A condition for a closed one-form to be exact

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Abstract. A condition for a closed one-form to be exact, the one-form having values in Euclidean space, on a compact surface without boundary, is given in the case where the surface has suitable differentiable automorphisms. Tori and hyperelliptic curves, with holomorphic automorphisms, are in this case. A local representation formula for surfaces in Euclidean space is then globalized. A condition for a local surface of constant mean curvature to be global, can be written using a harmonic Gauss map.

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1. Introduction

The theory of de Rham cohomology is related to the theory of surfaces in Euclidean space. A surface is a smooth map f, from a smooth orientable connected two-dimensional manifold M without boundary, to an n-dimensional Euclidean space $\mathbb{R}^n$. The differential df is an exact one-form on M with values in $\mathbb{R}^n$. Hence, a surface represents a boundary of the first de Rham cohomology group of one-forms on M, with values in $\mathbb{R}^n$. The f is reconstructed from its differential by the integral formula

$$f(p) = \int_\gamma df + f(p_0),$$

with a curve γ starting at $p_0 \in M$ and ending at $p \in M$.

If M is simply connected, then a one-form is exact if and only if the one-form is closed. In this case, there are several ways to construct differentials of surfaces. For example, the Weierstrass-Enneper representation formula for minimal surfaces in $\mathbb{R}^3$ constructs a differential of a minimal surface in $\mathbb{R}^3$,

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from a meromorphic function and a holomorphic one-form on a Riemann surface (see Osserman [13]). The Kenmotsu formula constructs a differential of a surface in $\mathbb{R}^3$, from a non-zero mean curvature function and a Gauss map (see Kenmotsu [8]). These researches have been taken over by many researchers and developed in a variety of methods (see, for example, Hoffman and Osserman [6], Konopelchenko [12], Taimanov [18], [17], [16], [15], Friedrich [4], Pedit and Pinkall [14], and Burstall, Ferus, Leschke, Pedit and Pinkall [2]). Except [6], these researches are related with the spinorial or twistorial formulation of the theory of surfaces (see, for example, Bryant [1], Friedrich [5], Kamberov, Norman, Pedit, and Pinkall [7]).

In the case where the topology of M is more complicated, these formulae do not construct an exact one-form in general, but a closed one-form. This motivates us to study a condition for a closed one-form to be exact.

We assume that M is compact without boundary and of genus g. We embed $\mathbb{R}^n$ into the Clifford algebra $\mathrm{C}\ell_n$. We denote the first de Rham cohomology group of $\mathrm{C}\ell_n$-valued one-forms on M by Rh^1, and the cohomology class where a closed one-form η belongs by $[\![\eta]\!]$. We define a pairing $(\ ,\ )\colon \mathrm{Rh}^1 \times \mathrm{Rh}^1 \to \mathrm{C}\ell_n$ by

$$([\![\eta]\!], [\![\xi]\!]) := \int_M \eta \wedge \xi.$$

The dimension of Rh^1 is $2g$. Let $\delta_1, \ldots, \delta_{2g}$ be a basis of Rh^1. It is well-known that a closed one-form ω on M is exact, if and only if

$$([\![\omega]\!], \delta_i) = 0$$

for all $\delta_1, \ldots, \delta_{2g} \in \mathrm{Rh}^1$. In general, it is difficult to find δ_1, ..., δ_g and calculate the above $2g$ integrals.

To ease this difficulty, we assume that M has suitable differentiable automorphisms. We denote by $H_1(M, \mathbb{Z})$ the first homology group of M with integer coefficients. For a closed curve γ in M, we denote by $[\![\gamma]\!]$ the homology class where γ belongs. Let a_1,..,a_g, b_1,..,b_g be closed curves in M such that $[\![a_1]\!]$,..,$[\![a_g]\!]$, $[\![b_1]\!]$,..,$[\![b_g]\!]$ is a canonical basis of $H_1(M, \mathbb{Z})$. We denote by E_g the g by g unit matrix. Let J_{2g} be the $2g$ by $2g$ matrix defined by

$$J_{2g} := \begin{pmatrix} O & E_g \\ -E_g & O \end{pmatrix}.$$

We denote by $\mathrm{Sp}(g, \mathbb{Z})$ the symplectic group of $2g$ by $2g$ matrices with entries in $\mathbb{Z}$. For a matrix N, we denote its transpose by N^T. Then $\mathrm{Sp}(g, \mathbb{Z}) = \left\{X \,\middle|\, X J_{2g} X^T = J_{2g}\right\}$. Let $\mathcal{A}$ be the group of differentiable automorphisms of M. A representation $h = (h_{jk})\colon \mathcal{A} \to \mathrm{Sp}(g, \mathbb{Z})$ is defined by the equation

$$\begin{aligned} &\left([\![\mu(a_1)]\!] \quad \ldots \quad [\![\mu(a_g)]\!] \quad [\![\mu(b_1)]\!] \quad \ldots \quad [\![\mu(b_g)]\!]\right) \\ &= \left([\![a_1]\!] \quad \ldots \quad [\![a_g]\!] \quad [\![b_1]\!] \quad \ldots \quad [\![b_g]\!]\right) h(\mu). \end{aligned}$$

We decompose the matrix $J_{2g}h^T - hJ_{2g}$ into a diagonal matrix $B(\mu) = (b_{ij}(\mu))$, and a matrix $C(\mu) = (c_{ij}(\mu))$ such that the entries of the main diagonal are zero. Then, $J_{2g}h^T(\mu) - h(\mu)J_{2g} = B(\mu) + C(\mu)$. Let $\tilde{C}(\mu) =$

$(\tilde{c}_{ij}(\mu)) = (|c_{ij}(\mu)|)$ and $\Phi\colon\ C\ell_n \to \mathbb{R}1$ be the projection. Then we have the following condition.

Theorem 1.1. *We assume that there exist* $\mu_1, \dots, \mu_m \in \mathcal{A}$ *and* $r_1, \dots, r_m \in \mathbb{R} \setminus \{0\}$*, satisfying one of the following conditions:*

1. $\sum_{l=1}^m \left(r_l B(\mu_l) - |r_l| \tilde{C}(\mu_l) \right)$ *is positive definite,*
2. $\sum_{l=1}^m \left(r_l B(\mu_l) + |r_l| \tilde{C}(\mu_l) \right)$ *is negative definite.*

Let ω *be a one-form on* M *with*

$$\sum_{l=1}^{m} r_l \left(\omega \wedge {\mu_l}^* \omega - {\mu_l}^* \omega \wedge \omega \right) \neq 0.$$

A one-form ω *with values in* $\mathbb{R}^n$ *is exact, if and only if* ω *is closed and*

$$\Phi \left(\sum_{l=1}^{m} r_l [([\![\omega]\!], [\![{\mu_l}^*\omega]\!]) - ([\![{\mu_l}^*\omega]\!], [\![\omega]\!])] \right) = 0.$$

If $\sum_{l=1}^m r_l \left(\omega \wedge {\mu_l}^* \omega - {\mu_l}^* \omega \wedge \omega \right)$ is a non-zero exact one-form, then we understand that ω is an exact one-form, without integration. We see examples in the case where M is a square torus (Corollary 4.1), a hexagonal torus (Corollary 4.2), and a hyperelliptic curve with affine plane model

$$\left\{ (z, w) \in \mathbb{C} \times \mathbb{C} \,\middle|\, w^2 = z^{2(g+1)} - 1 \right\}$$

(Corollary 5.1).

We identify $\mathbb{R}^4$ with quaternions $\mathbb{H}$. For $a \in \mathbb{H}$, we denote its conjugate by $\bar{a}$. Then we have the following similar condition.

Theorem 1.2. *We assume that there exist* $\mu_1, \dots, \mu_m \in \mathcal{A}$ *and* $r_1, \dots, r_m \in \mathbb{R} \setminus \{0\}$*, satisfying one of the following conditions:*

1. $\sum_{l=1}^m \left(r_l B(\mu_l) - |r_l| \tilde{C}(\mu_l) \right)$ *is positive definite,*
2. $\sum_{l=1}^m \left(r_l B(\mu_l) + |r_l| \tilde{C}(\mu_l) \right)$ *is negative definite.*

Let ω *be a one-form on* M *with*

$$\sum_{l=1}^{m} r_l \left(\omega \wedge {\mu_l}^* \bar{\omega} - {\mu_l}^* \omega \wedge \bar{\omega} \right) \neq 0.$$

A one-form ω *with values in* $\mathbb{H}$ *is exact, if and only if* ω *is closed and*

$$\sum_{l=1}^{m} r_l [([\![\omega]\!], [\![{\mu_l}^*\bar{\omega}]\!]) - ([\![{\mu_l}^*\omega]\!], [\![\bar{\omega}]\!])] = 0.$$

By Theorem 1.2, we have a property of a period of a one-form. We denote the inner product of $\mathbb{R}^4$ by $\langle\ ,\ \rangle$. Let η be a one-form with values in $\operatorname{Im}\mathbb{H}$, such that

$$\int_{a_1} \eta \neq 0, \quad \int_{a_2} \eta = \cdots = \int_{a_g} \eta = \int_{b_1} \eta = \cdots = \int_{b_g} \eta = 0.$$

We can consider η as a differential of a singly-periodic surface in $\mathbb{R}^3$.

Corollary 1.3. *We assume that there exist $\mu \in \mathcal{A}$ and $r \in \mathbb{R} \setminus \{0\}$, satisfying one of the following conditions:*

1. *$rB(\mu) - |r|\tilde{C}(\mu)$ is positive definite,*
2. *$rB(\mu) + |r|\tilde{C}(\mu)$ is negative definite.*

Let η be an $\operatorname{Im}\mathbb{H}$-valued one-form such that

$$\int_{a_1} \eta \neq 0, \quad \int_{a_2} \eta = \cdots = \int_{a_g} \eta = \int_{b_1} \eta = \cdots = \int_{b_g} \eta = 0.$$

We assume that

$$r(\eta \wedge \mu^* \bar{\eta} - \mu^* \eta \wedge \bar{\eta}) \neq 0.$$

Then

$$\left\langle \int_{a_1} \eta, \int_{b_1} \mu^* \eta \right\rangle \neq 0.$$

Returning the initial motivation, we combine Theorem 1.2 and the construction of a closed one-form for a surface in $\mathbb{R}^4$ in [2]. We denote the complex structure of M by J. For a one-form ω on M, we define $*\omega := \omega \circ J$. We denote the set of real parts of quaternions by $\operatorname{Re}\mathbb{H}$ and the set of imaginary parts of quaternions by $\operatorname{Im}\mathbb{H}$. If f is an immersion, then there exists a map $N\colon M \to S^2 \subset \operatorname{Im}\mathbb{H}$ such that $*df = N\,df$. We call a non-constant map $f\colon M \to \mathbb{R}^4$ a *surface* if there exists $N\colon M \to S^2 \subset \operatorname{Im}\mathbb{H}$ such that $*df = N\,df$. Then we have the following condition.

Corollary 1.4. *We assume that there exist $\mu_1, \ldots, \mu_m \in \mathcal{A}$ and $r_1, \ldots, r_m \in \mathbb{R} \setminus \{0\}$, satisfying one of the following conditions:*

1. *$\sum_{l=1}^{m} \left(r_l B(\mu_l) - |r_l|\tilde{C}(\mu_l) \right)$ is positive definite,*
2. *$\sum_{l=1}^{m} \left(r_l B(\mu_l) + |r_l|\tilde{C}(\mu_l) \right)$ is negative definite.*

We assume that maps $N\colon M \to S^2 \subset \operatorname{Im}\mathbb{H}$ and $\mathcal{H}\colon M \to \mathbb{H} \setminus \{0\}$, and a non-zero one-form ω on M satisfy

$$-2\,\omega\,\bar{\mathcal{H}} = *\,dN + N\,dN, \quad 2\,\omega \wedge d\bar{\mathcal{H}} = d * dN + dN \wedge dN,$$

$$\sum_{l=1}^{m} r_l \left(\omega \wedge {\mu_l}^* \bar{\omega} - {\mu_l}^* \omega \wedge \bar{\omega} \right) \neq 0,$$

$$\sum_{l=1}^{m} r_l [(\llbracket \omega \rrbracket, \llbracket {\mu_l}^* \bar{\omega} \rrbracket) - (\llbracket {\mu_l}^* \omega \rrbracket, \llbracket \bar{\omega} \rrbracket)] = 0.$$

Then there exists a surface $f\colon M \to \mathbb{R}^4$ with mean curvature vector field $\mathcal{H}$, such that $df = \omega$ and $*df = N\,df$.

If f takes values in $\operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$, then $*df = N\,df = -df\,N$. Applying Corollary 1.2 to surfaces of constant mean curvature, we have the following condition.

Corollary 1.5. *We assume that there exist $\mu_1, \ldots, \mu_m \in \mathcal{A}$ and $r_1, \ldots, r_m \in \mathbb{R} \setminus \{0\}$, satisfying one of the following conditions:*

1. $\sum_{l=1}^m \left(r_l B(\mu_l) - |r_l|\tilde{C}(\mu_l)\right)$ *is positive definite,*
2. $\sum_{l=1}^m \left(r_l B(\mu_l) + |r_l|\tilde{C}(\mu_l)\right)$ *is negative definite.*

We assume that a non-conformal harmonic map $N\colon M \to S^2 \subset \operatorname{Im}\mathbb{H}$

$$\sum_{l=1}^m r_l \left(N * dN \wedge {\mu_l}^*(N * dN) - {\mu_l}^*(N * dN) \wedge (N * dN)\right) \neq 0,$$

$$\sum_{l=1}^m r_l[(\llbracket N * dN\rrbracket, \llbracket {\mu_l}^*(N * dN)\rrbracket) - (\llbracket {\mu_l}^*(N * dN)\rrbracket, \llbracket N * dN\rrbracket)] = 0.$$

Then there exists a surface $f\colon M \to \operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$ *of non-zero constant mean curvature, such that* $*df = N\,df = -df\,N$.

2. Clifford algebra-valued one-forms

Throughout this paper, we assume that all manifolds, maps, and differential forms are smooth. In this section, we show an analog of a relation between the periods of two complex-valued, closed one-forms (see Farkas and Kra [3], III.2.3. Proposition). Then we prove Theorem 1.1.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$. The Clifford algebra $C\ell_n$ is the algebra generated by $e_1, \ldots, e_n$ subject to the relation

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$

We denote the projection $C\ell_n \to \mathbb{R}^1$ by Φ. Let $\langle u, v\rangle$ be the inner product of u and $v \in \mathbb{R}^n$ and $|u| := \langle u, u\rangle^{1/2}$ the norm of $\mathbb{R}^n$. Let $u := \sum_{i=1}^n u_i e_i$, $v := \sum_{i=1}^n v_i e_i \in \mathbb{R}^n \subset C\ell_n$ $(u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{R})$. Then

$$uv = -\langle u, v\rangle + \sum_{i<j} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix} e_i e_j$$

We identify the Clifford algebra and the exterior algebra in a natural manner. Then $uv = -\langle u, v\rangle + u \wedge v$. Hence u and v are linearly independent over $\mathbb{R}$, if and only if $uv - vu \neq 0$. Let $\alpha\colon C\ell_n \to C\ell_n$ be the automorphism which extends $\alpha(u) = -u$ for $u \in \mathbb{R}^n$. The map $\delta\colon C\ell_2 \to \mathbb{H}$, which extends

$$\delta(1) = 1,\ \ \delta(e_1) = i,\ \ \delta(e_2) = j,\ \ \delta(e_1 e_2) = k,$$

is an isomorphism between $C\ell_2$ and $\mathbb{H}$.

Let M be a compact oriented two-dimensional manifold without boundary. We assume that the genus of M is g. For closed curves γ_1 and γ_2 in M, we denote $\gamma_1 \cdot \gamma_2$ the intersection number of γ_1 and γ_2. Let $\pi_1(M) = \pi_1(M, p_0)$ be the fundamental group of M with base point $p_0 \in M$. For a closed curve u with initial and end point p_0 in M, we denote the inverse curve of u by u^{-1}, and the homotopy class that u represents by $[u]$. The map from $\pi_1(M)$ to $H_1(M, \mathbb{Z})$ defined by $[u] \mapsto \llbracket u\rrbracket$ is a surjective group homomorphism. We fix simple closed curves

$$\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$$

in M with initial and end point $p_0 \in M$, such that all curves are disjoint from each other, except p_0, that

$$a_i \cdot a_j = b_i \cdot b_j = 0, \ \ a_i \cdot b_j = -b_j \cdot a_i = \delta_{ij} \ \ (i, j = 1, \dots, g),$$

and that

$$[a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}] = 1.$$

The ordered cycles

$$[\![a_1]\!], \dots, [\![a_g]\!], [\![b_1]\!], \dots, [\![b_g]\!]$$

form a canonical basis of $H_1(M, \mathbb{Z})$. We assume that $[\![\gamma]\!] \in H_1(M, \mathbb{Z})$ and $[\![\eta]\!] \in \mathrm{Rh}^1$. Then we can define a map $V_{[\![\gamma]\!]} \colon \mathrm{Rh}^1 \to \mathrm{C}\ell_n$ by

$$V_{[\![\gamma]\!]}([\![\eta]\!]) := \int_\gamma \eta.$$

We set

$$\begin{aligned} P(\eta) &= \begin{pmatrix} P_1(\eta) & \dots & P_g(\eta) & P_{g+1}(\eta) & \dots & P_{2g}(\eta) \end{pmatrix} \\ &:= \begin{pmatrix} V_{[\![a_1]\!]}([\![\eta]\!]) & \dots & V_{[\![a_g]\!]}([\![\eta]\!]) & V_{[\![b_1]\!]}([\![\eta]\!]) & \dots & V_{[\![b_g]\!]}([\![\eta]\!]) \end{pmatrix}. \end{aligned}$$

Lemma 2.1. *Let η and ξ be closed one-forms on M with values in $\mathrm{C}\ell_n$. Then*

$$([\![\eta]\!], [\![\xi]\!]) = P(\eta) J_{2g} P(\xi)^T \tag{2.1}$$

Proof. Let $\psi \colon U \to M$ be the universal covering. Then there exists a simply connected set $\tilde{M}$ with boundary $\partial\tilde{M}$ in U such that

$$\partial\tilde{M} = \tilde{a}_1 \tilde{b}_1 \tilde{a}_1^- \tilde{b}_1^- \cdots \tilde{a}_g \tilde{b}_g \tilde{a}_g^- \tilde{b}_g^-,$$

$$\psi(\tilde{a}_i) = a_i, \ \ \psi(\tilde{a}_i^-) = a_i^{-1}, \ \ \psi(\tilde{b}_i) = b_i, \ \ \psi(\tilde{b}_i^-) = b_i^{-1} \ \ (i = 1, \dots, g).$$

(see Figure 1).

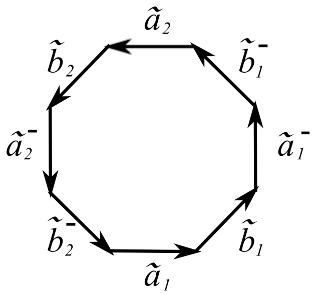

Figure 1. $\tilde{M}$ in the case where $g = 2$.

We fix a point $z_0 \in \tilde{M}$ such that $\psi(z_0) = p_0$. We denote a curve with initial point z_0 and end point z in $\tilde{M}$ by γ. We can define a map $f \colon \tilde{M} \to \mathrm{C}\ell_n$ by

$$f(z) := \int_\gamma \psi^* \eta.$$

By Stokes' theorem, we have

$$\begin{aligned}(\llbracket\eta\rrbracket,\llbracket\xi\rrbracket) &= \int_M \eta\wedge\xi = \int_{\tilde M}\psi^*(\eta\wedge\xi) = \int_{\tilde M} df\wedge\psi^*\xi = \int_{\partial\tilde M} f\psi^*\xi \\ &= \sum_{m=1}^{g}\int_{a_m+b_m+a_m^-+b_m^-} f\psi^*\xi.\end{aligned} \tag{2.2}$$

For a point $z\in\tilde a_m$, we define $z'\in\tilde a_m^-$ by $\psi(z)=\psi(z')$. Let $\tilde\gamma$ be a curve in $\tilde M$ with initial point z_0 and end point z'. Then

$$\begin{aligned}\int_{\tilde a_m+\tilde a_m^-} f\psi^*\xi &= \int_{\tilde a_m}\left(\int_\gamma \psi^*\eta - \int_{\tilde\gamma}\psi^*\eta\right)\psi^*\xi \\ = -\int_{\tilde b_m}\psi^*\eta\int_{\tilde a_m}\psi^*\xi &= -\int_{b_m}\eta\int_{a_m}\xi = -P_{g+m}(\eta)P_m(\xi).\end{aligned}$$

Similarly, for a point $z\in\tilde b_m$, we define $z'\in\tilde b_m^-$ by $\psi(z)=\psi(z')$. Then

$$\begin{aligned}\int_{\tilde b_m+\tilde b_m^-} f\psi^*\xi &= \int_{\tilde b_m}\left(\int_\gamma \psi^*\eta - \int_{\tilde\gamma}\psi^*\eta\right)\psi^*\xi \\ = \int_{\tilde a_m}\psi^*\eta\int_{\tilde b_m}\psi^*\xi &= \int_{a_m}\eta\int_{b_m}\xi = P_m(\eta)P_{g+m}(\xi).\end{aligned}$$

By (2.2), we have (2.1). □

Proof of Theorem 1.1. For a closed curve γ in M, we have

$$V_{\llbracket\gamma\rrbracket}(\llbracket\tau^*\omega\rrbracket) = \int_\gamma \tau^*\omega = \int_{\tau(\gamma)}\omega = V_{\llbracket\tau(\gamma)\rrbracket}(\llbracket\omega\rrbracket).$$

Hence

$$P({\mu_l}^*\omega) = P(\omega)h(\mu_l).$$

Then

$$(\llbracket\omega\rrbracket,\llbracket{\mu_l}^*\omega\rrbracket) = P(\omega)J_{2g}P({\mu_l}^*\omega)^T = P(\omega)J_{2g}h(\mu_l)^TP(\omega)^T.$$

We have

$$(P(\omega)J_{2g}h(\mu_l)^TP(\omega)^T)^T = P(\omega)h(\mu_l)J_{2g}^TP(\omega) = -(\llbracket{\mu_l}^*\omega\rrbracket,\llbracket\omega\rrbracket).$$

Let $\tilde P(\omega) = \begin{pmatrix}|P_1(\omega)| & \dots & |P_{2g}(\omega)|\end{pmatrix}$.

We assume that $\sum_{l=1}^m\left(r_lB(\mu_l) - |r_l|\tilde C(\mu_l)\right)$ is positive definite. Then

$$\begin{aligned}0 &= \Phi\left(\sum_{l=1}^m r_l[(\llbracket\omega\rrbracket,\llbracket{\mu_l}^*\omega\rrbracket) - (\llbracket{\mu_l}^*\omega\rrbracket,\llbracket\omega\rrbracket)]\right) \\ &= \Phi\left(P(\omega)\sum_{l=1}^m r_l\left[J_{2g}h(\mu_l)^T - h(\mu_l)J_{2g}\right]P(\omega)^T\right) \\ &= \Phi\left(P(\omega)\sum_{l=1}^m r_l\left[B(\mu_l)+C(\mu_l)\right]P(\omega)^T\right)\end{aligned}$$

$$= -\sum_{l=1}^{m}\left(\sum_{i=1}^{2g} r_l b_{ii}(\mu_l)|P_i(\omega)|^2 + \sum_{i,j=1}^{2g} r_l c_{ij}(\mu_l)\langle P_i(\omega), P_j(\omega)\rangle\right)$$
$$\leq -\sum_{l=1}^{m}\left(\sum_{i=1}^{2g} r_l b_{ii}(\mu_l)|P_i(\omega)|^2 - \sum_{i,j=1}^{2g} |r_l||c_{ij}(\mu_l)||P_i(\omega)||P_j(\omega)|\right)$$
$$= -\tilde{P}(\omega)\left(\sum_{l=1}^{m}\left(r_l B(\mu_l) - |r_l|\tilde{C}(\mu_l)\right)\right)\tilde{P}(\omega)^T \leq 0.$$

Hence $\tilde{P}(\omega) = 0$.

We assume that $\sum_{l=1}^{m}\left(r_l B(\mu_l) + |r_l|\tilde{C}(\mu_l)\right)$ is negative definite. Then

$$0 = \Phi\left(\sum_{l=1}^{m} r_l[(\llbracket\omega\rrbracket, \llbracket{\mu_l}^*\omega\rrbracket) - (\llbracket{\mu_l}^*\omega\rrbracket, \llbracket\omega\rrbracket)]\right)$$
$$= -\sum_{l=1}^{m}\left(\sum_{i=1}^{2g} r_l b_{ii}(\mu_l)|P_i(\omega)|^2 + \sum_{i,j=1}^{2g} r_l c_{ij}(\mu_l)\langle P_i(\omega), P_j(\omega)\rangle\right)$$
$$\geq -\sum_{l=1}^{m}\left(\sum_{i=1}^{2g} r_l b_{ii}(\mu_l)|P_i(\omega)|^2 + \sum_{i,j=1}^{2g} |r_l||c_{ij}(\mu_l)||P_i(\omega)||P_j(\omega)|\right)$$
$$= -\tilde{P}(\omega)\left(\sum_{l=1}^{m}\left(r_l B(\mu_l) + |r_l|\tilde{C}(\mu_l)\right)\right)\tilde{P}(\omega)^T \geq 0.$$

Hence $\tilde{P}(\omega) = 0$.

A one-form ω is exact if and only if $\tilde{P}(\omega) = 0$. Hence, Theorem 1.1 holds. $\square$

3. Quaternionic-valued one-forms

We have a similar condition for quaternionic-valued one-forms.

Proof of Theorem 1.2. We have

$$(\llbracket\omega\rrbracket, \llbracket{\mu_l}^*\bar{\omega}\rrbracket) = P(\omega)J_{2g}P({\mu_l}^*\bar{\omega})^T = P(\omega)J_{2g}h(\mu_l)^T P(\bar{\omega})^T,$$
$$\overline{(\llbracket\omega\rrbracket, \llbracket{\mu_l}^*\bar{\omega}\rrbracket)} = \overline{(P(\omega)J_{2g}P({\mu_l}^*\bar{\omega})^T)}^T$$
$$= -P(\omega)h(\mu_l)J_{2g}P(\bar{\omega})^T = -(\llbracket{\mu_l}^*\omega\rrbracket, \llbracket\bar{\omega}\rrbracket).$$

Then

$$2\,\mathrm{Re}(\llbracket\omega\rrbracket, \llbracket{\mu_l}^*\bar{\omega}\rrbracket) = (\llbracket\omega\rrbracket, \llbracket{\mu_l}^*\bar{\omega}\rrbracket) + \overline{(\llbracket\omega\rrbracket, \llbracket{\mu_l}^*\bar{\omega}\rrbracket)}$$
$$= P(\omega)J_{2g}h(\mu_l)^T P(\bar{\omega})^T - P(\omega)h(\mu_l)J_{2g}P(\bar{\omega})^T$$
$$= P(\omega)\left(J_{2g}h(\mu_l)^T - h(\mu_l)J_{2g}\right)P(\bar{\omega})^T$$
$$= P(\omega)(B(\mu_l) + C(\mu_l))P(\bar{\omega})^T.$$

We assume that $\sum_{l=1}^{m}\left(r_l B(\mu_l) - |r_l|\tilde{C}(\mu_l)\right)$ is positive definite. Since $\langle u, v\rangle = (u\bar{v} + \bar{v}u)/2$ for u, $v \in \mathbb{H}$, we have

$$\begin{aligned}
0 &= \sum_{l=1}^{m} r_l[(\llbracket\omega\rrbracket, \llbracket{\mu_l}^*\bar{\omega}\rrbracket) - (\llbracket{\mu_l}^*\omega\rrbracket, \llbracket\bar{\omega}\rrbracket)] \\
&= P(\omega)\sum_{l=1}^{m} r_l\left[B(\mu_l) + C(\mu_l)\right] P(\bar{\omega})^T \\
&= -\sum_{l=1}^{m}\left(\sum_{i=1}^{2g} r_l b_{ii}(\mu_l)|P_i(\omega)|^2 + \sum_{i,j=1}^{2g} r_l c_{ij}(\mu_l)\langle P_i(\omega), P_j(\omega)\rangle\right) \\
&\leq -\sum_{l=1}^{m}\left(\sum_{i=1}^{2g} r_l b_{ii}(\mu_l)|P_i(\omega)|^2 - \sum_{i,j=1}^{2g} |r_l||c_{ij}(\mu_l)||P_i(\omega)||P_j(\omega)|\right) \\
&= -\tilde{P}(\omega)\left(\sum_{l=1}^{m}\left(r_l B(\mu_l) - |r_l|\tilde{C}(\mu_l)\right)\right)\tilde{P}(\bar{\omega})^T \leq 0.
\end{aligned}$$

Hence $\tilde{P}(\omega) = 0$.

We assume that $\sum_{l=1}^{m}\left(r_l B(\mu_l) + |r_l|\tilde{C}(\mu_l)\right)$ is negative definite. Then

$$\begin{aligned}
0 &= \sum_{l=1}^{m} r_l[(\llbracket\omega\rrbracket, \llbracket{\mu_l}^*\bar{\omega}\rrbracket) - (\llbracket{\mu_l}^*\omega\rrbracket, \llbracket\bar{\omega}\rrbracket)] \\
&= -\sum_{l=1}^{m}\left(\sum_{i=1}^{2g} r_l b_{ij}(\mu_l)|P_i(\omega)|^2 + \sum_{i,j=1}^{2g} r_l c_{ij}(\mu_l)\langle P_i(\omega), P_j(\omega)\rangle\right) \\
&\geq -\sum_{l=1}^{m}\left(\sum_{i=1}^{2g} r_l b_{ij}(\mu_l)|P_i(\omega)|^2 + \sum_{i,j=1}^{2g} |r_l||c_{ij}(\mu_l)||P_i(\omega)||P_j(\omega)|\right) \\
&= -\tilde{P}(\omega)\left(\sum_{l=1}^{m}\left(r_l B(\mu_l) + |r_l|\tilde{C}(\mu_l)\right)\right)\tilde{P}(\bar{\omega})^T \geq 0.
\end{aligned}$$

Hence $\tilde{P}(\omega) = 0$.

A one-form ω is exact if and only if $\tilde{P}(\omega) = 0$. Hence, Theorem 1.2 holds. □

Proof of Corollary 1.3. Since η is not exact, we have

$$\int_M r\left(\eta \wedge \mu^*\overline{\eta} - \mu^*\eta \wedge \overline{\eta}\right) \neq 0.$$

On the other hand,

$$\int_M r\left(\eta \wedge \mu^*\overline{\eta} - \mu^*\eta \wedge \overline{\eta}\right) = -r\left(\int_{a_1}\eta\int_{b_1}\mu^*\eta + \int_{b_1}\mu^*\eta\int_{a_1}\eta\right)$$

$$= -2r \left\langle \int_{a_1} \eta, \int_{b_1} \mu^* \eta \right\rangle$$

by Lemma 2.1. Hence Corollary 1.3 holds. □

4. One-forms on tori

We review the classification of tori and their holomorphic automorphisms, and consider Theorem 1.1 and Theorem 1.2 in the case where M is a torus.

Let M be a torus. We consider M as a Riemann surface. Then M is biholomorphic to an orbit space $\mathbb{C}/\Lambda_\lambda$ with a lattice

$$\Lambda_\lambda := \mathbb{Z} + \mathbb{Z}\lambda, \ \operatorname{Im}\lambda > 0, \ -\frac{1}{2} < \operatorname{Re}\lambda \leq \frac{1}{2}, \ \begin{cases} |\lambda| \geq 1 & (\operatorname{Re}\lambda \geq 0), \\ |\lambda| > 1 & (\operatorname{Re}\lambda < 0). \end{cases}$$

The torus $\mathbb{C}/\Lambda_i$ is called a *square torus*. The torus $\mathbb{C}/\Lambda_{e^{\pi i/3}}$ is called a *hexagonal torus*. The projection $\psi_\lambda \colon \mathbb{C} \to \mathbb{C}/\Lambda_\lambda$ is the universal covering. We define $\tilde{a} \colon [0,1] \to \mathbb{C}$ and $\tilde{b} \colon [0,1] \to \mathbb{C}$ by $\tilde{a}(t) := t$ and $\tilde{b}(t) := \lambda t$ respectively. Then $a := \psi_\lambda \circ \tilde{a}$ and $b := \psi_\lambda \circ \tilde{b}$ are closed curves in $\mathbb{C}/\Lambda_\lambda$ subject to the relation $aba^{-1}b^{-1} = 1$. The fundamental group $\pi_1(\mathbb{C}/\Lambda_\lambda, \psi_\lambda(0))$ is generated by $[a]$ and $[b]$.

A map $\tau \colon \mathbb{C}/\Lambda_\lambda \to \mathbb{C}/\Lambda_\lambda$ is a holomorphic automorphism such that τ^2 is the identity map, if and only if $(\tau \circ \psi_\lambda)(z) = \psi_\lambda(\pm z)$. There exists a holomorphic automorphism τ such that τ^2 is not the identity map, if and only if $\mathbb{C}/\Lambda_\lambda$ is a square torus or a hexagonal torus. In fact, we define $\tilde{\tau}_{m,n} \colon \mathbb{C} \to \mathbb{C}$ by

$$\tilde{\tau}_{m,n}(z) = e^{2\pi m i/n} z \ \ (n = 4, 6, \ \ m = 0, 1, \ldots, n-1).$$

Then $\tau_{m,6} \colon \mathbb{C}/\Lambda_{e^{\pi i/3}} \to \mathbb{C}/\Lambda_{e^{\pi i/3}}$ is defined by $\tau_{m,6} \circ \psi_{e^{\pi i/3}} := \psi_{e^{\pi i/3}} \circ \tilde{\tau}_{m,6}$. A map $\tau_{m,6}$ is a holomorphic automorphism of a hexagonal torus. Similarly, $\tau_{m,4} \colon \mathbb{C}/\Lambda_i \to \mathbb{C}/\Lambda_i$ is defined by $\tau_{m,4} \circ \psi_i := \psi_i \circ \tilde{\tau}_{m,4}$. A map $\tau_{m,4}$ is a holomorphic automorphism of a square torus.

Corollary 4.1. *Let M be a square torus and ω a one-form on M with values in $\mathbb{R}^n \subset C\ell_n$, satisfying $\omega \wedge {\tau_{1,4}}^* \bar{\omega} - {\tau_{1,4}}^* \omega \wedge \bar{\omega} \neq 0$. A one-form ω is exact, if and only if ω is closed and*

$$\Phi\left(([\![\omega]\!], [\![{\tau_{1,4}}^*\omega]\!]) - ([\![{\tau_{1,4}}^*\omega]\!], [\![\omega]\!])\right) = 0.$$

Proof. Let M be a square torus. We see that

$$h(\tau_{1,4}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $m = 1$, $r_1 = 1$, and $\mu_1 = \tau_{1,4}$. Then

$$J_2 h(\tau_{1,4})^T - h(\tau_{1,4}) J_2 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = B(\tau_{1,4}),$$

$$C(\tau_{1,4}) = \tilde{C}(\tau_{1,4}) = 0,$$

$$B(\tau_{1,4}) + \tilde{C}(\tau_{1,4}) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then Corollary 4.1 holds by Theorem 1.1. □

Corollary 4.2. *Let M be a hexagonal torus and ω a one-form on M with values in $\mathbb{R}^n \subset C\ell_n$, satisfying $\omega \wedge {\tau_{1,6}}^*\bar{\omega} - {\tau_{1,6}}^*\omega \wedge \bar{\omega} \neq 0$. A one-form ω is exact, if and only if*

$$\Phi\left((\llbracket\omega\rrbracket, \llbracket{\tau_{1,6}}^*\omega\rrbracket) - (\llbracket{\tau_{1,6}}^*\omega\rrbracket, \llbracket\omega\rrbracket)\right) = 0.$$

Proof. Let M be a hexagonal torus. We see that

$$h(\tau_{1,6}) = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Let $m = 1$, $r_1 = 1$, and $\mu_1 = \tau_{1,6}$. Then

$$J_2 h(\tau_{1,6})^T - h(\tau_{1,6}) J_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},$$

$$B(\tau_{1,6}) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad C(\tau_{1,6}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{C}(\tau_{1,6}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$B(\tau_{1,6}) + \tilde{C}(\tau_{1,6}) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

Then Corollary 4.2 holds by Theorem 1.1. □

We collect statements for quaternionic-valued one-forms, which are obtained in a similar fashion as above. We omit their proof.

Corollary 4.3. *Let M be a square torus and ω a one-form on M with values in $\mathbb{R}^4 \cong \mathbb{H}$, satisfying $\omega \wedge {\tau_{1,4}}^*\bar{\omega} - {\tau_{1,4}}^*\omega \wedge \bar{\omega} \neq 0$. A one-form ω is exact, if and only if ω is closed and*

$$(\llbracket\omega\rrbracket, \llbracket{\tau_{1,4}}^*\bar{\omega}\rrbracket) - (\llbracket{\tau_{1,4}}^*\omega\rrbracket, \llbracket\bar{\omega}\rrbracket) = 0.$$

Corollary 4.4. *Let M be a hexagonal torus and ω a one-form on M with values in $\mathbb{R}^4 \cong \mathbb{H}$, satisfying $\omega \wedge {\tau_{1,6}}^*\bar{\omega} - {\tau_{1,6}}^*\omega \wedge \bar{\omega} \neq 0$. A one-form ω is exact, if and only if*

$$(\llbracket\omega\rrbracket, \llbracket{\tau_{1,6}}^*\bar{\omega}\rrbracket) - (\llbracket{\tau_{1,6}}^*\omega\rrbracket, \llbracket\bar{\omega}\rrbracket) = 0.$$

5. One-forms on a hyperelliptic curve

We review a hyperelliptic curve and its automorphisms, and prove Corollary 5.1.

Let M be the hyperelliptic curve of genus g with affine model (5.1) and τ be a holomorphic automorphism of M defined by (5.2). If $g = 1$, then M is a square torus and $\tau = \tau_{1,4}$.

We label two copies of a sphere, which is identified with $\Sigma = \mathbb{C} \cup \{\infty\}$, sheet I and sheet II. On each sheet, we draw a smooth curve, joining $e^{(2k-1)\pi i/(g+1)}$ and $e^{2k\pi i/(g+1)}$ $(k = 1, \ldots, g+1)$. These curves are called cuts. We assume that these cuts do not intersect each other. Each cut has two banks, called the N-bank and the S-bank. The surface M is constructed by joining every S-bank on sheet I to an N-bank of the corresponding cut on sheet II, and joining every N-bank on sheet I to an S-bank of the corresponding cut on sheet II. We draw a simple closed curve a_k, winding counterclockwise once, around the cuts joining $e^{(2k-1)\pi i/(g+1)}$ and $e^{2k\pi i/(g+1)}$ on sheet I ($k = 1, \dots, g$). We choose a curve b_k starting from a point on the cut from $e^{(2g+1)\pi i/(g+1)}$ to 1, going on sheet I, to a point on the cut from $e^{(2k-1)\pi i/(g+1)}$ to $e^{2k\pi i/(g+1)}$, and returning on the sheet II ($k = 1, \dots, g$). A map $\tau\colon M \to M$ defined by $\tau(w, z) = (-w, e^{\pi i/(g+1)} z)$ is a holomorphic automorphism of M. We have a situation similar to a square torus, when a Riemann surface is hyperelliptic. Let M be a hyperelliptic curve of genus g with affine plane model

$$\left\{ (z, w) \in \mathbb{C} \times \mathbb{C} \,\middle|\, w^2 = z^{2(g+1)} - 1 \right\}. \tag{5.1}$$

A holomorphic automorphism $\tau\colon M \to M$ is defined by

$$\tau(z, w) := \left(e^{\pi i/(g+1)} z, -w \right). \tag{5.2}$$

We have

$$h(\tau) = \begin{pmatrix} O & Q(\tau) \\ R(\tau) & O \end{pmatrix},$$

$$Q(\tau) = \begin{pmatrix} -1 & \cdots\cdots & & -1 \\ 0 & -1 & \dots & -1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 \end{pmatrix},$$

$$R(\tau) = \begin{pmatrix} 1 & 0 & \cdots\cdots\cdots & & & 0 \\ -1 & 1 & 0 & \cdots\cdots & & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 1 & 0 \\ 0 & \cdots\cdots & & 0 & -1 & 1 \end{pmatrix}.$$

Corollary 5.1. *Let M be a hyperelliptic curve of genus two with affine model* (5.1)*, and τ a holomorphic automorphism defined by* (5.2)*. Let ω be a one-form with values in $\mathbb{R}^n \subset C\ell_n$, satisfying $\omega \wedge \tau^*\omega - \tau^*\omega \wedge \omega \neq 0$. A one-form ω is exact, if and only if ω is closed and*

$$\Phi\left((\llbracket \omega \rrbracket, \llbracket \tau^*\omega \rrbracket) - (\llbracket \tau^*\omega \rrbracket, \llbracket \omega \rrbracket) \right) = 0.$$

Proof. We have

$$h(\tau) = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}.$$

Let $m = 1$, $r_1 = 1$, and $\mu_1 = \tau_{1,6}$. Then

$$J_4 h(\tau)^T - h(\tau) J_4 = \begin{pmatrix} -2 & -1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix},$$

$$B(\tau) = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad C(\tau) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\tilde{C}(\tau) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B(\tau) + \tilde{C}(\tau) = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Then Corollary 4.2 holds by Theorem 1.1. □

As in the previous section, we have a similar statement as above for quaternionic-valued one-forms. We omit the proof.

Corollary 5.2. *Let M be a hyperelliptic curve of genus two with affine model* (5.1)*, and τ a holomorphic automorphism defined by* (5.2)*. Let ω be a one-form with values in $\mathbb{R}^4 \cong \mathbb{H}$, satisfying $\omega \wedge \tau^*\bar{\omega} - \tau^*\omega \wedge \bar{\omega} \neq 0$. A one-form ω is exact, if and only if ω is closed and*

$$([\![\omega]\!], [\![\tau^*\bar{\omega}]\!]) - ([\![\tau^*\omega]\!], [\![\bar{\omega}]\!]) = 0.$$

6. Surfaces in Euclidean four-space

We prove Corollary 1.4 and Corollary 1.5.

Firstly, we review the theory of surfaces in terms of quaternions (see [2]). Let $f\colon M \to \mathbb{H}$ be a surface. Then there exists $N\colon M \to S^2 \subset \operatorname{Im}\mathbb{H}$ such that $*\,df = N\,df$. Let $\mathcal{H}\colon M \to \mathbb{H}$ be the mean curvature vector of f. Then the following equation holds.

$$-2\,df\,\bar{\mathcal{H}} = *\,dN + N\,dN.$$

Differentiating this equation, we have

$$2\,df \wedge d\bar{\mathcal{H}} = d * dN + dN \wedge dN.$$

Proof of Corollary 1.4*.* We assume that N, $\mathcal{H}$, and ω satisfy the assumption. Then

$$2\,d\omega\,\bar{\mathcal{H}} = 2\,d(\omega\bar{\mathcal{H}}) + 2\omega \wedge d\bar{\mathcal{H}} = 0.$$

Hence ω is a closed one-form. By Theorem 1.1, the one-form ω is exact. Then there exists a map $f\colon M \to \mathbb{H}$ with $df = \omega$. By the assumption for ω, we have $*\,df = N\,df$. Hence, f is a surface with $*\,df = N\,df$ and mean curvature vector field $\mathcal{H}$. □

The following is the case where a surface takes values in $\mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$.

Corollary 6.1. *We assume that there exist* $\mu_1, \dots, \mu_m \in \mathcal{A}$ *and* $r_1, \dots, r_m \in \mathbb{R} \setminus \{0\}$*, satisfying one of the following conditions:*

1. $\sum_{l=1}^m \left(r_l B(\mu_l) - |r_l| \tilde{C}(\mu_l) \right)$ *is positive definite,*
2. $\sum_{l=1}^m \left(r_l B(\mu_l) + |r_l| \tilde{C}(\mu_l) \right)$ *is negative definite.*

We assume that maps $N\colon M \to S^2 \subset \operatorname{Im}\mathbb{H}$ *and* $H\colon M \to \mathbb{R}$*, and a non-zero one-form* ω *on* M *satisfy*

$$2\,\omega\, H = N * dN - dN, \quad -2\,\omega \wedge (dH\, N + H\, dN) = d * dN + dN \wedge dN,$$

$$\sum_{l=1}^m r_l \left(\omega \wedge {\mu_l}^* \bar{\omega} - {\mu_l}^* \omega \wedge \bar{\omega} \right) \neq 0,$$

$$\sum_{l=1}^m r_l [(\llbracket \omega \rrbracket, \llbracket {\mu_l}^* \bar{\omega} \rrbracket) - (\llbracket {\mu_l}^* \omega \rrbracket, \llbracket \bar{\omega} \rrbracket)] = 0.$$

Then there exists a surface $f\colon M \to \operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$ *with mean curvature* H*, such that* $df = \omega$ *and* $*\,df = N\,df = -df\,N$*.*

Proof. For a surface $f\colon M \to \operatorname{Im}\mathbb{H}$, we have $*\,df = N\,df = -df\,N$ with Gauss map $N\colon M \to S^2 \subset \operatorname{Im}\mathbb{H}$. The mean curvature vector of f is $\mathcal{H} = HN$ with mean curvature function H. Then this corollary follows from Corollary 1.4. □

Proof of Corollary 1.5. A harmonic map $N\colon M \to S^2 \subset \operatorname{Im}\mathbb{H}$ satisfies the equation

$$d * dN = N\, dN \wedge * dN.$$

Let $2\omega\, H = N * dN - dN$ $(H \in \mathbb{R} \setminus \{0\})$. Then

$$\begin{aligned} -2\omega \wedge (dH\, N + H\, dN) &= -(N * dN - dN) \wedge dN \\ = -N * dN \wedge dN + dN \wedge dN &= d * dN + dN \wedge dN. \end{aligned}$$

Then Corollary 1.5 follows from Corollary 6.1. □

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