\ L \end{array}$ *is the canonical projection.*

Proof. Given an object $\begin{array}{c} M \\ \downarrow \\ L \end{array}$ in the slice category $\mathcal{K}/L$, we note the following:

1. We can naturally identify the slice category
$$(\mathcal{K}/L) / \left(\begin{array}{c} M \\ \downarrow \\ L \end{array}\right)$$

with the slice category

$$\mathcal{K}/M$$

for which the reader is referred, say, to Page 8 of [9].

2. We have

$$(\mathbf{T}_L)_{M\to L} = \mathbf{T}_M,$$

since the diagram

$$\begin{array}{ccccc} \mathbf{T}^W(M) & \underrightarrow{\alpha_{W\to k}(M)} & \mathbf{T}^k(M) & \underrightarrow{\alpha_{k\to W}(M)} & \mathbf{T}^W(M) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{T}^W(L) & \overrightarrow{\alpha_{W\to k}(L)} & \mathbf{T}^k(L) & \overrightarrow{\alpha_{k\to W}(L)} & \mathbf{T}^W(L) \end{array}$$

is commutative.

3. It is easy to see that

$$\left(\alpha^L\right)^{M\to L} = \alpha^M$$

4. We have

$$\begin{aligned} & M \times_L (L \times \mathbb{R}) \\ & = M \times \mathbb{R} \end{aligned}$$

Therefore the localization

$$\left((\mathcal{K}/L) / \begin{pmatrix} M \\ \downarrow \\ L \end{pmatrix}, (\mathbf{T}_L)_{M\to L}, \left(\alpha^L\right)^{M\to L}, \begin{array}{ccc} M \times_L (L \times \mathbb{R}) & \to & L \times \mathbb{R} \\ \downarrow & & \downarrow \\ M & \to & L \end{array} \right)$$

of the DG-category

$$\left(\mathcal{K}/L, \mathbf{T}_L, \alpha^L, \begin{array}{c} L \times \mathbb{R} \\ \downarrow \\ L \end{array} \right)$$

with respect to

$$\begin{array}{c} M \\ \downarrow \\ L \end{array}$$

is no other than the localization

$$\left(\mathcal{K}/M, \mathbf{T}_M, \alpha^M, \begin{array}{c} M \times \mathbb{R} \\ \downarrow \\ M \end{array} \right)$$

of the DG-category

$$(\mathcal{K}, \mathbf{T}, \alpha, \mathbb{R})$$

with respect to M, so that the desired conclusion follows readily. ■

References

[1] Baez, John C. and Hoffnung, Alexander E.:Convenient categories of smooth spaces, Trans. Amer. Math. Soc. **363** (2011), 5789-5825.

[2] Chen, K.-T.:Iterated integrals of differential forms and loop space homology, Ann. Math. **97** (1973), 217-246.

[3] Chen, K.-T.:Iterated integrals, fundamental groups and covering spaces, Trans. Amer. Math. Soc. **206** (1975), 83-98.

[4] Chen, K.-T.:Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977), 831-879.

[5] Chen, K.-T.:On differentiable spaces, Categories in Continum Physics, Lecture Notes in Math. **1174**, Springer, Berlin, (1986), 38-42.

[6] Frölicher, Alfred:Smooth structures, Lecture Notes in Math. **962**, pp.69-81, Springer, Berlin and Heidelberg, 1982.

[7] Frölicher, Alfred and Kriegl, Andreas: Linear Spaces and Differentiation Theory, John Wiley and Sons, Chichester, 1988.

[8] Jacobs Bart:Categorical Logic and Type Theory, Elsevier, Amsterdam, 1999.

[9] Johnstone, Peter T.:Sketches of an Elephant:a Topos Theory Compendium, Volumes 1 & 2, Oxford University Press, New York, 2002.

[10] Kock, A.: Synthetic Differential Geometry, 2nd edition, London Mathematical Society Lecture Note Series, **333**, Cambridge University Press, Cambridge, 2006.

[11] Kolář, Ivan, Michor, Peter W. and Slovák, Jan: Natural Operations in Differential Geometry, Springer-Verlag, Berlin and Heidelberg, 1993.

[12] Kriegl, Andreas and Michor, Peter W.: The Convenient Setting of Global Analysis, American Mathematical Society, Rhode Island, 1997.

[13] Lavendhomme, R.: Basic Concepts of Synthetic Differential Geometry, Kluwer, Dordrecht, 1996.

[14] MacLane, Saunders:Categories for the Working Mathematician, Springer, New York, 1971.

[15] Nishimura, Hirokazu: A much larger class of Frölicher spaces than that of convenient vector spaces may embed into the Cahiers topos, Far East Journal of Mathematical Sciences, **35** (2009), 211-223.

[16] Nishimura, Hirokazu:Microlinearity in Frölicher spaces -beyond the regnant philosophy of manifolds-, International Journal of Pure and Applied Mathematics, **60** (2010), 15-24.

[17] Nishimura, Hirokazu:Axiomatic differential geometry I, arXiv 1203.3911.

[18] Nishimura, Hirokazu:Axiomatic differential geometry II-1, arXiv 1204.5230.

[19] Nishimura, Hirokazu:Axiomatic differential geometry II-2, arXiv 1207.5121.

[20] Nishimura, Hirokazu:Axiomatic differential geometry II-3, arXiv 1208.1894.

[21] Nishimura, Hirokazu:Axiomatic differential geometry III-1, Model theory I, arXiv 1209.1247.

[22] Nishimura, Hirokazu:Axiomatic differential geometry III-3, The old kingdom of differential geometers, in preparation.

[23] Nishimura, Hirokazu:Axiomatic differential geometry III-4, The fascinating kingdom of supergeometry, in preparation.

[24] Nishimura, Hirokazu:Axiomatic differential geometry III-5, The promising kingdom of braided differential geometry, in preparation.

[25] Nishimura, Hirokazu:Axiomatic differential geometry III-6, The burgeoning kingdom of arithmetical differential geometry, in preparation.

[26] Nishimura, Hirokazu:Axiomatic differential geometry III-7, The futuristic kingdom of noncommutative differential geometry, in preparation.

[27] Schubert, Horst:Categories, Springer, 1972, Berlin Heidelberg.

[28] Souriau, J. M.:Groupes differentiels, in Differential Geometrical Methods in Mathematical Physics (Proc. Conf. Aix-en-Province/Salamanca, 1979), Lecture Notes in Math. **836**, Springer, Berlin, 1980, pp.91-128.

[29] Stacey, Andrew:Comparative smootheology, Theory and Applications of Categories **25** (2011), 64-117.

[30] Steenrod, N. E.: A convenient category of topological spaces, Michigan Math. J., **14** (1967), 133-152.

[31] Weil, André: Théorie des points proches sur les variétés différentiables, Colloques Internationaux du Centre National de la Reserche Scientifique, Strassbourg, pp.111-117, 1953.