A comment on pure-strategy Nash equilibria in competitive diffusion games

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Abstract

In [N. Alon, M. Feldman, A. D. Procaccia, and M. Tennenholtz, A note on competitive diffusion through social networks, Inform. Process. Lett. 110 (2010) 221-225], the authors introduced a game-theoretic model of diffusion process through a network. They showed a relation between the diameter of a given network and existence of pure Nash equilibria in the game. Theorem 1 of their paper says that a pure Nash equilibrium exists if the diameter is at most two. However, we have an example which does not admit a pure Nash equilibrium even if the diameter is two. Hence we correct the statement of Theorem 1 of their paper.

Keywords: Graph algorithm, Algorithmic game theory, Nash equilibria

1. Introduction

In their interesting paper [1] on a competitive facility location game, Alon et al. addressed a diffusion game on an undirected graph $G = \langle V, E\rangle$ with a set of players $N = \{1, 2, \ldots, n\}$. Each player i has an individual color c_i, which is neither white nor gray. In this game, initially, all vertices are colored by white. At time one, each player i selects one vertex on a given graph, and colors the vertex by c_i. If a vertex is selected by more than two players, it is colored by gray. At time $t + 1$, each white vertex is colored in c_i ($i \in N$), if it is adjacent to vertices colored by c_i, but is not adjacent to vertices colored

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by c_j for any $j \in N \setminus \{i\}$. A white vertex that has two neighbors colored by two distinct colors c_i and c_j $(i, j \in N, i \neq j)$ is colored by gray. When there is no vertex whose color is changed from white in this time, the process terminates. A strategy profile $\boldsymbol{x}(\in V^N)$ stands for a vertex selected by each player at time one. The utility of player i, denoted by $U_i(\boldsymbol{x})$, is given by the number of vertices colored by c_i at the end of the process.

Given a strategy profile $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $x' \in V$, we denote by $(x', \boldsymbol{x}_{-i})$ a vector equals to $\boldsymbol{x}$ but with the ith component replaced by x', that is, $(x_1, \ldots, x_{i-1}, x', x_{i+1}, \ldots, x_n)$. A strategy profile $\boldsymbol{x}$ is a pure Nash equilibrium of this game, if $U_i(x', \boldsymbol{x}_{-i}) \leq U_i(\boldsymbol{x})$ holds for any $i \in N$ and $x' \in V$.

The paper [1] discussed a relationship between existence of pure Nash equilibria of this game and the diameter of the given graph. The diameter of a graph is the maximum distance between a pair of vertices. They showed that if the diameter is at most two then a pure Nash equilibrium exists, whereas if the diameter is greater than two then an equilibrium is not guaranteed to exist. In this paper, we show an example which does not admit a pure Nash equilibrium even though the diameter is two.

2. Counterexample

We have an instance which does not have a pure Nash equilibrium for two players even though the diameter of the graph is two. For a diffusion game induced by the graph shown in Figure 1 and two players, the utility bimatrix is given by Table 1. From this bimatrix, we can observe the following facts:

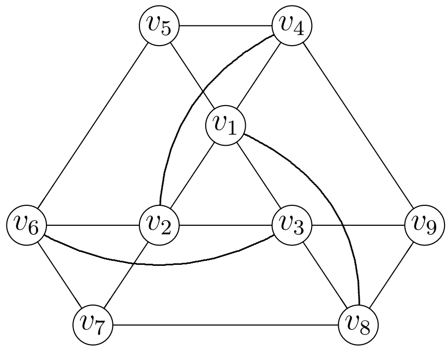

Figure 1: A graph that does not admit a pure Nash equilibrium for two players.

Table 1: Utility bimatrix for the diffusion game induced by the graph in Figure 1 and two players. Element (i, j) implies $(U_1(v_i, v_j), U_2(v_i, v_j))$.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	$(0,0)$	$(4,3)$	$(3,4)$	$(5,2)$	$(5,2)$	$(4,2)$	$(5,2)$	$(4,3)$	$(5,1)$
v_2		$(0,0)$	$(4,3)$	$(4,3)$	$(5,1)$	$(5,2)$	$(5,2)$	$(4,2)$	$(5,2)$
v_3			$(0,0)$	$(4,2)$	$(5,2)$	$(4,3)$	$(5,1)$	$(5,2)$	$(5,2)$
v_4				$(0,0)$	$(4,2)$	$(3,3)$	$(4,3)$	$(3,3)$	$(4,3)$
v_5					$(0,0)$	$(3,4)$	$(3,3)$	$(3,4)$	$(3,3)$
v_6						$(0,0)$	$(4,2)$	$(3,3)$	$(4,3)$
v_7							$(0,0)$	$(3,4)$	$(3,3)$
v_8								$(0,0)$	$(4,2)$
v_9									$(0,0)$

- For any $x_1 \in V$ and $x_2 \in \{v_4, v_5, v_6, v_7, v_8, v_9\}$, there exists $v \in \{v_1, v_2, v_3\}$ such that $U_2(x_1, x_2) < U_2(x_1, v)$. Thus, any vertex in $\{v_1, v_2, v_3\}$ always beats a vertex in $\{v_4, v_5, v_6, v_7, v_8, v_9\}$.
- We have $U_2(v_1, v_2) < U_2(v_1, v_3)$, $U_2(v_2, v_3) < U_2(v_2, v_1)$, and $U_2(v_3, v_1) < U_2(v_3, v_2)$.

Hence, we can see this game does not admit a pure Nash equilibrium.

A point to notice in this example is that the diffusion process may repeat until time three. Let N_v be the neighborhood of vertex v including v. For instance, we consider a strategy profile $\boldsymbol{x} = (v_1, v_9)$. Since $(N_{v_9} \setminus \{v_9\}) \subset N_{v_1}$, any vertex is not colored by c_2 at time two. On the other hand, v_2 and v_5 are colored by c_1 at time two. Finally, at time three, vertices v_6 and v_7 are colored by c_1. In this case, the key equation in the proof of [1]

$$U_i(\boldsymbol{x}) = |N_{x_i}| - |\bigcup_{j \neq i} (N_{x_i} \cap N_{x_j})| + \chi_{A_i}(\boldsymbol{x}), \tag{1}$$

where $\chi_{A_i}(\boldsymbol{x})$ is the indicator function for $A_i = \{\boldsymbol{x} \mid \exists j \in N \setminus \{i\}, x_j \in N_{x_i}\}$, does not hold for $i = 1$. Indeed, the right-hand-side of (1) becomes $|N_{v_1}| - |(N_{v_1} \cap N_{v_9})| + \chi_{A_1}((v_1, v_9)) = 6 - 3 - 0 = 3$, although $U_1((v_1, v_9)) = 5$.

Eq. (1) holds when $N_v \cup N_u = V$ for any pair of vertices v and u. Hence we correct the argument of Theorem 1 in [1] as follows:

Theorem 1. *When $N_u \cup N_v = V$ holds for any pair of vertices u and v, the diffusion game with any number of players admits a pure Nash equilibrium. Furthermore, an equilibrium can be found in polynomial time.*

Finally, we show another condition such that Eq. (1) holds when the number of players is two. For a couple of pairs of vertices (v, u) and (z, w), let $D_{u,v}(z, w)$ be the distance between z and w in the graph deleting $(N_v \cap N_u) \setminus \{u, v\}$. Eq. (1) holds for any $\boldsymbol{x} \in V \times V$, when the following condition holds.

restricted equivalent distance condition: For any pair of vertices (v, u) and any vertex $z \in V \setminus (N_v \cup N_u)$, $D_{u,v}(u, z) = D_{u,v}(v, z)$ holds.

This condition implies that a vertex colored at time three is colored by only gray.

Theorem 2. *When the restricted equivalent distance condition holds, the diffusion game with two players admits a pure Nash equilibrium. Furthermore, an equilibrium can be found in polynomial time.*

Note that the restricted equivalent distance condition is a necessary and sufficient condition for Eq. (1) when there are two players. It is a future work to show necessary conditions for Eq. (1) for any number of players.

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[1] N. Alon, M. Feldman, A. D. Procaccia, and M. Tennenholtz, A note on competitive diffusion through social networks, *Information Processing Letters*, 110, (2010) 221-225.