Department of Social Systems and Management

Discussion Paper Series

No.1292

The Structure of the Competitive Equilibria in an Assignment Market

by

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May 2012

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Tsukuba, Ibaraki 305-8573
JAPAN

The Structure of the Competitive Equilibria in an Assignment Market

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May 30, 2012

Abstract

We consider the structure of the set of competitive equilibria in a generalized assignment market. In an assignment market with homogeneous indivisible goods, we can observe the result of non-simultaneous multiplicity that if there are multiple competitive prices, the equilibrium quantity is unique; equivalently, if there are multiple equilibrium quantities, the competitive price is unique. We show that non-simultaneous multiplicity holds separately for each type of indivisible good in a market with heterogeneous indivisible goods. Based on this result, for each good, we evaluate the sizes of the sets of competitive prices and quantities. These results are proved only under the basic assumptions on the generalized assignment market. As an application, we give a sufficient condition for large assignment markets so that the set of competitive prices shrinks to a unique price when the markets get larger and denser.

1 Introduction

We consider the structure of the set of competitive equilibria in an assignment market. This market consists of two types of economic agents: Sellers and buyers. Objects of trade are multiple types of indivisible goods and a perfectly divisible good (money). Each seller may provide multiple units of indivisible goods, but each buyer demands at most one unit of an indivisible good. We adopt the model of an generalized assignment market (GAM) due to Kaneko [2].

The GAM model is a generalization of Shapley-Shubik's [10] assignment market model in that each seller may provide multiple units of indivisible goods and furthermore quasi-linearity is not assumed on a utility function of each buyer. Kaneko [2] proved the existence of a competitive equilibrium in the GAM model.

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[†]This paper is based on my master thesis, which was written under the supervision of Professor M.Kaneko. I thank him for many valuable suggestions and discussions.

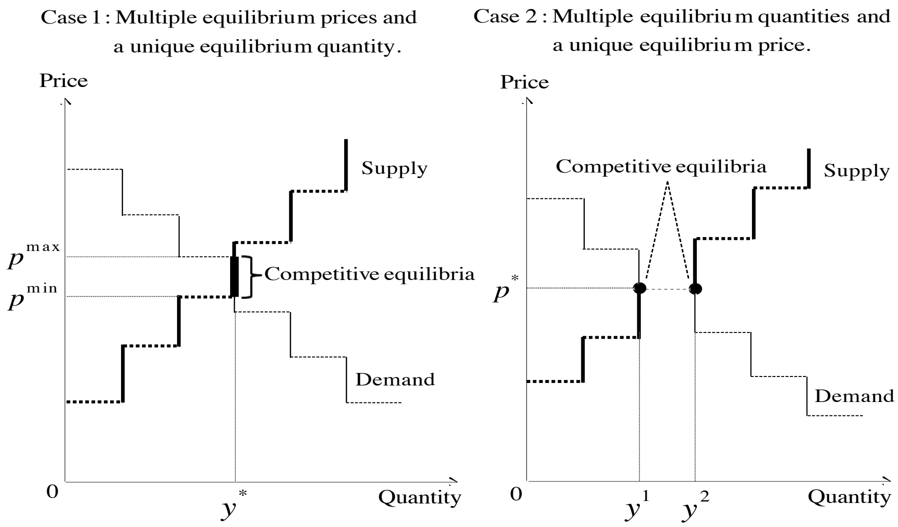

Figure 1: The case of homogeneous goods.

The GAM model targets economic problems of indivisible goods such as houses, cars, and labors. This is a salient difference from the standard general equilibrium model with perfectly divisible goods (cf., Mas-Colell-Whinston-Green [7]). The structure of the set of competitive equilibria for GAM also differs considerably from that for the standard model. It is a contribution of this paper to show this difference: Although the market itself is interactive with all the indivisible goods, we can analyze the set of competitive equilibria in GAM separately for each indivisible good. Furthermore, we can evaluate the sizes of competitive prices and allocations separately for each indivisible good.

We will give three theorems, Theorem 3.1-3.3, on the structure of competitive equilibria in GAM, which will be given in Section 3. Let t ($t = 1, \dots, T$) be the type of indivisible goods. Theorem 3.1 is stated as follows:

Theorem 3.1. If there are multiple competitive prices (equilibrium quantities) for good t, then the equilibrium quantity (competitive price) of good t is unique.

This result is obtained for each good t, separately from other goods. This is equivalent to excluding the existence of multiple competitive prices and multiple equilibrium quantities.

Theorem 3.1 is better understood by recalling Böhm-Bawerk's horse market, where all indivisible goods are homogeneous (the number of types of goods $T = 1$). There are two possible cases for the structure of the competitive equilibria, which are illustrated in Figure 1. In case 1 of Figure 1, there are multiple equilibrium prices and a unique equilibrium quantity, and in case 2, there are multiple equilibrium quantities and a unique equilibrium price. Returning to Theorem 3.1, we find that the same results holds separately for each indivisible good in the GAM model.

Since we can consider the competitive equilibria separately for each indivisible good t, the size of competitive prices for good t may be characterized by the marginal costs of the seller. We will give this characterization by Theorem 3.2. When a competitive price (equilibrium quantity) is uniquely determined, we denote it by p_t^* (y_t^*).

Theorem 3.2.(1): If there are multiple competitive prices for good t, then all competitive prices of good t are bounded by the marginal costs of good t at $y_t^* - 1$ and y_t^*.

(2): If there are multiple equilibrium quantities for good t, then p_t^* coincides with the marginal cost of good t at the minimum equilibrium quantity.

As for Theorem 3.1, this theorem can be understood by Figure 1. Indeed, assertion (1) corresponds to Case 1 of Figure 1, and (2) to Case 2.

Based on the above two theorems, we will obtain a convergence result of competitive prices for a large GAM. Shapley-Shubik [10] observed, for Böhm-Bawerk's market, when a market becomes large and dense, the set of equilibrium prices becomes small. When the market is "dense", the demand and the supply curves, which are step functions, become approximately continuous, which case has a unique competitive price. They expected that this also holds in the general case (the number of types of goods $T > 1$), but also stated a difficulty caused by the increase of the dimensionality of the set of equilibria.

Our Theorem 3.1 and 3.2 will enable us to answer the conjecture by Shapley-Shubik. We give a sufficient condition for the convergence of the competitive prices in our GAM model.

Theorem 3.3. The competitive prices of good t converges to the unique value as the number of sellers of good t (but their cost functions are similar to each others) increases.

Theorem 3.3 is a direct generalization of Shapley-Shubik's observation for a "dense" Böhm-Bawerk's market.

For notational simplicity, we assume throughout the paper that for each t, goods of type-t are provided by a single seller. However, this assumption can be made without loss of generality in the consideration of competitive equilibrium. This aggregation result will be discussed in Section 4.

The format of the paper is as follows. Section 2 presents the GAM model. Section 3 presents two main theorems about the structure of competitive equilibria. Also we give, as an application of the theorems, the convergence theorem on the competitive prices in a large GAM. Section 4 shows the invariance result on competitive equilibria for the GAM models with/without aggregations of sellers. Section 5 presents the conclusion and some remarks.

2 Generalized Assignment Markets

We denote the generalized assignment market model by (M, N), where $M = \{1', \ldots, m'\}$ is the set of buyers and $N = \{1, \ldots, n\}$ is the set of sellers. There are T-types of indivisible goods to be traded for a perfectly divisible good, called "money".

The *consumption set* for a buyer is given as $X := \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\} \times \mathbb{R}_+$, where for $t \neq 0$, $\mathbf{e}^t$ is the T-dimensional unit vector with t-th component 1 and $\mathbf{e}^0 = \mathbf{0}$, and $\mathbb{R}_+$ is the set of nonnegative real numbers. A consumption vector $(\mathbf{e}^t, d) \in X$ means that a buyer consumes one unit of indivisible good t and d amount of money. For $t = 0$, he consumes no indivisible goods. The *initial endowment* of each buyer $i \in M$ is given as $(\mathbf{e}^0, I_i)$ with $I_i > 0$. That is, buyer $i \in M$ has initially no indivisible goods but money I_i. Each buyer wants to buy at most one unit of indivisible goods by paying some part of I_i.

We define buyer i's *utility function* as $u_i : X \to \mathbb{R}$. We assume the following A1 and A2 for u_i: For each $i \in M$,

Assumption A1 (*Continuity and Monotonicity*). For each $x_i \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}$, $u_i(x_i, d)$ is a continuous and strictly monotone increasing function with respect to d.

Assumption A2 (*Boundary condition*). $u_i(\mathbf{e}^0, I_i) > u_i(\mathbf{e}^t, 0)$ for $t = 1, \ldots, T$.

A1 would need no explanation. A2 means that a buyer prefers the initial endowment to consuming an indivisible good by paying all his income I_i.

Each seller $j \in N$ provides indivisible goods of exactly one type, but each may provide more than one units. We divide the set N into $N_1, \ldots, N_T$, where N_t is the set of all sellers who provide indivisible good t.

We define the *cost function* of seller $j \in N_t$ $(t = 1, \ldots, T)$ as $c_j : \mathbb{Z}_+ \to \mathbb{R}_+$, where $\mathbb{Z}_+$ is the set of nonnegative integers. $c_j(y_j)$ represents the cost (in terms of money) of producing y_j units of indivisible good t. For each $j \in N_t$, we define the *marginal cost* $mc_j(y_j) := c_j(y_j + 1) - c_j(y_j)$ for $y_j \in \mathbb{Z}_+$. We assume the following B1 and B2 for c_j :

Assumption B1 (*No fixed cost*). $c_j(0) = 0$ and $c_j(0) < c_j(1)$.

Assumption B2 (*Convexity*). $mc_j(y_j) \leq mc_j(y_j + 1)$ for all $y_j \in \mathbb{Z}_+$.

The first assumption means that no fixed costs are required, but that a positive cost is required for production. Assumption B2 is a discrete version of convexity, meaning that a marginal cost of one additional unit is increasing.

The model given in Shapley-Shubik [10] can be regarded as a special case of the above model of GAM. Shapley-Shubik assumed that each buyer $i \in M$ wants to buy at most one unit of indivisible good and has a quasi-linear utility function, i.e., $u_i(\mathbf{e}^t, d) = u_i(\mathbf{e}^t, 0) + d$ for all $(\mathbf{e}^t, d) \in X$; and each seller $j \in N$ has one unit of an indivisible good for sale with reservation price $r_j > 0$. Let us see that this model is a special case of the GAM model. In A1 and A2, we do not assume quasi-linearity; we allow income effects in buyers' behavior. A seller in Shapley-Shubik's model is expressed in our model as a seller having the cost function $c_j(y_j)$ with $c_j(1) = r_j$ and $c_j(y_j) =$"large" for $y_j \geq 2$.

For notational simplicity, we assume that the set N_t of sellers of type t $(t = 1, ..., T)$ consists of one seller, i.e.,

$$N_t = \{t\} \text{ for all } t \in N. \tag{2.1}$$

This is interpreted as meaning that the sellers of type t can be represented by one "aggregated seller". This assumption can be made *without lose of generality*, as far as the competitive equilibrium is concerned. This will be shown in Section 4.

In the GAM model, we consider the concept of a competitive equilibrium. Let (p, x, y) $= ((p_1, \ldots, p_T), (x_{1'}, \ldots, x_{m'}), (y_1, \ldots, y_T))$ be a triple of $p \in \mathbb{R}^T_+$, $x \in \left\{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\right\}^{m'}$ and $y \in \mathbb{Z}^T_+$.

Definition 2.1 (*Competitive Equilibrium*). We say that (p, x, y) is a *competitive equilibrium* iff:

(1) Utility Maximization under the Budget Constraint. For all $i \in M$,
(i): $I_i \geq px_i$, where $px_i = \sum_{t=1}^{T} p_t x_{it}$;
(ii): $u_i(x_i, I_i - px_i) \geq u_i(x'_i, I_i - px'_i)$ for all $x'_i \in \{\mathbf{e}^0, \mathbf{e}^1, \ldots, \mathbf{e}^T\}$ with $I_i \geq px'_i$.

(2) Profit Maximization. For all $t \in N$,
$p_t y_t - c_t(y_t) \geq p_t y'_t - c_t(y'_t)$ for all $y'_t \in \mathbb{Z}_+$.

(3) Balance of the Total Demand and Supply. $\sum_{i \in M} x_i = \sum_{t=1}^{T} y_t \mathbf{e}^t$.

Notice that since each x_i is a T-dimensional vector and each y_t is a scalar, we need to multiply y_t by $\mathbf{e}^t$. We abbreviate *competitive equilibrium* as *c.e.*

The existence of a c.e. in (M, N) is proved in Kaneko [2] and Kaneko-Yamamoto [5].

Theorem 2.1 (*Existence*). *There exists a c.e.* (p, x, y) *in* (M, N).

In Shapley-Shubik's model without the quasi-liniarity assumption for buyers and without (2.1) for sellers, the core coincides with the set of competitive allocations. In general, the latter is included in the core in the GAM model (M, N). The converse does not necessarily hold. Kaneko [2] gave a sufficient condition for the equivalence between the core and the set of competitive allocations. It is the condition that for each seller $j \in N$, there is another seller j' who is the same type as j (Theorem 10, p. 227).

We denote the *set of all c.e.'s* in (M, N) by $\mathbb{C}$. We say that a pair (x, y) is a *competitive allocation* iff $(p, x, y) \in \mathbb{C}$ for some $p \in \mathbb{R}^T_+$. Let $\mathbb{A}_C$ be the *set of all competitive allocations* in (M, N). We say that p is a *competitive price vector* iff $(p, x, y) \in \mathbb{C}$ for some $(x, y) \in \mathbb{A}_C$. Let $\mathbb{P}_C$ be the *set of all competitive price vectors* in (M, N).

Various authors studied the structure of the set of c.e.'s in assignment markets with and without quasi-linearity for buyers (e.g. Shapley-Shubik [10], Mishra-Talman [8], Miyake [9]). It is known in their models that the set of competitive price vectors has a lattice structure. This guarantees the existence of the maximum and the minimum competitive price vectors. Our GAM model also holds for the set of competitive price vectors having the lattice structure. From this, we have the maximum and minimum price vectors.

3 The Structure of the Competitive Equilibria

In Section 3.1, we give two theorems. The first theorem states that for each good t, the multiplicity of competitive prices implies a unique equilibrium quantity. The second is about the size of the set of competitive prices and allocations. By both theorems, we can evaluate separately the c.e.'s for each indivisible good. As an application of the theorems, we consider, in Section 3.2, the convergence of competitive prices for large assignment markets. The proofs of the first two theorems will be given in Section 3.3.

3.1 The size of the competitive equilibria

For each $t = 1, \ldots, T$, we denote the sizes of the sets of the t-th competitive prices and t-th competitive allocations by

$$\begin{aligned} \delta_t(\mathbb{P}_C) \quad &: \quad = \max\{|p_t - p'_t| : p, p' \in \mathbb{P}_C\}; \\ \delta_t(\mathbb{A}_C) \quad &: \quad = \max\{|y_t - y'_t| : (x, y), (x', y') \in \mathbb{A}_C\}. \end{aligned}$$

When $\delta_t(\mathbb{P}_C) > 0$, there are at least two different competitive prices for good t, and when $\delta_t(\mathbb{P}_C) = 0$, good t has unique competitive price. The cases of $\delta_t(\mathbb{A}_C) > 0$ and $\delta_t(\mathbb{A}_C) = 0$ are similarly interpreted.

The first theorem is about the possible cases of $\delta_t(\mathbb{P}_C)$ and $\delta_t(\mathbb{A}_C)$. The proof will be given in Section 3.3.

Theorem 3.1 (*Non-simultaneous Multiplicity for Competitive Equilibria*). *For each* $t = 1, \ldots, T$, *either* (1), (2) *or* (3) *holds:*

(1): $\delta_t(\mathbb{P}_C) > 0$ *and* $\delta_t(\mathbb{A}_C) = 0$.

(2): $\delta_t(\mathbb{P}_C) = 0$ *and* $\delta_t(\mathbb{A}_C) > 0$.

(3): $\delta_t(\mathbb{P}_C) = 0$ *and* $\delta_t(\mathbb{A}_C) = 0$.

The assertion of the theorem is equivalent to that $\delta_t(\mathbb{P}_C) > 0$ implies $\delta_t(\mathbb{A}_C) = 0$, ($\delta_t(\mathbb{A}_C) > 0$ implies $\delta_t(\mathbb{P}_C) = 0$), which was adopted in Section 1. Nevertheless, this does not exclude (3): $\delta_t(\mathbb{P}_C) = 0$ and $\delta_t(\mathbb{A}_C) = 0$.

In an assignment market without commodity differentiation, i.e., $T = 1$, the above result of Non-simultaneous Multiplicity is straightforward, which is seen from Figure 1. Theorem 3.1 states that even if we allow commodity differentiation, i.e., $T > 1$, Non-simultaneous Multiplicity holds separately for each good.

Although (1) and (2) of Theorem 3.1 are dual, we would expect the case (1) as more likely than (2). Indeed, (2) is special in that at the equilibrium quantity, the marginal cost coincides with marginal demand for good t.

Theorem 3.1 enables us to evaluate the size of the set $\mathbb{P}_C$ and $\mathbb{A}_C$ for each good t. When $\delta_t(\mathbb{P}_C) > 0$, the equilibrium quantity for good t is uniquely determined, and is denoted by y_t^*. When $\delta_t(\mathbb{A}_C) > 0$, the competitive price for good t is uniquely determined, and is denoted by p_t^*. The proof will be given in Section 3.3.

Theorem 3.2 (*The Size of the Set of Competitive Equilibria*). *For each* $t = 1, \ldots, T$,

(1): $\delta_t(\mathbb{P}_C) > 0$ *implies* $mc_t(y_t^* - 1) \leq p_t \leq mc_t(y_t^*)$ *for all* $p \in \mathbb{P}_C$.

(2): $\delta_t(\mathbb{A}_C) > 0$ *implies* $\delta_t(\mathbb{A}_C) \leq |\{y_t \in \mathbb{Z}_+ : p_t^* = mc_t(y_t)\}|$ *and* $p_t^* = mc_t(\widehat{y}_t)$, where $\widehat{y}_t := \min\{y_t \in \mathbb{Z}_+ : (x, y) \in \mathbb{A}_C\}$[1].

[1] $|X|$ is the cardinality of the set X.

Assertion (1) states that if there are multiple competitive prices for good t, then all the competitive price of good t are bounded by two marginal costs $mc_t(y_t^* - 1)$ and $mc_t(y_t^*)$. (2) states that if there are multiple equilibrium quantities for good t, then $\delta_t(\mathbb{A}_C)$ is restricted by the condition of seller t's marginal costs. Although (2) allows multiple equilibrium quantities for good t, the magnitude of multiplicity is expected rather small. For example, if the cost function c_t is strictly convex, then (2) implies $\delta_t(\mathbb{A}_C) \leq 1$. The additional $p_t^* = mc_t(\widehat{y}_t)$ means that the competitive price of good t is the marginal cost $mc_t(\widehat{y}_t)$. In sum, even if competitive prices or quantities are multiple, they are not distantly located.

As an application of Theorems 3.1 and 3.2, we will consider a convergence of the set of competitive prices for a large GAM. For this application, here we restate Theorem 3.2.(1). Let $p_t^{\max} := \max\{p_t : p \in \mathbb{P}_C\}$ and $p_t^{\min} := \min\{p_t : p \in \mathbb{P}_C\}$[2]. Then by the definition of $\delta_t(\mathbb{P}_C)$, we have $\delta_t(\mathbb{P}_C) = p_t^{\max} - p_t^{\min}$. Then, Theorem 3.2.(1) is restated as follows:

$$\text{If } \delta_t(\mathbb{P}_C) > 0, \text{ then } \delta_t(\mathbb{P}_C) = p_t^{\max} - p_t^{\min} \leq mc_t(y_t^*) - mc_t(y_t^* - 1). \tag{3.1}$$

3.2 Shrink of competitive prices

We start with the following passage from Shapley-Shubik [10] (pp. 127-128):

> "If the number of traders is increased, on both sides of the market, in such a way that their valuations for the products brought to market more and more diverse (but remain bounded in a suitable sense), then the core (*the set of competitive price vectors* in the present context) will tend to shrink in size."

They observe that this shrinkage could be easily obtained in homogeneous goods case. But they continue

> "In the more general model, however, the increasing dimensionality of the solution and the space in which it is defined makes a precise discussion of the shrinkage phenomenon more difficult."

Here, we analyze their observation by using Theorems 3.1 and 3.2.

Formally we keep the assumption, (2.1), that there is only one seller for each good t. However, it would be easier to interpret the following argument without (2.1). The elimination of (2.1) will be discussed in Section 4.

We would like to consider a large economy where the numbers of buyers and sellers are large. We formally consider a sequence $\{(M^\nu, N^\nu)\}_{\nu=1}^{+\infty}$ of GAM's with $|M^\nu| \to +\infty$ as $\nu \to +\infty$. We would like to assume that (M^ν, N^ν) is densely populated with buyers and sellers for large ν. This idea is formulated as follows. Let $\mathbb{P}_C^\nu$ be the set of all competitive price vectors in (M^ν, N^ν). Let $t = 1, \ldots, T$.

[2] Since the set of competitive price vector $\mathbb{P}_C$ is a compact set, these maximum and minimum are well defined.

Condition D_t (*Denseness of Marginal costs*). There are some positive constants α_t and β_t ($\alpha_t < \beta_t$) such that for any ν, (M^ν, N^ν) satisfies

(1): $\alpha_t \leq mc_t^\nu(y_t) \leq \beta_t$ for all $y_t \leq |M^\nu|$.

(2): $\max_{1 \leq y_t \leq |M^\nu|}[mc_t^\nu(y_t) - mc_t^\nu(y_t - 1)] \leq (\beta_t - \alpha_t)/|M^\nu|$.

The first condition states that the relevant marginal costs are in the same interval, though the size of (M^ν, N^ν) becomes large. The second condition states that the marginal costs are densely distributed.

Theorem 3.3 (*Shrink of Competitive Prices*)[3]. *Let* $\{(M^\nu, N^\nu)\}_{\nu=1}^{+\infty}$ *be a sequence of GAM. Suppose that* $\{(M^\nu, N^\nu)\}_{\nu=1}^{+\infty}$ *satisfies Condition* D_t. *Then* $\delta_t(\mathbb{P}_C^\nu) \to 0$ *as* $\nu \to \infty$.

Proof. Let ν be an arbitrary natural number. Let $\delta_t(\mathbb{P}_C^\nu) > 0$ and $y_t^{\nu*}$ be an (unique) equilibrium quantity of good t in (M^ν, N^ν). By (3.1) and $y_t^{\nu*} \leq |M^\nu|$, we have $\delta_t(\mathbb{P}_C^\nu) \leq mc_t(y_t^{\nu*}) - mc_t(y_t^{\nu*} - 1) \leq \max_{1 \leq y_t \leq |M^\nu|}[mc_t^\nu(y_t) - mc_t^\nu(y_t - 1)]$. This inequality and Condition D_t implies $\delta_t(\mathbb{P}_C^\nu) \leq (\beta_t - \alpha_t)/|M^\nu|$ and the right hand side of the inequality tends to zero as $\nu \to \infty$. Therefore, we obtain the assertion of the theorem. □

Theorem 3.3 states the convergence result for a given good t. When we assume Condition D_t for $t = 1, ..., T$, we have the convergence result for all the goods.

Theorem 3.3 is an extension of the result for the "dense" Böhm-Bawerk's market mentioned by Shapley-Shubik [10]. As quoted above, they gave a caution about higher dimensionality for heterogeneous goods. However, Theorem 3.1 guarantees our separate treatment of each indivisible good.

In fact, a much weaker condition than D_t is enough to obtain the shrinkage result: For any $\varepsilon > 0$, there is some ν_0 such that $mc_t^\nu(y_t^{\nu*}) - mc_t^\nu(y_t^{\nu*} - 1) < \varepsilon$ for all $\nu \geq \nu_0$, where $y_t^{\nu*}$ is an equilibrium quantity for good t in (M^ν, N^ν). Using (3.1), we obtain the shrinkage result. To avoid the reference to equilibria and to have a clear picture of a large market, we formulated Condition D_t and Theorem 3.3.

3.3 Proofs of Theorems 3.1 and 3.2

Here, we give proofs of Theorems 3.1 and 3.2. We start with the following lemmas.

Lemma 3.1. *Let* $(p, x, y), (p', x', y') \in \mathbb{C}$ *and* $t = 1, \ldots, T$. *Then* $p_t < p'_t$ *implies* $y_t \leq y'_t$.

Proof. We have $p_t y_t - c_t(y_t) \geq p_t y'_t - c_t(y'_t)$ and $p'_t y'_t - c_t(y'_t) \geq p'_t y_t - c_t(y_t)$ by seller t's profit maximization condition. By these inequations, we have $p_t y_t + p'_t y'_t \geq p_t y'_t + p'_t y_t$. Hence we obtain $y_t(p_t - p'_t) \geq y'_t(p_t - p'_t)$. This inequality and $p_t < p'_t$ imply $y_t \leq y'_t$. □

Lemma 3.2. *Let* $(p, x, y), (p', x', y') \in \mathbb{C}$ *and* $k, l = 1, \ldots, T$ *with* $k \neq l$. *Suppose* $p_t \geq p'_t$ *and* $p_u < p'_u$. *Then there is no* $i \in M$ *such that* $x_i = \mathbf{e}^t$ *and* $x'_i = \mathbf{e}^u$.

Proof. Let $p_t \geq p'_t$. We suppose that $x_i = \mathbf{e}^t$ and $x'_i = \mathbf{e}^u$ for some $i \in M$. It suffices to show $p_u \geq p'_u$. By utility maximization for i, we have

$$u_i(\mathbf{e}^t, I_i - p_t) \geq u_i(\mathbf{e}^u, I_i - p_u) \text{ and } u_i(\mathbf{e}^u, I_i - p'_u) \geq u_i(\mathbf{e}^t, I_i - p'_t). \quad (3.2)$$

[3] I thank Professor Kaneko for suggesting this specific formulation.

Since $p_t \geq p'_t$,we have, by Assumption A1, $u_i(\mathbf{e}^t, I_i - p'_t) \geq u_i(\mathbf{e}^t, I_i - p_t)$. This together with the the first inequality of (3.2) implies $u_i(\mathbf{e}^t, I_i - p'_t) \geq u_i(\mathbf{e}^u, I_i - p_u)$. Also the second inequality implies $u_i(\mathbf{e}^u, I_i - p'_u) \geq u_i(\mathbf{e}^u, I_i - p_u)$. By Assumption A1, we have $p_u \geq p'_u$. □

We have the following lemma by Lemmas 3.1 and 3.2.

Lemma 3.3. *Let* $(p, x, y), (p', x', y') \in \mathbb{C}$ *and* $t = 1, \ldots, T$. *Then* $p_t \neq p'_t$ *implies* $y_t = y'_t$.

Proof. Suppose $p_t < p'_t$. We show $y_t = y'_t$. Let

$$K = \{k : 1 \leq k \leq T,\ p_k < p'_k\},\ L = \{1, ..., T\} - K.$$

It follows from Lemma 3.1 that $y_k \leq y'_k$ for all $k \in K$. Hence, $\sum_{k \in K} y_k \leq \sum_{k \in K} y'_k$. If the converse of this inequality holds, then $y_k = y'_k$ should be the case for all $k \in K$. Hence, it suffices to show $\sum_{k \in K} y_k \geq \sum_{k \in K} y'_k$.

Now, let

$$M(K) = \{i \in M : x_i = \mathbf{e}^k \text{ for some } k \in K\};$$
$$M(L) = \{i \in M : x_i = \mathbf{e}^l \text{ for some } l \in L\};$$
$$M'(K) = \{i \in M : x'_i = \mathbf{e}^k \text{ for some } k \in K\}.$$

Then $\{M(K), M(L), \{i \in M : x_i = \mathbf{e}^0\}\}$ is a partition of M. Now, let us show $M(K) \supseteq M'(K)$. By Lemma 3.2, for any $l \in L$ and $k \in K$, there is no $i \in M$ such that $x_i = \mathbf{e}^l$ and $x'_i = \mathbf{e}^k$, i.e., $M(L) \cap M'(K) = \emptyset$. Also by Assumption A1 and $p_k < p'_k$ for any $k \in K$, there is no $i \in M$ such that $x_i = \mathbf{e}^0$ and $x'_i = \mathbf{e}^k$, i.e., $\{i \in M : x_i = \mathbf{e}^0\} \cap M'(K) = \emptyset$. Since $M(K) \cup M(L) \cup \{i \in M : x_i = \mathbf{e}^0\} = M$ is a partition of M, we have $M(K) \supseteq M'(K)$.

Since the balance of total demand and supply in (p, x, y) and (p', x', y'), we have $|M(K)| = \sum_{k \in K} y_k$ and $|M'(K)| = \sum_{k \in K} y'_k$. By the inclusion result, we have $\sum_{k \in K} y_k = |M(K)| \geq |M'(K)| = \sum_{k \in K} y'_k$. □

Now we are in a state to prove Theorem 3.1.

Proof of Theorem 3.1. We prove the following equivalent assertion: $\delta_t(\mathbb{P}_C) > 0$ implies $\delta_t(\mathbb{A}_C) = 0$. Suppose $\delta_t(\mathbb{P}_C) = p_t^{\max} - p_t^{\min} > 0$. Let $(p^1, x^1, y^1), (p^2, x^2, y^2) \in \mathbb{C}$ with $p_t^1 = p_t^{\max}$ and $p_t^2 = p_t^{\min}$. By Lemma 3.3, we have $y_t^1 = y_t^2$. Let $(p, x, y) \in \mathbb{C}$ with $p_t^{\max} > p_t > p_t^{\min}$. Again, by Lemma 3.3, $y_t = y_t^1 = y_t^2$. Thus, an equilibrium quantity of good t is unique, i.e., $\delta_t(\mathbb{A}_C) = 0$. □

We next prove Theorem 3.2 by using Theorem 3.1.

Proof of Theorem 3.2.(1): Since $\delta_t(\mathbb{A}_C) = 0$ by Theorem 3.1.(2), an equilibrium quantity of good t is unique. We denote it by y_t^*. Let $p \in \mathbb{P}_C$. Then we have $mc_t(y_t^* - 1) \leq p_t \leq mc_t(y_t^*)$ by seller t's profit maximization condition.

(2): Since $\delta_t(\mathbb{P}_C) = 0$ by Theorem 3.1.(2), a competitive price of good t is unique. We denote it by p_t^*. Suppose, on the contrary, $\delta_t(\mathbb{A}_C) > |\{y_t \in \mathbb{Z}_+ : mc_t(y_t) = p_t^*\}|$. Let $n = |\{y_t \in \mathbb{Z}_+ : mc_t(y_t) = p_t^*\}|$. Let $(x, y), (x', y') \in \mathbb{A}_C$ with $y'_t + n + 1 \leq y_t$. Then we have $mc_t(y_t - 1) \leq p_t^* \leq mc_t(y'_t)$ by seller t's profit maximization condition.

This and Assumption B2 imply $p_t^* = mc_t(y_t') = \cdots = mc_t(y_t' + n)$. Thus, we obtain $|\{y_t \in \mathbb{Z}_+ : mc_t(y_t) = p_t^*\}| \geq n + 1$, which contradicts the definition of n.

We next prove $p_t^* = mc_t(\widehat{y}_t)$. Let $\widehat{y}_t = \min\{y_t \in \mathbb{Z}_+ : (x, y) \in \mathbb{A}_C\}$. Then we have $p_t^* \leq mc_t(\widehat{y}_t)$ by seller t's profit maximization condition. On the other hand, let $(x^1, y^1) \in \mathbb{A}_C$ with $y_t^1 \geq \widehat{y}_t + 1$. Similarly, we have $mc_t(y_t^1 - 1) \leq p_t^*$. This and Assumption B2 imply $mc_t(\widehat{y}_t) \leq p_t^*$. □

4 Aggregation of Sellers

We have assumed that the set N_t of sellers is represented by the aggregated seller, i.e., (2.1): $N_t = \{t\}$ for each $t = 1, \ldots, T$. As far as competitive equilibrium is concerned, this assumption does not loose any generality. Indeed, let (M, N) be the original assignment market without (2.1), and let $N^o = \{1, ..., T\}$ be the set of "aggregated" sellers. Now, we show how the set N_t of sellers for good t is aggregated into one seller $\{t\}$. Our problem is to define the cost function $\overline{c}_t$ of each aggregated seller $t \in N^o$ preserving the structure of c.e.'s in (M, N).

The following theorem explains the equivalence of c.e.'s between (M, N) and (M, N^o).

Theorem 4.1 (*Aggregation of Sellers*)[4]. *Let (M, N) be an assignment market without requiring* (2.1). *There are cost functions $\overline{c}_t : \mathbb{Z}_+ \to \mathbb{R}_+$ for $t \in N^o$ such that,*

(1): *if (p, x, y) is a c.e. in (M, N), then there is a $z \in \mathbb{Z}_+^T$ such that (p, x, z) is a c.e. in (M, N^o).*

(2): *if (p, x, z) is a c.e. in (M, N^o), then there is a $y \in \mathbb{Z}_+^n$ such that (p, x, y) is a c.e. in (M, N).*

As stated, the set N_t for good t is aggregated into $\{t\}$: The essential part is to define the aggregated cost function for seller t, which is the cost function $\overline{c}_t : \mathbb{Z}_+ \to \mathbb{R}_+$ stated in the theorem. Once this $\overline{c}_t$ is appropriately defined, (1) the supplies $\{y_j\}_{j \in N_t}$ are aggregated into $z_t = \sum_{j \in N_t} y_j$ and this aggregation makes a c.e. in (M, N^o). The second assertion (2) needs a more careful consideration, since we should consider how the aggregated z_t will be divided into the sellers in N_t. This "disaggregation" needs various definitions and lemmas: the aggregated cost function $\overline{c}_t : \mathbb{Z}_+ \to \mathbb{R}_+$ is defined so that the correspondence between the supplies $\{y_j\}_{j \in N_t}$ in (M, N) and the supply z_t in (M, N^o) is preserved in a one-one way with $z_t = \sum_{j \in N_t} y_j$.

Now, let us fix an arbitrary $t \in N^o$, and recall that $mc_j(y_j)$ is the marginal cost of c_j at y_j, and $\overline{mc}_t(z_t)$ is the marginal cost of $\overline{c}_t$ at z_t. We stipulate $mc_j(-1) = 0$ for $j \in N_t$ and $\overline{mc}_t(-1) = 0$. We focus on the set of the sequences of possible marginal costs $\{\{mc_j(y_j) : y_j \in \mathbb{Z}_+\} : j \in N_t\}$. Now, we will combine this set into one sequence $\{\overline{mc}_t(z_t) : z_t \in \mathbb{Z}_+\}$. Once this is done, we can obtain the *aggregated cost function* $\overline{c}_t$ by:

$$\overline{c}_t(z_t) = \sum_{k \leq z_t - 1} \overline{mc}_t(k) \text{ for all } z_t \in \mathbb{Z}_+. \quad (4.1)$$

[4] This theorem is due to Professor Kaneko.

Hence, our problem is to construct $\{\overline{mc}_t(z_t) : z_t \in \mathbb{Z}_+\}$ from $\{\{mc_j(y_j) : y_j \in \mathbb{Z}_+\} : j \in N_t\}$.

The idea of construction is as follows. Recall that each seller $j \in N_t$ has the convex cost function $c_j : \mathbb{Z}_+ \to \mathbb{R}_+$ with $c_j(0) = 0$ and $c_j(1) > 0$. To define $\{\overline{mc}_t(z_t) : z_t \in \mathbb{Z}_+\}$, we order all the components of $\{mc_j(y_j) : y_j \in \mathbb{Z}_+\}, j \in N_t$ in the ascending order: In the start, we choose the smallest one in $\{mc_j(0) : j \in N_t\}$. If we choose $mc_{j_0}(0)$ here, next we choose the smallest one from $\{mc_j(0) : j \in N_t \text{ and } j \neq j_0\} \cup \{mc_{j_0}(1)\}$, and repeat this process. To formalize this idea rigorously, we need functions $\varphi : \mathbb{Z}_+^{|N_t|} \to 2^{N_t}$ and $\psi : \mathbb{Z}_+ \to \mathbb{Z}_+^{|N_t|}$[5].

We define $\varphi : \mathbb{Z}_+^{|N_t|} \to 2^{N_t}$ by

$$\varphi(y^{N_t}) = \{j \in N_t : mc_j(y_j) \leq mc_{j'}(y_{j'}) \text{ for all } j' \in N_t\}. \tag{4.2}$$

The value $\varphi(y^{N_t})$ is the set of sellers who have the smallest marginal costs at their y_j's, which is nonempty. For example, when $y^{N_t} = (0, ..., 0)$, $\varphi(y^{N_t})$ is the set of sellers with the smallest $mc_j(0)$.

Now, we define the function $\psi : \mathbb{Z}_+ \to \mathbb{Z}_+^{|N_t|}$ inductively as follows: $\psi(0) = (0, ..., 0)$ and for any $z_t \geq 1$, we suppose that $\psi(z_t) = \{\psi_j(z_t)\}_{j \in N_t} = y^{N_t}(z_t)$ is already defined. Then, we define $\psi(z_t + 1)$ as follows:

$$\psi_j(z_t + 1) = \begin{cases} \psi_j(z_t) + 1 & \text{if } j \in \varphi(y^{N_t}(z_t)) \text{ and } j \leq j' \text{ for } j' \in \varphi(y^{N_t}(z_t)), \\ \psi_j(z_t) & \text{otherwise.} \end{cases} \tag{4.3}$$

The vector $\psi(z_t + 1)$ gives an additional unit to the "first" seller having the smallest marginal cost. Since $\varphi(y^{N_t}(z_t)) \neq \emptyset$, $\psi(z_t + 1)$ is uniquely determined from $\psi(z_t)$ by (4.3).

The marginal cost $\overline{mc}_t(z_t)$ is defined to be the smallest marginal cost in $\{mc_j(y_j) : j \in N_t\}$, where $y^{N_t} = \psi(z_t)$. Using φ and ψ, it is expressed as

$$\overline{mc}_t(z_t) = mc_j(\psi_j(z_t)), \text{ where } j \in \varphi(\psi(z_t)). \tag{4.4}$$

From the above definitions of φ and ψ, we have the following assertions: In each step from $\psi(z_t)$ to $\psi(z_t+1)$, one unit is added to $y_j = \psi_j(z_t)$ for some $j \in N_t$. This observation implies (1) and (2) of Lemma 4.1, and also (3) is obtained from (2) and (4.4).

Lemma 4.1. **(1)**: *$z_t = \sum_{j \in N_t} \psi_j(z_t)$ for all $z_t \in \mathbb{Z}_+$.*

(2): *$\psi(z_t)$ is (weakly) increasing, i.e., $\psi_j(z_t) \leq \psi_j(z_t + 1)$ for all $z_t \in \mathbb{Z}_+$ and $j \in N_t$;*

(3): *$\{\overline{mc}_t(z_t) : z_t \in \mathbb{Z}_+\}$ is (weakly) increasing, i.e., $\overline{mc}_t(z_t) \leq \overline{mc}_t(z_t + 1)$ for all $z_t \in \mathbb{Z}_+$.*

The first assertion states that the function ψ correctly disaggregates z_t into $y^{N_t} = \psi(z_t)$. The third assertion guarantees the convexity of the aggregated cost function $\overline{c}_t$.

It remains to show that the profit maximization with $\overline{c}_t$ is equivalent that with $\{c_j\}_{j \in N_t}$. We give the following lemma.

[5] 2^X is the power set of X.

Lemma 4.2. *Let* $z_t \in \mathbb{Z}_+$. *Then*

(1): $\overline{mc}_t(z_t) \leq mc_j(\psi_j(z_t))$ *for all* $j \in N_t$;

(2): $mc_j(\psi_j(z_t) - 1) \leq \overline{mc}_t(z_t)$ *for all* $j \in N_t$.

Proof. (1): By (4.4) and (4.2), there is $j_1 \in \varphi(\psi(z_t))$ such that $\overline{mc}_t(z_t) = mc_{j_1}(\psi_{j_1}(z_t))$ and $mc_{j_1}(\psi_{j_1}(z_t)) \leq mc_j(\psi_j(z_t))$ for all $j \in N_t$. Thus we have $\overline{mc}_t(z_t) \leq mc_j(\psi_j(z_t))$ for all $j \in N_t$.

(2): We prove this by induction over $z_t \in \mathbb{Z}_+$. Let $z_t = 0$. Since $\psi(0) = (0, \ldots, 0)$, we have $mc_j(\psi_j(0) - 1) = mc_j(-1) = 0 \leq \overline{mc}_t(0)$ for all $j \in N_t$.

Suppose the induction hypothesis that (2) holds for $z_t \in \mathbb{Z}_+$. we now show $mc_j(\psi_j(z_t+1) - 1) \leq \overline{mc}_t(z_t + 1)$ for all $j \in N_t$. By (4.4), there is $j_1 \in \varphi(\psi(z_t + 1))$ such that $j_1 = \min \varphi(\psi(z_t+1))$ and $\overline{mc}_t(z_t+1) = mc_{j_1}(\psi_{j_1}(z_t+1))$. By the convexity of c_{j_1} (Assumption B2), $mc_{j_1}(\psi_{j_1}(z_t + 1) - 1) \leq mc_{j_1}(\psi_{j_1}(z_t + 1)) = \overline{mc}_t(z_t + 1)$. Consider any $j \in N_t \backslash \{j_1\}$. Since $\psi_j(z_t + 1) = \psi_j(z_t)$ by (4.3), we have $mc_j(\psi_j(z_t + 1) - 1) = mc_j(\psi_j(z_t) - 1)$. By the induction hypothesis and Lemma 4.1.(3), we have $mc_j(\psi_j(z_t + 1) - 1) \leq \overline{mc}_t(z_t) \leq \overline{mc}_t(z_t + 1)$. Thus, we have shown that $mc_j(\psi_j(z_t + 1) - 1) \leq \overline{mc}_t(z_t + 1)$ for all $j \in N_t$. □

The following lemma is obtained from Lemma 4.2.

Lemma 4.3. *Let* $z_t \in \mathbb{Z}_+$, $\{y_j\}_{j \in N_t} = \psi(z_t)$ *and* $p_t \in \mathbb{R}_+$. *Then*

$\overline{mc}_t(z_t - 1) \leq p_t \leq \overline{mc}_t(z_t)$ *if and only if* $mc_j(y_j - 1) \leq p_t \leq mc_j(y_j)$ *for all* $j \in N_t$.

Proof. (*Only if*): Suppose $\overline{mc}_t(z_t - 1) \leq p_t \leq \overline{mc}_t(z_t)$. We first show $p_t \leq mc_j(y_j)$ for all $j \in N_t$. Lemma 4.2.(1) implies $\overline{mc}_t(z_t) \leq mc_j(\psi_j(z_t)) = mc_j(y_j)$ for all $j \in N_t$. Thus we have $p_t \leq mc_j(y_j)$ for all $j \in N_t$.

Next we show $mc_j(y_j - 1) \leq p_t$ for all $j \in N_t$. By (4.4), there is $j_1 \in \varphi(\psi(z_t - 1))$ such that $j_1 = \min \varphi(\psi(z_t - 1))$ and $\overline{mc}_t(z_t - 1) = mc_{j_1}(\psi_{j_1}(z_t - 1))$. By (4.3), $\psi_{j_1}(z_t) = \psi_{j_1}(z_t - 1) + 1$ and $\psi_j(z_t) = \psi_j(z_t - 1)$ for $j \in N_t \backslash \{j_1\}$. For j_1, $mc_{j_1}(y_{j_1} - 1) = mc_{j_1}(\psi_{j_1}(z_t) - 1) = mc_{j_1}(\psi_{j_1}(z_t - 1)) = \overline{mc}_t(z_t - 1)$. On the other hand, for $j \in N_t \backslash \{j_1\}$, $mc_j(y_j - 1) = mc_j(\psi_j(z_t) - 1) = mc_j(\psi_j(z_t - 1) - 1) \leq \overline{mc}_t(z_t - 1)$ by Lemma 4.2.(2). Thus, we have $mc_j(y_j - 1) \leq p_t$ for all $j \in N_t$.

(*If*): Suppose $mc_j(y_j - 1) \leq p_t \leq mc_j(y_j)$ for all $j \in N_t$. We first show $p_t \leq \overline{mc}_t(z_t)$. By (4.4), $\overline{mc}_t(z_t) = mc_j(\psi_j(z_t)) = mc_j(y_j)$ for some $j \in N_t$. Thus, we have $p_t \leq \overline{mc}_t(z_t)$.

Next we show $\overline{mc}_t(z_t - 1) \leq p_t$. By (4.4), $\overline{mc}_t(z_t - 1) = mc_j(\psi_j(z_t) - 1) = mc_j(y_j - 1)$ for some $j \in N_t$. Thus, we have $\overline{mc}_t(z_t - 1) \leq p_t$. □

It follows from Lemma 4.3 that for given price p_t, the aggregated seller t satisfies profit maximization with production z_t if and only if each seller $j \in N_t$ satisfies profit maximization with production $y_j = \psi_j(z_t)$.

Here, we return to the proof of Theorem 4.1. Suppose that (p, x, y) is a c.e. in (M, N). By the above observation, for each $t \in N^o$, profit maximization holds with production $z_t = \sum_{j \in N_t} y_j$. Since $\sum_{i \in M} x_i = \sum_{t=1}^{T} \sum_{j \in N_t} y_j \mathbf{e}^t = \sum_{t=1}^{T} z_t \mathbf{e}^t$, the total demand coincides with the total supply. Thus, (p, x, z) where $z = (z_1, \ldots, z_T)$ is a competitive equilibrium in (M, N^o).

Conversely, suppose that (p, x, z) is a c.e. in (M, N^o). As observed above, for each $j \in N_t$, $t = 1, \ldots, T$, profit maximization holds with $\psi_j(z_t)$. Since $\sum_{i \in M} x_i = \sum_{t=1}^{T} z_t \mathbf{e}^t = \sum_{t=1}^{T} \sum_{j \in N_t} \psi_j(z_t) \mathbf{e}^t$ by Lemma 4.1.(1), the total demand coincides with the total supply. Thus, (p, x, y) where $\{y_j\}_{j \in N_t} = \psi(z_t)$ for each $t = 1, \ldots, T$ is a competitive equilibrium in (M, N).

5 Conclusions

We studied the structure of the set of c.e.'s in the GAM model. It is Theorem 3.1 that if there are multiple competitive prices for good t, then the t-th equilibrium quantity is uniquely determined. We should emphasize that this result enables us to study the relationship between the competitive prices and quantities for each indivisible good t. Using this fact, we evaluated the entire sizes of the sets of the t-th competitive prices and quantities. As an application of these results, we presented the theorem on the shrink of competitive prices when the market becomes large and dense with sellers.

As stated in Section 2, the core of the GAM model often coincides with the set of competitive equilibria, in particular, for a large and dense market. Therefore, we also obtain the shrinkage result for the core.

We may apply the GAM model to rental housing markets and/or second-hand automobile markets. Those markets are typically thick in the sense that the numbers of sellers and buyers are large and moreover there are many similar sellers and buyers. Kaneko [3] and Kaneko-Ito-Osawa [4] adopted the GAM model for the analysis of rental housing markets, making some assumptions specific and special to their studies. By our third result, the competitive prices are restrictive; hence their studies may be done under more general assumptions.

Finally, we give a remark on the relation between our results and gross substitutability. Kelso-Crawford [6] first introduced the gross substitutes (GS) condition for producers' preferences to show the existence of competitive equilibria. Gul-Stacchetti [1] considered a GS condition for buyers under the assumption of quasi-linear utilities. It is relevant to our study that unit demand preferences satisfy their GS. Our results, in particular Lemma 3.2, may be interpreted as implying that our model is related to the GS condition, but this still remains an open problem.

References

[1] Gul, F., E, Stacchetti (1999), Walrasian Equilibrium with Gross Substitutes, *Journal of Economic Theory* 87, 95-124.

[2] Kaneko, M. (1982), The central assignment game and the assignment markets, *Journal of Mathematical Economics* 10, 205-232.

[3] Kaneko, M. (1983), Housing market with indivisibilities, *Journal of Urban Economics* 13, 22-50.

[4] Kaneko, M., T. Ito and Y. Osawa (2006), Duality in comparative statics in rental housing markets with indivisibilities, *Journal of Urban Economics* 59, 142-170.

[5] Kaneko, M. and Y. Yamamoto (1986), The existence and computation of competitive equilibria in markets with an indivisible commodity, *Journal of Economic Theory* 38, 118-136.

[6] Kelso, A.S. and V.P.Crawford (1982), Job matching, coalition formation, and gross substitutes, *Econometrica* 50, 1483-1504.

[7] Mas-Colell, A., M. Whinston and J. Green (1995), Microeconomic Theory, *Oxford University Press.*

[8] Mishra, D. and D. Talman (2010), Characterization of the Walrasian equilibria of the assignment model, *Journal of Mathematical Economics* 46, 6-20.

[9] Miyake, M. (1994), Comparative statics of assignment markets with general utilities, *Journal of Mathematical Economics* 23, 519-531.

[10] Shapley, L.S.and M. Shubik (1972), Assignment game I: The core, *International Journal of Game Theory* 1, 111-130.