Non-vanishing elements in finite groups

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Abstract

We show that if A is an elementary abelian normal p-subgroup of a finite group G and P is a Sylow p-subgroup of G, then no irreducible character of G vanish on any element of $Z(P) \cap A$.

Let G be a finite group. A well-known result of W. Burnside asserts that for each non-linear irreducible character χ of G, then there is an element $g \in G$ such that $\chi(g) = 0$. It is a natural question to determine elements of G which vanish for some irreducible character. Equivalently, we want to determine elements which do not vanish on any irreducible characters of G. Such an element is called a *non-vanishing* element. Nontrivial non-vanishing elements are not common. If G has a p-block of defect zero, then the order of confining element is prime to p. For example, none of simple group of Lie type has nontrivial non-vanishing elements.

Recently, I. M. Isaacs, G. Navarro and T. R. Wolf [1] have shown that if G is solvable, then the images of non-vanishing elements in $G/F(G)$ are 2-elements. They have also shown that if G has a normal Sylow p-subgroup P, then all elements of $Z(P)$ are non-vanishing in G as an easy result. We slightly extend this result.

Theorem 1 *If a finite group G possesses a nontrivial elementary abelian normal p-subgroup A and P is a Sylow p-subgroup of G, then all elements of $Z(P) \cap A$ are non-vanishing in G.*

Proof Let χ be an arbitrary irreducible character of G. Then we have

$$\chi_{|A} = e(\zeta_1 + \zeta_2 + \cdots + \zeta_r).$$

with some integer e, where $\chi_{|A}$ is the restriction of χ to A and $\{\zeta_1, \zeta_2, \ldots, \zeta_r\} \subset \mathrm{Irr}(A)$ is the set of all distinct conjugate characters for some $\zeta = \zeta_1 \in \mathrm{Irr}(A)$ under the action of G. Let $H = \{g \in G | \zeta^g = \zeta\}$ be the inertia subgroup of ζ, where we define $\zeta^g(x) = \zeta(gxg^{-1})$. We have $r = |G : H|$.

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Now, by way of contradiction, suppose $\chi(x) = 0$ for $x \in Z(P) \cap A$. Note that $\deg(\zeta_j) = 1$ and the character values $\zeta_j(x)$ are p-th roots of 1. Pick a primitive p-root of unity ξ. It is well known that any $p-1$ elements subset of $\{1, \xi, \ldots, \xi^{p-1}\}$ is linearly independent over $\mathbb{Q}$ and $1 + \xi + \cdots + \xi^{p-1} = 0$. Then, in the expression :

$$\chi(x) = e(\zeta_1(x) + \zeta_2(x) + \cdots + \zeta_r(x))$$

all p roots of $X^p - 1 = 0$ must appear with the same multiplicity $\frac{r}{p} = p^{m-1}q$. In particular, r is divisible by p.

We next consider the action of P on the set $\{\zeta^g | g \in H\backslash G\} = \{\zeta_1, \zeta_2, \ldots, \zeta_r\}$. For each $g \in H\backslash G$, the size of the P-orbit of ζ^g is

$$|P : P \cap H^g| = |G : P \cap H^g|_p = (|G : H^g||H^g : P \cap H^g|)_p = |G : H|_p |H^g : P \cap H^g|_p,$$

which is divisible by $|G : H|_p$, where $|S|_p$ denotes the highest power of the prime p that divides $|S|$.

Finally we consider the value $\zeta^g(x)$ on each orbit of P. That $x \in Z(P)$ implies $\zeta^{gy}(x) = \zeta^g(yxy^{-1}) = \zeta^g(x)$ for all $y \in P$. This implies that on each P-orbit of $\{\zeta^g | g \in H\backslash G\}$, the element x takes on the same value $\zeta^g(x)$. Hence the multiplicity $\dfrac{r}{p}$ must be divisible by $|G : H|_p$, which is obviously impossible. This completes the proof.

As a corollary, we have

Corollary 2 *If $G \neq \{1\}$ is solvable, then G possesses a nontrivial non-vanishing element.*

Proof Let A be a minimal normal subgroup of G. Then A is an elementary abelian p-group for some prime p. Also, $A \subseteq P$, where P is any Sylow p-subgroup of G. Now, $A \lhd P$, $Z(P) \cap A > 1$ and every $x \in Z(P) \cap A$ is non-vanishing.

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References

[1] M. Isaacs, G. Navarro and T. Wolf, Finite group elements where no irreducible character vanishes, J. Algebra 222 (1999) 413–423.