PAPER

Structure Properties of Punctured Convolutional Codes and Their Applications

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SUMMARY This paper presents the generator polynomial matrices and the upper bound on the constraint length of punctured convolutional codes (PCCs), respectively. By virtue of these properties, we provide the puncturing realizations of the good known nonsystematic and systematic high rate CCs.
key words: *polynomial generator matrix, punctured convolutional codes, constraint length*

1. Introduction

Convolutional codes (CCs) are ones of the most powerful forward error correcting (FEC) codes, which are widely used in communication systems. Unfortunately, the use of these codes is primarily restricted to the low rate $(n, 1, K)$ CCs (the rate $R = \frac{1}{n}$, K is constraint length) or high rate, short constraint length CCs ($K \leq 7$). But, in many applications, their transmission rates must be high while each bandwidth is strictly limited, such as the wireless channel. For compromising the power and bandwidth efficiency, the high rate CCs are needed, whose decoding becomes complex.

The punctured CCs (PCCs) were suggested in 1979 [1] to make high rate codes from low rate ones simply. The punctured high rate CCs ($R = \frac{l}{nl-m}$) are produced by being periodically (nl bits) punctured (m bits) from $R = \frac{1}{n}$ low rate CCs (called as the original code). Some of PCCs were shown to be almost as good as the best known regular codes. For example, puncturing the initial code reduces its free distance, however, this distance of $R = \frac{l}{nl-m}$ PCCs is as large as that can be achieved with the ones of any $R = \frac{l}{nl-m}$ code. Thus, in this case no loss in minimum distance is caused by using a punctured code. Besides they have two other advantages:

- Simplifying the Viterbi decoder for high rate CCs. In the meantime, PCCs can be advantageously decoded by sequential decoding too [2].
- Being able to implement a multirate (or rate-compatible) CC encoder/decoder [3], which is very useful in multimedia communication systems.

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The basic steps being punctured from $R = \frac{1}{n}$ CC are shown in Fig. 1. Firstly, by the good $R = \frac{1}{n}$ CC encoder, the input data sequence is changed into the original coded data sequence. Then, the original coded data sequence is periodically (nl bits) punctured m bits, according to the map of deleting bits which indicates deleting bit positions.

From the point of minimum bit error probability, Yasuda et al. [4] have shown a set of good PCCs with different rates, which can be obtained from $R = \frac{1}{2}$ encoder. Lee [5] found new rate $R = \frac{l}{l+1}$ PCCs that minimize the required signal-to-noise ratio for a target BER of 10^{-9}. Kim [6] derived a group of good high rate systematic PCCs by analyzing their weight spectra and BEP simulation. However, though some algebraic properties have been found [2], [7], [8], no systematic construction method for good PCCs is known yet. This limits the exhaustive search for good PCCs. To give indications for guiding the search for good PCCs,

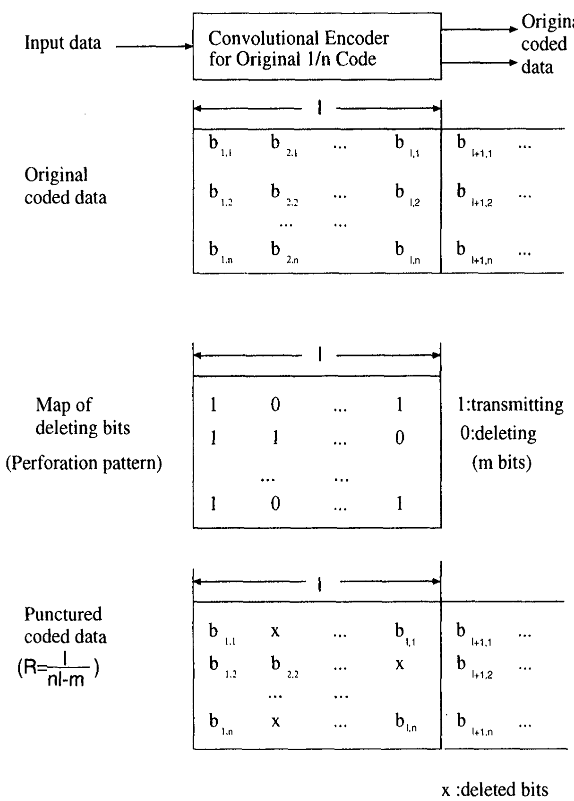

Fig. 1 The basic steps being punctured from $R = \frac{1}{n}$ CC.

this paper shows the polynomial generator matrix of the PCCs in Sect. 2 and the constraint length of PCCs in Sect. 3. By virtue of these properties, we give the puncturing realizations of good known nonsystematic high rate (n, l) CCs from nonsystematic $(n, 1)$ CCs and good known systematic high rate CCs from $(2, 1)$ systematic CCs in Sect. 4.

2. The Polynomial Generator Matrix of the Punctured CCs

The polynomial generator matrix is very important for the construction of CCs. In this subsection, we put forward Theorem 1 about the generator polynomial matrix $J(D)$ of the $R = \frac{l}{nl}$ CCs.

Theorem 1: Suppose the polynomial generator matrix of $R = \frac{1}{n}$ CC be:

$$G(D) = [G_1(D), G_2(D), \ldots, G_n(D)], \tag{1}$$

where D is the delay operator in the shift register. Then, the polynomial generator matrix $J(D)$ of $R = \frac{l}{nl}$ CC can be expressed by:

$$J(D) = \begin{pmatrix} J_{1,1}(D) & J_{1,2}(D) & \ldots & J_{1,n}(D) & \cdots & J_{1,(i-1)n+1}(D) & J_{1,(i-1)n+2}(D) & \ldots & J_{1,in}(D) & \cdots & J_{1,(l-1)n+1}(D) & J_{1,(l-1)n+2}(D) & \ldots & J_{1,ln}(D) \\ \vdots & \vdots & \ddots & \vdots & & \vdots & & \ddots & \vdots & & \vdots & & \ddots & \vdots \\ J_{j,1}(D) & J_{j,2}(D) & \ldots & J_{j,n}(D) & \cdots & J_{j,(i-1)n+1}(D) & J_{j,(i-1)n+2}(D) & \ldots & J_{j,in}(D) & \cdots & J_{j,(l-1)n+1}(D) & J_{j,(l-1)n+2}(D) & \ldots & J_{j,ln}(D) \\ \vdots & \vdots & \ddots & \vdots & & \vdots & & \ddots & \vdots & & \vdots & & \ddots & \vdots \\ J_{l,1}(D) & J_{l,2}(D) & \ldots & J_{l,n}(D) & \cdots & J_{l,(i-1)n+1}(D) & J_{l,(i-1)n+2}(D) & \ldots & J_{l,in}(D) & \cdots & J_{l,(l-1)n+1}(D) & J_{l,(l-1)n+2}(D) & \ldots & J_{l,ln}(D) \end{pmatrix} \tag{2}$$

where, for $i, j = 1, 2, \ldots, l$;

$$J_{j,(i-1)n+s}(D) = D^{\frac{j-i}{l}} G_{s,h}(D^{\frac{1}{l}}), \tag{3}$$

and, for $h = l + i - j \bmod l$, $s = 1, 2, \ldots, n$, $G_{s,h}(D)$ is construction part of $G_s(D)$, expressed by:

$$G_{s,0}(D) + G_{s,1}(D) + \cdots + G_{s,l-1}(D) = \sum_{t=0}^{\infty} (g_{s,0} + g_{s,1}D + \cdots + g_{s,l-1}D^{l-1})D^{tl}.$$

Proof: Refer to Appendix. □

The polynomial generator matrix $Q(D)$ of the $R = \frac{l}{nl-m}$ PCCs is derived by being punctured m columns from $J(D)$ in terms of the perforation matrix.

For example, let the original code be (3,1,7) CC, which has a generator polynomial matrix: $G(D) = [1 + D + D^2 + D^3 + D^5, 1 + D^2 + D^3 + D^4 + D^5 + D^6, 1 + D + D^3 + D^5]$.

From Theorem 1, the polynomial generator matrix $J(D)$ of (9, 3) CC is as follows:

$$\begin{pmatrix} 1+D & 1+D+D^2 & 1+D & 1 & D & 1 & 1+D & 1+D & D \\ D+D^2 & D+D^2 & D^2 & 1+D & 1+D+D^2 & 1+D & 1 & D & 1 \\ D & D^2 & D & D+D^2 & D+D^2 & D^2 & 1+D & 1+D+D^2 & 1+D \end{pmatrix}.$$

Let the corresponding perforation matrix be:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

then, the polynomial generator matrix $Q(D)$ of (4, 3) PCC is as follows:

$$\begin{pmatrix} 1+D & 1+D & 1 & 1+D \\ D+D^2 & D^2 & 1+D & D \\ D & D & D^2 & 1+D+D^2 \end{pmatrix}. \tag{4}$$

In the most practical cases, the original (2, 1) CCs are selected. Suppose the generator polynomial matrix of $R = \frac{1}{2}$ original CC is:

$$G(D) = [G_1(D), G_2(D)]. \tag{5}$$

By virtue of Theorem 1, the polynomial generator matrix $J(D)$ of $R = \frac{l}{2l}$ CC can be expressed by:

$$J(D) = \begin{pmatrix} J_{1,1}(D) & J_{1,2}(D) & \ldots & J_{1,2i-1}(D) & J_{1,2i}(D) & \ldots & J_{1,2l-1}(D) & J_{1,2l}(D) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J_{j,1}(D) & J_{j,2}(D) & \ldots & J_{j,2i-1}(D) & J_{j,2i}(D) & \ldots & J_{j,2l-1}(D) & J_{j,2l}(D) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ J_{l,1}(D) & J_{l,2}(D) & \ldots & J_{l,2i-1}(D) & J_{l,2i}(D) & \ldots & J_{l,2l-1}(D) & J_{l,2l}(D) \end{pmatrix} \tag{6}$$

where, for $i, j = 1, 2, \ldots, l$;

$$J_{j,2i-1}(D) = D^{\frac{j-i}{l}} G_{1,h}(D^{\frac{1}{l}}),$$
$$J_{j,2i}(D) = D^{\frac{j-i}{l}} G_{2,h}(D^{\frac{1}{l}}),$$

Table 1 The relationship between (2, 1) good original CCs and (4, 3) PCCs.

(2,1) good original CCs		(4,3) PCCs				P
K_1	$G_1(D)$ $G_2(D)$	$Q_{11}(D)$ $Q_{21}(D)$ $Q_{31}(D)$	$Q_{12}(D)$ $Q_{22}(D)$ $Q_{32}(D)$	$Q_{13}(D)$ $Q_{23}(D)$ $Q_{33}(D)$	$Q_{14}(D)$ $Q_{24}(D)$ $Q_{34}(D)$	
		1	1	1	1	101
3	$1+D^2$	D	D	1	0	110
	$1+D+D^2$	0	D	D	1	
		$1+D$	$1+D$	1	1	110
4	$1+D+D^3$	0	D	$1+D$	1	101
	$1+D+D^2+D^3$	D	D	0	1+D	
		$1+D$	1	$1+D$	0	101
5	$1+D^3+D^4$	0	D	1	D	110
	$1+D+D^2+D^4$	D^2	$D+D^2$	D	$1+D$	
		1	$1+D$	1	$1+D$	100
6	$1+D^2+D^4+D^5$	$D+D^2$	$D+D^2$	$1+D$	1	111
	$1+D+D^2+D^3+D^5$	D^2	D	$D+D^2$	$1+D$	
		$1+D+D^2$	$1+D+D^2$	0	1	110
7	$1+D^2+D^3+D^5+D^6$	$D+D^2$	D	$1+D+D^2$	1	101
	$1+D+D^2+D^3+D^6$	0	D	$D+D^2$	$1+D+D^2$	
		$1+D^2$	$1+D$	D^2	1	110
8	$1+D^2+D^5+D^6+D^7$	$D+D^2$	D	$1+D^2$	$1+D+D^2$	101
	$1+D+D^2+D^3+D^4+D^7$	D^3	$D+D^2+D^3$	$D+D^2$	$1+D$	
		$1+D$	$1+D$	D	$1+D^2$	111
9	$1+D^2+D^3+D^4+D^8$	$D+D^3$	$D+D^2+D^3$	$1+D$	D	100
	$1+D+D^2+D^3+D^5+D^7+D^8$	D^2	$D+D^3$	$D+D^3$	$1+D$	

and $h = l + i - j \bmod l$.

Similarly, the corresponding polynomial generator matrix $Q(D)$ of $R = \frac{l}{2l-m}$ is derived by being punctured m columns from $J(D)$ in terms of the perforation matrix. For example, Table 1 gives the polynomial generator matrices $Q(D)$ of $(4,3)$ PCCs, being punctured by perforation matrices P from good $(2,1,K_1)$ CC, where the constraint length K_1 of original $(2,1)$ CC is from 3 to 9. The choosing of original codes and perforation matrices are referred from [4].

3. The Constraint Length of the Punctured CCs

The constraint length is a very important parameter of CC. For a CC, the greater its constraint length is, the better its performance is. From this point, we induce Theorem 2.

Theorem 2: The constraint length of high rate CC $(nl-m, l)$ punctured from low rate $(n,1)$ CC is :

$$K_l \leq \lceil (K_1 - 1)/l \rceil + 1, \tag{7}$$

where K_1 is the constraint length of $(n,1)$ CC, and $\lceil x \rceil$ indicates the minimal integer which is larger than or equal to x.

Proof: Let M_1 and M_l be the highest dimension of the $G_s(D)(s = 1,2,\cdots,n)$ and $J_{ji}(D)(j = 1,\ldots,l; i = 1,\ldots,nl)$, respectively, so $M_1 = K_1 - 1$ and $M_l = K_l - 1$.
If $M_1 = tl$, then, $t = \frac{M_1}{l}$.
If $M_1 = tl + l_1 (1 \leq l_1 \leq l-1)$, then,
$t = \frac{M_1 - l_1}{l} = \frac{M_1 + l - l_1}{l} - 1 = \lceil M_1/l \rceil - 1$.
From Theorem 1, according to the generality of ensuring M_l as large as possible, we have the following conclusions:
If $M_1 = tl$, then, $M_l = \frac{tl}{l} = t = \frac{M_1}{l}$.
If $M_1 \neq tl$, then,
$M_l = \frac{j-i}{l} + \frac{tl+l+i-j}{l} = \frac{(t+1)l}{l} = t+1 = \lceil M_1/l \rceil$.
That is to say: $M_l = \lceil M_1/l \rceil$.
In general, we have the conclusion as follows:
$K_l \leq \lceil (K_1 - 1)/l \rceil + 1$. □

From Theorem 2, we have the upper bound of constraint length of high rate PCC $(nl-m, l)$, which is related to the constraint length K_1 of original $(n,1)$ CC and the periodical length l, but no relationship with n and m.

For example, the constraint length of $(4,3)$ high rate CC punctured from low rate $(3,1,7)$ CC is :

$$K_3 = \lceil (7-1)/3 \rceil + 1 = 3, \tag{8}$$

which is in good agreement with (4).

4. The Puncturing Realization of Good Known High Rate CCs

Up to now, many good PCCs are obtained [4]–[6] on the basis of one general constructing method, which includes two steps as follows:
Step 1 Selecting the best known $(n,1)$ CCs of a given constraint length as an original code.
Step 2 Determining the perforation matrices that will yield the best PCCs for different coding rates.

But, by this method, all good PCCs may not always correspond with the best high rate CCs. In order to produce the same PCCs as the good known high rate

Table 2 (4, 1) Original codes that yield the good known (4, 3) CCs.

	(4,1) Original CCs	(4,3) PCCs, i.e. good known (4,3) CCs		
K_1	$G_1(D)$ $G_2(D)$ $G_3(D)$ $G_4(D)$	$Q_{11}(D)$ $Q_{12}(D)$ $Q_{13}(D)$ $Q_{14}(D)$	$Q_{21}(D)$ $Q_{22}(D)$ $Q_{23}(D)$ $Q_{24}(D)$	$Q_{31}(D)$ $Q_{32}(D)$ $Q_{33}(D)$ $Q_{34}(D)$
7	1 $1+D+D^2+D^3$ $1+D^2+D^4+D^6$ $1+D+D^2+D^6$	1 1 1 1	0 $1+D$ D 1	0 D $1+D^2$ $1+D^2$
9	$1+D+D^2+D^3+D^5$ $1+D^2+D^5+D^7$ $1+D^3+D^6+D^7$ $1+D+D^2+D^4+D^5+D^8$	$1+D$ D^2 0 $1+D+D^2$	$D+D^2$ 1 D^2 $1+D$	D $D+D^2$ $1+D+D^2$ 1
11	$1+D^3+D^5+D^6+D^7+D^8$ $1+D^4+D^5+D^6+D^7$ $1+D+D^3+D^5+D^6+D^9$ $1+D+D^2+D^7+D^{10}$	$1+D+D^2$ $D+D^2$ D 1	D^2+D^3 $1+D^2$ 1 $1+D^2+D^3$	D^3 D^2 $1+D+D^2+D^3$ 1

CCs, we need select available original CCs and perforation matrices. In this section, we give the polynomial generator matrices of systematic and nonsystematic original CCs, which can produce the good known high rate CCs by the available perforation matrices.

4.1 The Puncturing Realization of Good Known Nonsystematic CCs

Suppose that the orthogonal perforation matrix [7] is used, where only one entry "1" is in each row, the puncturing bits $m = nl - n$, and the $R = \frac{l}{n}$ PCC is obtained. Then, the corresponding polynomial generator matrix $Q(D)$ can be expressed by:

$$Q(D) = \begin{pmatrix} Q_{1,1}(D) & Q_{1,2}(D) & \dots & Q_{1,n}(D) \\ Q_{2,1}(D) & Q_{2,2}(D) & \dots & Q_{2,n}(D) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{l,1}(D) & Q_{l,2}(D) & \dots & Q_{l,n}(D) \end{pmatrix}.$$

From Theorem 1, $Q_{j,s}(D)$ is one of n entries: $J_{j,(s-1)n+1}(D), J_{j,(s-1)n+2}(D), \dots, J_{j,sn}(D)$, which is only related to $G_s(D)$, for $s = 1, \dots, n$ and $j = 1, \dots, l$. This makes all the original code (i.e., different generators polynomial) to be used fully. On the other hand, this makes it to be possible to yield original codes from the known good high rate codes by some orthogonal perforation matrix.

By virtue of these conclusions, we have obtained the low-rate (4, 1) original codes corresponding to good (4, 3) CCs [9], as listed in Table 2. It is worth noting that Table 2 is identical to the conclusions in [7], which are based on the general PCCs constructing method, as we mentioned at the beginning of this section. However, Table 2 is induced by different method, which is based on how the polynomial generator matrices of original low rate convolutional code are constructed from best high rate convolutional code.

For Table 2, K_1 is the constraint length of $(4,1)$ CC and $m = 4 \times 3 - 4 = 8$, the orthogonal perforation matrix is:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using this method, any good known (n, l) high rate CCs can be constructed by one original $(n, 1)$ CCs and one orthogonal perforation matrix.

4.2 The Puncturing Realization of Good Known Systematic CCs

Since there is a large body of research devoted to the class of $(2, 1)$ CCs, the constructing of PCCs from $(2, 1)$ original CCs is very useful. The determinate generator polynomial matrix of original $(2, 1)$ CCs can not be obtained from the polynomial generator matrix of (3, 2) and (4, 3) PCCs as in the previous subsection. But we can find the good systematic PCCs [10] from systematic $(2, 1)$ original CCs for their determinate relationship as follows.

Suppose the polynomial generator matrix of systematic $R = \frac{1}{2}$ be:

$$G(D) = [1, G_2(D)]. \tag{9}$$

If the polynomial generator matrix of systematic (3, 2) CCs is:

$$\begin{pmatrix} 1 & 0 & Q_{13}(D) \\ 0 & 1 & Q_{23}(D) \end{pmatrix},$$

from Theorem 1, its determinate relationship with $G(D)$ is:

$$G_2(D) = Q_{23}(D^2) + DQ_{13}(D^2). \tag{10}$$

If the polynomial generator matrix of systematic (4, 3) CCs is:

Table 3 Systematic original codes that yield the good known systematic (3, 2) CCs.

Systematic original (2, 1) CCs	Systematic punctured (3,2) CCs		
$G_2(D)$	K_2	$Q_{13}(D)$	$Q_{23}(D)$
$1+D+D^3$	2	$1+D$	1
$1+D+D^3+D^4$	3	$1+D$	$1+D^2$
$1+D+D^3+D^4+D^6$	4	$1+D$	$1+D^2+D^3$
$1+D+D^3+D^4+D^6+D^8+D^9$	5	$1+D+D^4$	$1+D^2+D^3+D^4$
$1+D+D^3+D^4+D^6+D^8+D^9+D^{10}+D^{11}$	6	$1+D+D^4+D^5$	$1+D^2+D^3+D^4+D^5$
$1+D+D^3+D^4+D^6+D^8+D^9+D^{10}+D^{11}+D^{13}$	7	$1+D+D^4+D^5+D^6$	$1+D^2+D^3+D^4+D^5$

Table 4 Systematic original codes that yield the good known systematic (4, 3) CCs.

Systematic original (2,1) CCs	Systematic punctured (4,3) CCs	
$G_2(D)$	K_3	$Q_{14}(D)$ $Q_{24}(D)$ $Q_{34}(D)$
$1+D+D^2+D^3+D^5+D^6+D^7$	3	$1+D$ $1+D^2$ $1+D+D^2$
$1+D+D^2+D^3+D^5+D^6+D^7+D^9+D^{10}+D^{11}$	4	$1+D+D^3$ $1+D^2+D^3$ $1+D+D^2+D^3$
$1+D+D^2+D^3+D^5+D^6+D^7+D^9+D^{10}+D^{11}+D^{13}+D^{14}$	5	$1+D+D^3+D^4$ $1+D^2+D^3+D^4$ $1+D+D^2+D^3$
$1+D+D^2+D^3+D^5+D^6+D^7+D^9+D^{10}+D^{11}+D^{13}+D^{14}+D^{19}$	7	$1+D+D^3+D^4$ $1+D^2+D^3+D^4+D^6$ $1+D+D^2+D^3$
$1+D+D^2+D^3+D^5+D^6+D^7+D^9+D^{10}+D^{11}+D^{13}+D^{14}+D^{19}+D^{21}$	8	$1+D+D^3+D^4$ $1+D^2+D^3+D^4+D^6$ $1+D+D^2+D^3+D^7$

$$\begin{pmatrix} 1 & 0 & 0 & Q_{14}(D) \\ 0 & 1 & 0 & Q_{24}(D) \\ 0 & 0 & 1 & Q_{34}(D) \end{pmatrix},$$

from Theorem 1, its determinate relationship with $G(D)$ is:

$$G_2(D) = Q_{34}(D^3) + DQ_{24}(D^3) + D^2Q_{14}(D^3). \quad (11)$$

Therefore, Tables 3 and 4 give the systematic original codes that yield the good systematic (3, 2) and (4, 3) CCs. For Table 3, K_2 is the constraint length of (3, 2) CCs, m=1, the corresponding perforation matrix is:

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For Table 4, K_3 is the constraint length of $(4,3)$ CCs, m=2, the corresponding perforation matrix is:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

5. Conclusion

This paper derives the polynomial generator matrices and upper bound on the constraint length of punctured convolutional codes. These are useful for constructing good PCCs, which are same as those good known non-systematic and systematic high rate CCs. As the good high rate CCs are found more and more, our method contributes for the puncturing realization of their good CCs, and is quite useful for the communication area.

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Appendix: Proof of Theorem 1

We prove Theorem 1 at Sect. 2 in the following three steps:

Step1. (The relationship of inputs of two encoders) Let inputs of the $R=\frac{1}{n}$ CCs and $R=\frac{l}{nl}$ CCs be $X(D)$ and $X^*(D)=[X_1^*(D),X_2^*(D),\cdots,X_l^*(D)]$, respectively. When same input sequences are encoded by these CCs, the successive l information bits of $R=\frac{1}{n}$ CCs are equal to one information byte of $R=\frac{l}{nl}$ CCs. So, we have:

$$X(D) = X_1^*(D^l)+DX_2^*(D^l)+\cdots+D^{l-1}X_l^*(D^l).$$

Step2.(The relationship of outputs of two encoders) Let outputs of the $R=\frac{1}{n}$ CCs and $R=\frac{l}{nl}$ CCs be $Y(D)=[Y_1(D),Y_2(D),\cdots,Y_n(D)]$, and,

$$\begin{aligned}Y^*(D) &= [Y_1^*(D),Y_2^*(D),\cdots,Y_n^*(D),\\ &\quad Y_{n+1}^*(D),Y_{n+2}^*(D),\cdots,Y_{2n}^*(D),\\ &\quad \cdots,\\ &\quad Y_{(l-1)n+1}^*(D),Y_{(l-1)n+2}^*(D),\cdots,Y_{nl}^*(D)],\end{aligned}$$

respectively, then the successive l encoded information bytes of $R=\frac{1}{n}$ CCs are equal to one encoded information byte of $R=\frac{l}{nl}$ CCs. So, we have:

$$\begin{aligned}Y_1(D) &= Y_1^*(D^l)+DY_{n+1}^*(D^l)+\cdots\\ &\quad +D^{l-1}Y_{(l-1)n+1}^*(D^l),\\ Y_2(D) &= Y_2^*(D^l)+DY_{n+2}^*(D^l)+\cdots\\ &\quad +D^{l-1}Y_{(l-1)n+2}^*(D^l),\cdots\\ Y_n(D) &= Y_n^*(D^l)+DY_{2n}^*(D^l)+\cdots+D^{l-1}Y_{nl}^*(D^l).\end{aligned}$$

Step3. (The relationship of $G(D)$ and $J(D)$) Since $Y(D)=X(D)G(D)$, we immediately have $Y_s(D)=X(D)G_s(D)$ for $R=\frac{1}{n}$ CCs and $s=1,2,\cdots,n$. Since $Y^*(D)=X^*(D)J(D)$, we have $Y_i^*(D)=X_1^*(D)J_{1,i}(D)+\cdots+X_l^*(D)J_{l,i}(D)$ for $R=\frac{l}{nl}$ CCs and i=1,2,...,nl. Therefore, $Y_1(D)$ can be expressed by:

$$\begin{aligned}&(X_1^*(D^l)+DX_2^*(D^l)+\cdots+D^{l-1}X_l^*(D^l))G_1(D)\\ &= Y_1^*(D^l)+DY_{n+1}^*(D^l)+\cdots\\ &\quad +D^{l-1}Y_{(l-1)n+1}^*(D^l)\\ &= X_1^*(D^l)J_{1,1}(D^l)+X_2^*(D^l)J_{2,1}(D^l)\\ &\quad +\cdots+X_l^*(D^l)J_{l,1}(D^l)\\ &\quad +DX_1^*(D^l)J_{1,n+1}(D^l)\\ &\quad +DX_2^*(D^l)J_{2,n+1}(D^l)\\ &\quad +\cdots+DX_l^*(D^l)J_{l,n+1}(D^l)+\cdots\\ &\quad +\cdots\\ &\quad +D^{l-1}X_1^*(D^l)J_{1,(l-1)n+1}(D^l)\\ &\quad +D^{l-1}X_2^*(D^l)J_{2,(l-1)n+1}(D^l)\\ &\quad +\cdots+D^{l-1}X_l^*(D^l)J_{l,(l-1)n+1}(D^l).\end{aligned}$$

Obviously,

$$\begin{aligned}&G_1(D)\\ &\quad = J_{1,1}(D^l)+DJ_{1,n+1}(D^l)+\cdots\\ &\qquad +D^{l-1}J_{1,(l-1)n+1}(D^l),\\ &DG_1(D)\\ &\quad = J_{2,1}(D^l)+DJ_{2,n+1}(D^l)+\cdots\\ &\qquad +D^{l-1}J_{2,(l-1)n+1}(D^l),\\ &\cdots\\ &D^{l-1}G_1(D)\\ &\quad = J_{l,1}(D^l)+DJ_{l,n+1}(D^l)+\cdots\\ &\qquad +D^{l-1}J_{l,(l-1)n+1}(D^l).\end{aligned}$$

So,

$$\begin{aligned}&G_{1,0}(D)+G_{1,1}(D)+\cdots+G_{1,l-1}(D)\\ &\quad = J_{1,1}(D^l)+DJ_{1,n+1}(D^l)+\cdots\\ &\qquad +D^{l-1}J_{1,(l-1)n+1}(D^l),\\ &D[G_{1,0}(D)+G_{1,1}(D)+\cdots+G_{1,l-1}(D)]\\ &\quad = J_{2,1}(D^l)+DJ_{2,n+1}(D^l)+\cdots\\ &\qquad +D^{l-1}J_{2,(l-1)n+1}(D^l),\\ &\cdots\\ &D^{l-1}[G_{1,0}(D)+G_{1,1}(D)+\cdots+G_{1,l-1}(D)]\\ &\quad = J_{l,1}(D^l)+DJ_{l,n+1}(D^l)+\cdots\\ &\qquad +D^{l-1}J_{l,(l-1)n+1}(D^l).\end{aligned}$$

Therefore, we have: $J_{1,1}(D)=G_{1,0}(D^{\frac{1}{l}})$,

$$\begin{aligned}&J_{1,n+1}(D)=D^{-\frac{1}{l}}G_{1,1}(D^{\frac{1}{l}}),\\ &\cdots\\ &J_{1,(l-1)n+1}(D)=D^{-\frac{l-1}{l}}G_{1,l-1}(D^{\frac{1}{l}}),\\ &J_{2,1}(D)=D^{\frac{1}{l}}G_{1,l-1}(D^{\frac{1}{l}}),\\ &J_{2,n+1}(D)=G_{1,0}(D^{\frac{1}{l}}),\\ &\cdots\end{aligned}$$

$J_{2,(l-1)n+1}(D) = D^{-\frac{l-2}{l}} G_{1,l-2}(D^{\frac{1}{l}})$,
...
$J_{l,1}(D) = D^{\frac{l-1}{l}} G_{1,1}(D^{\frac{1}{l}})$,
$J_{l,n+1}(D) = D^{\frac{l-2}{l}} G_{1,2}(D^{\frac{1}{l}})$,
...
$J_{l,(l-1)n+1}(D) = G_{1,0}(D^{\frac{1}{l}})$.

For $i, j = 1, \cdots, l$ and $h = l + i - j \bmod l$, then the above equations can be expressed generally by:

$J_{j,(i-1)n+1}(D) = D^{\frac{j-i}{l}} G_{1,h}(D^{\frac{1}{l}})$.

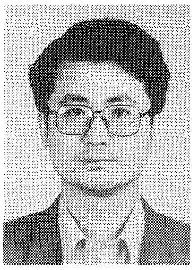
By the same process, we have:

$J_{j,(i-1)n+2}(D) = D^{\frac{j-i}{l}} G_{2,h}(D^{\frac{1}{l}})$,
...
$J_{j,in}(D) = D^{\frac{j-i}{l}} G_{n,h}(D^{\frac{1}{l}})$.

That is to say,

$J_{j,(i-1)n+s}(D) = D^{\frac{j-i}{l}} G_{s,h}(D^{\frac{1}{l}})$,

where $s = 1, 2, \cdots, n$. □



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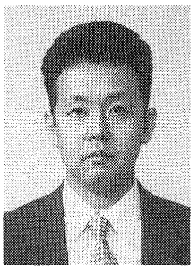

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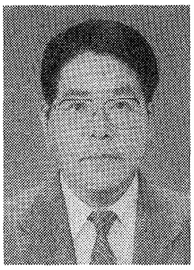

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