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The Equity Premium and Market Crises in a Static Tree Economy

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Abstract

In a static tree economy, we explore the effects of two types of rare events, economic crashes and crises, on the equilibrium prices of the market portfolios. Using the data of U.S. stock market and GDP growth rates, our calibrations show that the introduction of economic crises in place of economic crashes would sufficiently increase magnitudes of the equity premium.

Keywords: Equity premium; static Lucas model; rare events; economic crash, economic crisis; equilibrium price

JEL classification: D58, G12, F30, E27

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I. Introduction

Two decades ago, just after the publication of the Mehra and Prescott (1985) article on the equity premium puzzle, Rietz (1988) proposed a solution for the puzzle simply by bringing in low-probability economic disasters. While Mehra and Prescott (1988) criticized Rietz's proposed solution, Barro (2006) recently made arguments for Rietz's basic reasoning and conducted empirical analysis for the measurement of the frequencies and sizes of the international economic disasters that occurred during the twentieth century.

This paper further elaborates Rietz's idea in two respects. The first is to introduce events of economic crises in place of economic crashes (or disasters). Although the real economy falls into depression, however badly it does, it may be commonly observed that some of the economic sectors are relatively active in markets. Thus we consider low-probability events of economic crises in which some of the economic sectors fall into disastrous states but the others are still in active states. The second is for simplicity to consider a static version of Lucas's tree economy as in Gollier (2000) and Gollier and Schlesinger (2002).

Assuming that the economy consists of two sectors of production, we analyze the data of U.S. stock market and GDP growth rates during 1970-2005 and estimate the magnitude of the equity premium. Our calibrations show that the impact of introduction of economic crises on the magnitude of the equity premium is much larger than the introduction of economic crashes when the total decline of the whole productions is similar. We verify that our simple static model could provide appropriate magnitudes of the equity premium for the relative risk aversion coefficient $\alpha \geq 4$ when crash probabilities of each economic sector are less than 0.02, and the total production declines between 11.2%~45.4% (27.6% on the average).

II. A Static Tree Economy and Economic Crises

Following Gollier (2000, Chapter 5), we consider a static Lucas's tree economy in which several identical (i.e., the same preference and endowment) agents live for one period. Consumption takes place at the end of the period. There are n different assets in the economy, each of them representing a specific technology to produce a homogeneous consumption good. Each agent is endowed with one unit of each asset. Let $\tilde{y}^i$ be a random variable representing the production per capita in sector i of the economy, i= 1,…,n, so that $\tilde{y} = \sum_{i=1}^{n} \tilde{y}^i$ is the GDP per capita. There is a simple financial market in this economy. Before the realization of the

$\tilde{y}^i$, agents can sell their property rights on the different sectors of the economy against the delivery of a certain quantity of the consumption good. The decision problem of each agent in the economy is to determine the proportion α_i of property rights he owns in sector i that he will retain, given price $P_i(\tilde{y})$ of those rights. The equilibrium prices of assets are shown to be,

$$(1 + r_f)P_i(\tilde{y}) = \frac{E\tilde{y}^i u'(\tilde{y})}{Eu'(\tilde{y})}, \quad i = 1, \cdots n,$$

where r_f is the risk-free rate in the economy and u is agents'

von Neumann-Morgenstern utility function. Since our model is static, we assume that $r_f = 0$. Let $P(\tilde{y}) = \sum_{i=1}^{n} P_i(\tilde{y}) = \frac{E\tilde{y}^i u'(\tilde{y})}{Eu'(\tilde{y})}$ denote the initial market value of the investors' portfolio. Then the equity premium is given by

$$\varphi(\tilde{y}) = \frac{E\tilde{y}}{P(\tilde{y})} - 1.$$

We consider low-probability events of economic crises which may not be observed from the real GDP growth data. To formalize the introduction of such events, for each $K \subseteq N = \{1, \ldots, n\}$, let $\tilde{y}_K$ denote the random variable which represents market outputs when every asset in K falls into crash but not for assets in $N \setminus K$. Thus $\tilde{y}_N$ means that all economic sectors fall into economic crash. On the other hand, $\tilde{y}_\emptyset$ means no such crisis and crash. Assume that $\tilde{y}_K \leq \tilde{y}_\emptyset$ for all nonempty $K \subseteq N$.

Let $\tilde{y}$ be a compound lottery given by

$$\tilde{y} = [(\lambda_\emptyset, \tilde{y}_\emptyset), (\lambda_{\{1\}}, \tilde{y}_{\{1\}}), \cdots, (\lambda_K, \tilde{y}_K), \cdots, (\lambda_N, \tilde{y}_N)],$$

which means yielding $\tilde{y}_K$ with probability $\lambda_K \geq 0$ for all subsets $K \subseteq N$, where $\Sigma_{K \subseteq N} \lambda_K = 1$. We shall compare $\varphi(\tilde{y}_\emptyset)$ with $\varphi(\tilde{y})$. When u is constant relatively risk averse, Appendix shows that there is an interval $[0, x^0)$ such that, for sufficiently small probabilities λ_K for all nonempty K, $\varphi(\tilde{y}_\emptyset) < \varphi(\tilde{y})$ whenever supports of $\tilde{y}_K$ are contained in the interval. However, it does not tell how large the difference $\varphi(\tilde{y}) - \varphi(\tilde{y}_\emptyset)$ is to explain the puzzle. In what follows, we shall verify it numerically.

III. Numerical Analyses

Assumptions

For simplicity, we assume that the economy consists of two sectors of production, i.e., manufacturing and other private sectors. Thus N={1,2}. Let $\tilde{y}^1$ and $\tilde{y}^2$ be random variables representing the production per capita in sector i of the economy, so that $\tilde{y}_\emptyset = \tilde{y}^1 + \tilde{y}^2$ is the GDP per capita in the private sectors.

Consider a change $\tilde{y}_\emptyset \to \tilde{y} = [(\lambda_\emptyset, \tilde{y}_\emptyset), (\lambda_{\{1\}}, \tilde{y}_{\{1\}}), (\lambda_{\{2\}}, \tilde{y}_{\{2\}}), (\lambda_{\{1,2\}}, \tilde{y}_{\{1,2\}})]$. We assume that $\tilde{y}_{\{1\}} = \kappa_1 w_1 + \tilde{y}^2$, $\tilde{y}_{\{2\}} = \kappa_2 w_2 + \tilde{y}^1$, and $\tilde{y}_{\{1,2\}} = \kappa_1 w_1 +$

$\kappa_2 w_2$, where w_i (i = 1,2) is sector i 's average share of the GDP per capita in the private sectors and $1-\kappa_i$ (i =1,2) is sector i 's level of decline when the sector i falls into economic disruption. To specify the probability density function f_ϕ of $\tilde{y}_\emptyset$, we introduce the random variables $\tilde{z}^i$ $(i = 1,2)$, which represent the growth rates of the production per capita in sector i and assume that $\tilde{y}^i = w_i \tilde{z}^i$ for i = 1,2. Once we know the joint probabilities $g(z^1,z^2)$ for pairs (z^1,z^2), we obtain that, for all y_ϕ, $f_\phi(y_\phi) = \Sigma\{ g(z^1,z^2) : y_\phi = w_1 z^1 + w_2 z^2 \}$. Furthermore, the probability density function $f_{\{i\}}$ of $\tilde{y}_{\{i\}}$ $(i = 1,2)$ is given by $f_{\{i\}}(y^i) = \Sigma\{ g(z^1,z^2) : y^i = w_i z^i \}$.

Data and Calibration Results

We use the time series of the growth rates of U.S. real GDP per capita in the private sectors (around 95% share of GDP) for the period from 1970 to 2005 (average 2.4%, standard deviation 4.2%, maximum value, 9.2% in 1972, minimum value, -6.0% in 1974 for the manufacturing sector; average 2.9%, standard deviation 1.7%, maximum value, 5.5% in 1997, minimum value, -0.5% in 1974 for the other private sectors). The average of GDP share of the manufacturing sector during the period is 21%. Thus we let w_1 = 0.21 and w_2 = 0.79. We assume that agents believe that each pair (z^1,z^2) of realized growth rates of sectors productions in the past will occur with equal probability, and that agents have a constant relative risk aversion (CRRA) α. Then using formula $\varphi(\tilde{y}_{\emptyset})$, we can calculate the equity premium in the static tree economy. Several empirical studies about U.S. stock markets show that the estimated values of α may be ranged over an interval [1,8]. Table 1 reports various equity premiums as a function of α.

Table 1. Equity premium with CRRA α and $\tilde{y}_{\emptyset}$ based on the actual growth rates of U.S. real GDP per capita, 1970-2005

CRRA(α)	Equity premium (%)	CRRA(α)	Equity premium (%)
1.0	0.04	8.0	0.38
4.0	0.19	9.0	0.43
5.0	0.23	10.0	0.48

Over the period 1970-2005, we take the real returns of S&P500 as a risky asset and TB data as a risk-free asset. The averages of the real return of S&P500 and the TB data are respectively 8.74% and 1.02% per year. Hence we obtain that the average of equity premiums over the period is around 7.72%. Clearly, this simple static version does not fit the data for all α levels.

To see that the introduction of economic disruptions might bridge the gap, we show two calibration results to see how large the magnitudes of the equity premiums for the small probabilities of economic crashes and crises might be to resolve the puzzle.

Calibration 1: ($\lambda_{\{1\}} = \lambda_{\{2\}} = 0$ and $\lambda_{\{1,2\}} \geq 0$: economic crash). Assume $\kappa = \kappa_1 + \kappa_2$. First, we consider $\kappa = 0.75$ as in Rietz (1988), which corresponds to around 25% decline of GDP in private sectors. Also, let $0 < \lambda_{\{1,2\}} \leq 0.02$, which means that the economic crash will occur at most one in fifty years. Let $0 < \alpha \leq 10$ as in Rietz (1988). Barro (2006) estimated from international time series data that GDP will decline between 15% and 64% with probability around 0.015 and 0.02. Our calibration result, which is similar to Rietz's calibration, is that α needs to be at least 10 to explain the puzzle with the probability of economic crash being between 0.015 and 0.02.

Since the equity premium gets larger for smaller κ, however unrealistic it may be, we show a calibration result in Figure 1 when $\kappa = 0.55$ to compare with the calibration 2 below. We may conclude that the effect of economic crash (i.e., 45% decline of GDP in private sectors of the economy) on the magnitude of the equity premium may be large enough to explain the puzzle for $\alpha \geq 4$. For each of $\alpha = 1$, 2, 3, we respectively need $\lambda_{\{1,2\}} = 0.263, 0.076, 0.033$ to obtain $\varphi(\tilde{y}) \geq 0.0772$.

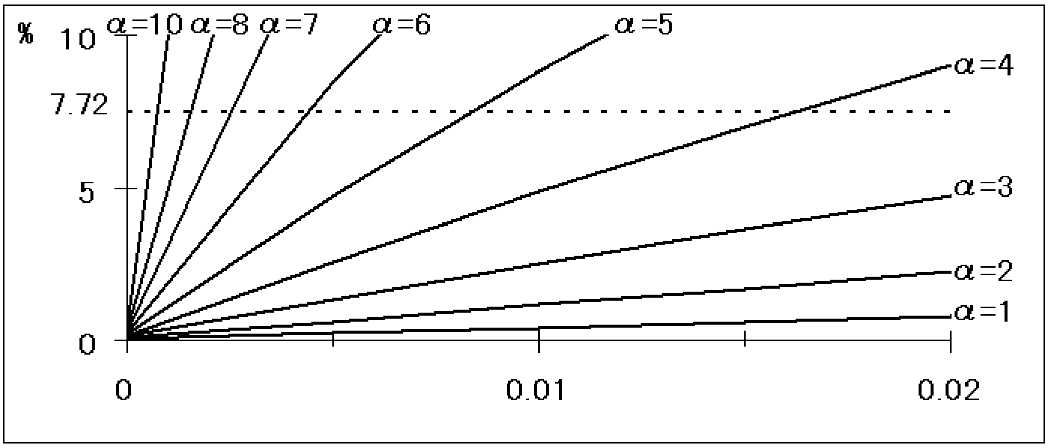

Figure 1: Equity premiums (vertical line) and probabilities of economic crashes (horizontal line)

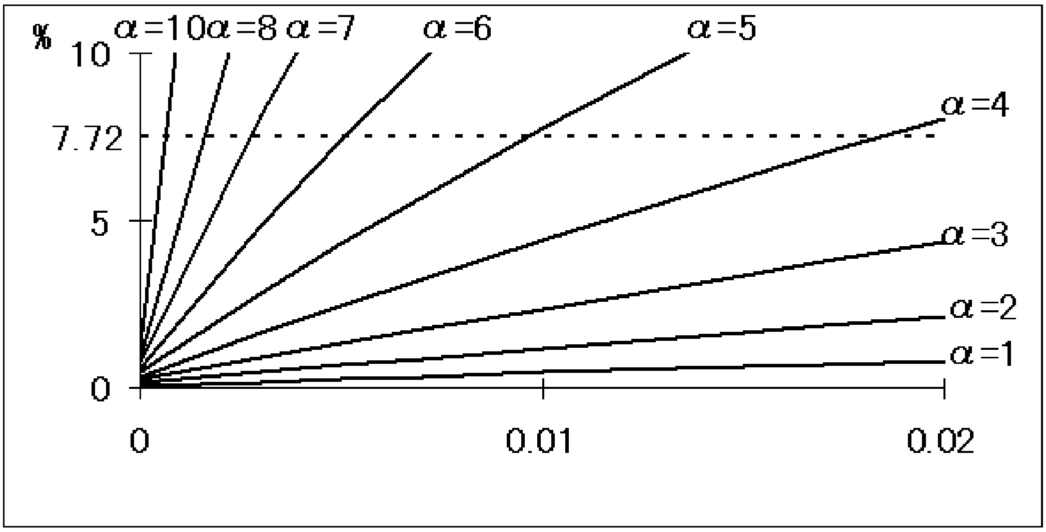

Figure 2: Equity premiums (vertical line) and probabilities of economic crisis of asset 2 (horizontal line)

Calibration 2: ($\lambda_{\{1\}} \geq 0$ $\lambda_{\{2\}} \geq 0$ and $\lambda_{\{1,2\}} = 0$: economic crisis). Let $0 < \lambda_{\{i\}} \leq 0.02$. Assume that $\kappa_1 = \kappa_2 = 0.4$. This means 11.2%~ 45.4% (27.6% on the average) decline of GDP in the private sectors of the economy. Fix $\lambda_{\{1\}} = 0.01$ and let $0 < \alpha \leq 10$. Although the average decline is similar to Rietz's case, Figure 2 shows the same effect as in economic crash of 45% decline in Figure 1. That is, the magnitudes of equity premium dramatically increase by the introduction of low-probability events of economic crisis. For each of $\alpha = 1, 2, 3$, we respectively need $\lambda_{\{2\}} = 0.290, 0.084, 0.038$ to obtain $\varphi(\tilde{y}) \geq 0.0772$.

IV. Conclusions

The aim of the paper is to further elaborate Rietz's idea by introducing two types of low-probability events, economic crash and economic crisis, in a static tree economy to see whether or not sizable magnitude of equity premium is achieved to resolve the puzzle. Our empirical finding is that the effect of economic crisis is much larger than economic crash. Although our model failed to obtain sizable effects for smaller coefficients $0 < \alpha \leq 4$, we believe that the individual behaviors of economic sectors may be more significant to understand the puzzle than the whole behavior of the economy.

Appendix

Consider a change $\tilde{y}_\emptyset \to \tilde{y}$, where $\tilde{y}$ is the compound lottery given by (1) in the text. $u(z) = \frac{z^{1-\alpha}}{(1-\alpha)}$. Then

$$E\tilde{y} = (1 - \sum_{K \neq \emptyset} \lambda_K) E\tilde{y}_\emptyset + \sum_{K \neq \emptyset} \lambda_K E\tilde{y}_K,$$

$$Eu'(\tilde{y}) = (1 - \sum_{K \neq \emptyset} \lambda_K) E\tilde{y}_\emptyset^{-\alpha} + \sum_{K \neq \emptyset} \lambda_K E\tilde{y}_K^{-\alpha},$$

$$E\tilde{y}u'(\tilde{y}) = (1 - \sum_{K \neq \emptyset} \lambda_K) E\tilde{y}_\emptyset^{1-\alpha} + \sum_{K \neq \emptyset} \lambda_K E\tilde{y}_K^{1-\alpha}.$$

The partial derivative of the equity premium $\varphi(\tilde{y})$ with respect to λ_K for a nonempty $K \subseteq N$ is given by

$$\frac{\partial \varphi(\tilde{y})}{\partial \lambda_K} = \frac{1}{(E\tilde{y}u'(\tilde{y}))^2} \{E\tilde{y}u'(\tilde{y})[Eu'(\tilde{y})(E\tilde{y}_K - E\tilde{y}_\emptyset) + E\tilde{y}(E\tilde{y}_K^{-\alpha} - E\tilde{y}_\emptyset^{-\alpha})]$$

$$- E\tilde{y}Eu'(\tilde{y})(E\tilde{y}_K^{1-\alpha} - E\tilde{y}_\emptyset^{1-\alpha})\}$$

Evaluating RHS at $\lambda_K = 0$ for all K, we get

$$\left.\frac{\partial \varphi(\tilde{y})}{\partial \lambda_K}\right|_{\lambda=0} = E\tilde{y}_K E\tilde{y}_\emptyset^{-\alpha} + E\tilde{y}_K^{-\alpha} E\tilde{y}_\emptyset - E\tilde{y}_K^{1-\alpha} \frac{E\tilde{y}_\emptyset E\tilde{y}_\emptyset^{-\alpha}}{E\tilde{y}_\emptyset^{1-\alpha}} - E\tilde{y}_\emptyset E\tilde{y}_\emptyset^{-\alpha},$$

where $\lambda = (\lambda_{\{1\}}, \cdots, \lambda_K, \cdots, \lambda_N)$, $2^n - 1$ dimensional real vector.

Consider a particular x in the support of $\tilde{y}_K$. Since $0 < x < \tilde{y}_\emptyset$, we have

$$x^\alpha E\tilde{y}_\emptyset^{1-\alpha} - E\tilde{y}_\emptyset = x^\alpha \int y(y^{-\alpha} - x^{-\alpha}) dF_\emptyset(y) < 0,$$

$$1 - x^\alpha E\tilde{y}_\emptyset^{-\alpha} = x^\alpha \int (x^{-\alpha} - y^{-\alpha}) dF_\emptyset(y) > 0.$$

Furthermore, since $E\tilde{y}_\emptyset^{-\alpha} E\tilde{y}_\emptyset \geq E\tilde{y}_\emptyset^{1-\alpha}$, we have

$$1 - x^\alpha E\tilde{y}_\emptyset^{-\alpha} \leq 1 - \frac{x^\alpha E\tilde{y}_\emptyset^{1-\alpha}}{E\tilde{y}_\emptyset}.$$

Hence, when $\tilde{y}_K = x$, positivity of $\left.\frac{\partial \varphi(\tilde{y})}{\partial \lambda_K}\right|_{\lambda=0}$ implies that

$$xE\tilde{y}_{\emptyset}^{-\alpha} + x^{-\alpha}E\tilde{y}_{\emptyset} - x^{1-\alpha}\frac{E\tilde{y}_{\emptyset}E\tilde{y}_{\emptyset}^{-\alpha}}{E\tilde{y}_{\emptyset}^{1-\alpha}} - E\tilde{y}_{\emptyset}\, E\tilde{y}_{\emptyset}^{-\alpha} > 0$$

$$\Leftrightarrow x^{-\alpha}\, E\tilde{y}_{\emptyset}E\tilde{y}_{\emptyset}^{1-\alpha}\left(1 - x^{\alpha}E\tilde{y}_{\emptyset}^{-\alpha}\right) > x^{1-\alpha}E\tilde{y}_{\emptyset}^{-\alpha}(E\tilde{y}_{\emptyset} - x^{\alpha}E\tilde{y}_{\emptyset}^{1-\alpha})$$

$$\Leftrightarrow \frac{E\tilde{y}_{\emptyset}^{1-\alpha}}{E\tilde{y}_{\emptyset}^{-\alpha}}\frac{1}{x} > \frac{1-\frac{x^{\alpha}E\tilde{y}_{\emptyset}^{1-\alpha}}{E\tilde{y}_{\emptyset}}}{1-x^{\alpha}E\tilde{y}_{\emptyset}^{-\alpha}} (\geq 1).$$

Therefore, there is a unique x^0 in $(0,1/E\tilde{y}_{\emptyset}^{-\alpha})$ such that, for all x in $(0,x^0)$, the above inequality obtains. This x^0 depends on α and the distribution of $\tilde{y}_{\emptyset}$. Hence, when the support of $\tilde{y}_K$ is contained in $(0,x^0)$, we have

$$\left.\frac{\partial\varphi(\tilde{y})}{\partial\lambda_K}\right|_{\lambda=0} > 0.$$

That is, a (sufficiently) small increase of probability with which assets in K go to economic disruption increases the magnitude of equity premium.

References

Barro, J.R. (2006) Rare disasters and asset markets in the twentieth century. *Quarterly Journal of Economics* **121**, 823-866.

Gollier, C. (2001) *The Economics of Risk and Time.* MIT Press, Cambridge, MA.

Gollier, C. and Schlesinger, H. (2002) Changes in risk and asset prices. *Journal of Monetary Economics* **49**, 747-760.

Mehra, R. and Prescott, E. (1985) The equity premium: a puzzle. *Journal of Monetary Economics* **15**, 145-161.

Mehra, R. and Prescott, E. (1988) The equity risk premium: a solution? *Journal of Monetary Economics* **22**, 133-136.

Rietz, T. (1988) The equity risk premium: a solution. *Journal of Monetary Economics* **22**, 117-132.