Auslander-Gorenstein resolution

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Abstract

We introduce the notion of Auslander-Gorenstein resolution and show that a noetherian ring is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over another Auslander-Gorenstein ring.

In this note, a noetherian ring A is a ring which is left and right noetherian, and a noetherian R-algebra A is a ring endowed with a ring homomorphism $R \to A$, with R a commutative noetherian ring, whose image is contained in the center of A and A is finitely generated as an R-module. Note that a noetherian algebra is a noetherian ring.

The main aim of this note is to provide a general method for constructing Auslander-Gorenstein rings (see Definition 3.2) from another one. Auslander-Gorenstein rings appear in various areas of current research. For instance, regular 3-dimensional algebras of type A in the sense of Artin and Schelter, Weyl algebras over fields of characteristic 0, enveloping algebras of finite dimensional Lie algebras and Sklyanin algebras are Auslander-Gorenstein rings (see [4], [10], [11] and [22], respectively). Also, consider the case where R is a commutative Gorenstein local ring and A is a noetherian R-algebra with $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq 0$. In case inj dim $A_A = \dim R$, such an algebra A is called a Gorenstein algebra and extensively studied in [16]. In particular, a Gorenstein algebra is an Auslander-Gorenstein ring. However, even if A is an Auslander-Gorenstein ring, it may happen that inj dim $A_A \neq \dim R$ (see examples in Section 4). Although we have many examples of Auslander-Gorenstein rings, it should be noted that there is a lack of general methods for constructing Auslander-Gorenstein rings.

One of such methods is given by the main theorem: A noetherian ring is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over another Auslander-Gorenstein ring (Theorem 3.6), where the notion of Auslander-Gorenstein resolution is introduced as follows. Let R, A be noetherian rings. A right resolution $0 \to A \to Q^0 \to \cdots \to Q^m \to 0$ in Mod-A is said to be an Auslander-Gorenstein resolution of A over R if the following conditions

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are satisfied: (1) every Q^i is an R-A-bimodule; (2) every $Q^i \in \text{Mod-}R^{\text{op}}$ is a finitely generated reflexive module with $\text{Ext}^j_R(\text{Hom}_{R^{\text{op}}}(Q^i, R), R) = 0$ for $j \neq 0$; (3) $\oplus_{i \geq 0} \text{Hom}_{R^{\text{op}}}(Q^i, R) \in \text{Mod-}A^{\text{op}}$ is faithfully flat; and (4) flat dim $Q^i \leq i$ in Mod-A for all $i \geq 0$. This notion formulates the following facts. Consider the case where R is a commutative Gorenstein local ring and A is a noetherian R-algebra with $\text{Ext}^i_R(A, R) = 0$ for $i \neq 0$. Set $\Omega = \text{Hom}_R(A, R)$. Then proj dim ${}_A\Omega < \infty$ and proj dim $\Omega_A < \infty$ if and only if Ω_A is a tilting module in the sense of [20] (see Remark 2.1). Assume that Ω_A is a tilting module. Take a projective resolution $P^\bullet \to \Omega$ in mod-A^{op} and set $Q^\bullet = \text{Hom}^\bullet_R(P^\bullet, R)$. Then we have a right resolution $0 \to A \to Q^0 \to \cdots \to Q^m \to 0$ in mod-A such that every $Q^i \in \text{mod-}R$ is a reflexive module with $\text{Ext}^j_R(\text{Hom}_R(Q^i, R), R) = 0$ for $j \neq 0$, $\oplus_{i \geq 0} \text{Hom}_R(Q^i, R) \in \text{mod-}A^{\text{op}}$ is a projective generator and proj dim $Q^i < \infty$ in mod-A for all $i \geq 0$ (Remark 2.8). Furthermore, A is an Auslander-Gorenstein ring if proj dim $Q^i \leq i$ in mod-A for all $i \geq 0$, the converse of which holds true if R is complete and $P^\bullet \to \Omega$ is a minimal projective resolution (Proposition 2.9).

This note is organized as follows. In Section 1, we will recall several basic facts which we need in later sections. In Section 2, we will study Auslander-Gorenstein algebras. In case R is a commutative Gorenstein local ring and A is a noetherian R-algebra with $\text{Ext}^i_R(A, R) = 0$ for $i \neq 0$, we will show that inj dim ${}_AA \leq \dim R + 1$ if and only if inj dim $A_A \leq \dim R + 1$ (Theorem 2.4). Also, we will prove the facts quoted above. In Section 3, we will introduce the notion of Auslander-Gorenstein resolution and prove the main theorem. In Section 4, we will provide several examples of Auslander-Gorenstein resolution.

We refer to [14] for topics on Auslander-Gorenstein rings. Also, we refer to [13] for standard homological algebra and to [19] for standard commutative ring theory.

1 Preliminaries

Let A be a ring. We denote by Mod-A the category of right A-modules and by mod-A the full subcategory of Mod-A consisting of finitely presented modules. We denote by $\mathcal{P}_A$ (resp., Inj-A) the full subcategory of mod-A (resp., Mod-A) consisting of projective (resp., injective) modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. In particular, we denote by $\text{Hom}_A(-,-)$ (resp., $\text{Hom}_{A^{\text{op}}}(-,-)$) the set of homomorphisms in Mod-A (resp., Mod-A^{op}). Sometimes, we use the notation M_A (resp., ${}_AM$) to stress that the module M considered is a right (resp., left) A-module. For a module $M \in \text{Mod-}A$ we denote by $E_A(M)$ an injective envelope and by $\text{rad}(M)$ the Jacobson radical. For each complex $X^\bullet$ we denote by $\text{Z}^i(X^\bullet)$, $\text{Z}'^i(X^\bullet)$, $\text{B}^i(X^\bullet)$ and $\text{H}^i(X^\bullet)$ the ith cycle, the ith cocycle, the ith boundary and the ith cohomology, respectively. We denote by $\text{Hom}^\bullet(-,-)$ (resp., $- \otimes^\bullet -$) the associated single complex of the double hom (resp., tensor) complex. As usual, we consider modules as complexes concentrated in degree zero. Finally, for an object X of an additive category $\mathcal{A}$ we denote by $\text{add}(X)$ the full subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of copies of X.

In this section, we recall several basic facts which are well-known but for the benefit of the reader we include direct proofs of some facts. Throughout this section, A stands for an arbitrary ring.

In the next lemma, we consider each complex $Q^\bullet$ as a double complex $Q^{\bullet\bullet}$ such that $Q^{\bullet 0} = Q^\bullet$ and $Q^{\bullet j} = 0$ unless $j = 0$. For each double complex $E^{\bullet\bullet}$ we denote by $s(E^{\bullet\bullet})$ the associated single complex, the nth term of which is given by $\oplus_{i+j=n} E^{ij}$.

Lemma 1.1. *Let $m \geq 0$ be an integer. Let $M \in \text{Mod-}A$ and $M \to Q^\bullet$ a right resolution with $Q^i = 0$ unless $0 \leq i \leq m$. Let $Q^\bullet \to E^{\bullet\bullet}$ be a homomorphism of double complexes with $E^{ij} = 0$ unless $0 \leq i \leq m$ and $j \geq 0$. Assume that $Q^i \to E^{i\bullet}$ is an injective resolution for all $0 \leq i \leq m$. Then the canonical homomorphism $M \to s(E^{\bullet\bullet})$ is an injective resolution.*

Proof. We may assume that $m \geq 1$. We make use of induction. In case $m = 1$, $s(E^{\bullet\bullet})$ is the (-1)-shift of the mapping cone of $E^{0\bullet} \to E^{1\bullet}$ and the assertion is obvious. Assume that $m > 1$. Denote by $Q'^\bullet$ the complex such that $Q'^i = Q^i$ unless $m-1 \leq i \leq m$, $Q'^{m-1} = \mathrm{Z}^{m-1}(Q^\bullet)$ and $Q'^m = 0$, and by $E'^{\bullet\bullet}$ the double complex such that $E'^{i\bullet} = E^{i\bullet}$ for $i < m-1$, $E'^{m-1\bullet}$ is the (-1)-shift of the mapping cone of $E^{m-1\bullet} \to E^{m\bullet}$ and $E'^{i\bullet} = 0$ for $i \geq m$. Then we have a right resolution $M \to Q'^\bullet$ and a homomorphism of double complexes $Q'^\bullet \to E'^{\bullet\bullet}$, so that by induction hypothesis $M \to s(E'^{\bullet\bullet})$ is an injective resolution. Since $s(E'^{\bullet\bullet}) \cong s(E^{\bullet\bullet})$, the assertion follows. $\square$

Definition 1.2 ([20])**.** A module $T \in \text{Mod-}A$ is said to be a tilting module if for some integer $m \geq 0$ the following conditions are satisfied:

(1) T admits a projective resolution $0 \to P^{-m} \to \cdots \to P^{-1} \to P^0 \to T \to 0$ in Mod-A with $P^{-i} \in \mathcal{P}_A$ for all $i \geq 0$.

(2) $\mathrm{Ext}^i_A(T, T) = 0$ for $i \neq 0$.

(3) A admits a right resolution $0 \to A \to T^0 \to T^1 \to \cdots \to T^m \to 0$ in Mod-A with $T^i \in \mathrm{add}(T)$ for all $i \geq 0$.

We refer to [21] for tilting complexes and derived equivalences.

Lemma 1.3. *For any tilting module $T \in \text{Mod-}A$ the following hold:*

(1) Take a projective resolution $0 \to P^{-m} \to \cdots \to P^{-1} \to P^0 \to T \to 0$ with $P^{-i} \in \mathcal{P}_A$ for all $0 \leq i \leq m$. Then the complex $P^\bullet$ is a tilting complex and $\oplus_{i\geq 0} P^{-i} \in \text{Mod-}A$ is a projective generator.

(2) Set $B = \mathrm{End}_A(T)$. Then $T \in \text{Mod-}B^{\mathrm{op}}$ is a tilting module with $A \cong \mathrm{End}_{B^{\mathrm{op}}}(T)^{\mathrm{op}}$ canonically and proj dim ${}_BT$ = proj dim T_A.

Proof. (1) Note that a module $M \in \text{mod-}A$ is a tilting module if and only if it admits a projective resolution $Q^\bullet \to M$ with $Q^\bullet$ a tilting complex (see e.g. [1, Proposition 3.9]). Let $M \in \text{Mod-}A$ with $\text{Hom}_A(P^{-i}, M) = 0$ for all $i \geq 0$. Then $\text{Ext}^i_A(T, M) = 0$ for all $i \geq 0$. Since we have a right resolution $0 \to A \to T^0 \to T^1 \to \cdots \to T^m \to 0$ in Mod-A with $T^i \in \text{add}(T)$ for all $i \geq 0$, applying $\text{Hom}_A(-, M)$ we have $M \cong \text{Hom}_A(A, M) = 0$.

(2) See [20, Theorem 1.5] for the first assertion. Set $m = \text{proj dim } T_A$. By symmetry, it suffices to show that proj dim ${}_BT \leq m$. We have a projective resolution $0 \to P^{-m} \to \cdots \to P^{-1} \to P^0 \to T \to 0$ in Mod-A with $P^{-i} \in \mathcal{P}_A$ for all $i \geq 0$ and hence, applying $\text{Hom}_A(-, T)$, we have a right resolution $0 \to B \to T^0 \to T^1 \to \cdots \to T^m \to 0$ in Mod-B^{op} with $T^i \in \text{add}(T)$ for all $i \geq 0$. Since $\text{Ext}^i_{B^{\text{op}}}(T, T) = 0$ for $i \neq 0$, applying $\text{Hom}_{B^{\text{op}}}(T, -)$ we have $\text{Ext}^i_{B^{\text{op}}}(T, B) = 0$ for $i > m$, so that proj dim ${}_BT \leq m$. □

Remark 1.4. Every projective generator is faithfully flat. Conversely, a finitely presented module is a projective generator if it is faithfully flat.

Lemma 1.5. *For any* $I \in \text{Inj-}A$ *and* $Q \in \text{mod-}A^{\text{op}}$ *we have a bifunctorial isomorphism*

$$\psi_{I,Q} : I \otimes_A Q \xrightarrow{\sim} \text{Hom}_A(\text{Hom}_{A^{\text{op}}}(Q, A), I), a \otimes x \mapsto (h \mapsto ah(x)).$$

Proof. Obviously, $\psi_{I,Q}$ is an isomorphism if $Q \in \mathcal{P}_{A^{\text{op}}}$. Since both $I \otimes_A -$ and $\text{Hom}_A(\text{Hom}_{A^{\text{op}}}(-, A), I)$ are right exact, it follows that $\psi_{I,Q}$ is an isomorphism for all $Q \in \text{mod-}A^{\text{op}}$. □

Lemma 1.6. *Assume that* A *is a left noetherian ring. Then for any* $I \in \text{Inj-}A$ *we have* flat dim $I_A \leq$ inj dim ${}_AA$, *where the equality holds if* I *is an injective cogenerator.*

Proof. It follows by Lemma 1.5 that $\text{Tor}^A_i(I, X) \cong \text{Hom}_A(\text{Ext}^i_{A^{\text{op}}}(X, A), I)$ for all $i \geq 0$ and $X \in \text{mod-}A^{\text{op}}$. □

Definition 1.7 ([7]). A module $M \in \text{Mod-}A$ is said to be reflexive if the canonical homomorphism

$$M \to \text{Hom}_{A^{\text{op}}}(\text{Hom}_A(M, A), A), x \mapsto (f \mapsto f(x))$$

is an isomorphism. In case A is a noetherian ring, a module $M \in \text{mod-}A$ is said to have Gorenstein dimension zero if it is reflexive and $\text{Ext}^i_A(M, A) = \text{Ext}^i_{A^{\text{op}}}(\text{Hom}_A(M, A), A) = 0$ for $i \neq 0$.

Lemma 1.8. *Assume that* A *is a noetherian ring. Then for any* $M \in \text{mod-}A$ *the following hold:*

(1) Assume that inj dim ${}_AA < \infty$. *Then* M *has Gorenstein dimension zero if* $\text{Ext}^i_A(M, A) = 0$ *for* $i \neq 0$.

(2) Assume that inj dim $A_A < \infty$. *Then* M *has Gorenstein dimension zero if it is reflexive and* $\text{Ext}^i_{A^{\text{op}}}(\text{Hom}_A(M, A), A) = 0$ *for* $i \neq 0$.

Corollary 1.9. *Assume that A is a noetherian ring and that* inj dim ${}_AA < \infty$. *Let $A \to I^\bullet$ be a minimal injective resolution in* Mod-A. *Then $\oplus_{j\geq 0} I^j \in$* Mod-A *is an injective cogenerator.*

Proof. For any $M \in$ mod-A with $\mathrm{Hom}_A(M, I^j) = 0$ for all $j \geq 0$, since we have $\mathrm{Ext}_A^j(M, A) = 0$ for all $j \geq 0$, it follows by Lemma 1.8(1) that $M = 0$. □

Lemma 1.10. *Let R be a noetherian ring and M an R-A-bimodule such that $M \in$* Mod-R^{op} *is finitely generated reflexive and* $\mathrm{Ext}_R^i(\mathrm{Hom}_{R^{\mathrm{op}}}(M, R), R) = 0$ *for $i \neq 0$. Then for any $X \in$* Mod-R *we have*

$$\text{flat dim } (X \otimes_R M)_A \leq \text{flat dim } X_R + \text{flat dim } M_A.$$

Proof. We may assume that flat dim $X_R = m < \infty$ and flat dim $M_A = n < \infty$. Take a projective resolution $P^\bullet \to \mathrm{Hom}_{R^{\mathrm{op}}}(M, R)$ in mod-R. Then we have a right resolution $M \to \mathrm{Hom}_R^\bullet(P^\bullet, R)$ in mod-R^{op} and hence $\mathrm{Tor}_j^R(X, M) \cong \mathrm{Tor}_{j+m}^R(X, \mathrm{Z}^m(\mathrm{Hom}_R^\bullet(P^\bullet, R))) = 0$ for $j > 0$. Next, let $N \in$ Mod-A^{op} and $Q^\bullet \to N$ a projective resolution. Then we have a left resolution in Mod-R^{op}

$$\cdots \to M \otimes_A Q^{-n-1} \to M \otimes_A Q^{-n} \to \mathrm{Z}'^{-n}(M \otimes_A^\bullet Q^\bullet) \to 0.$$

Note that for any $j \neq 0$, since $\mathrm{Tor}_j^R(X, M) = 0$, $\mathrm{Tor}_j^R(X, M \otimes_A Q^{-i}) = 0$ for all $i \geq n$. Thus applying $X \otimes_R -$ we have

$$\begin{aligned}\mathrm{Tor}_k^A(X \otimes_R M, N) &\cong \mathrm{H}^{-k}(X \otimes_R^\bullet M \otimes_A^\bullet Q^\bullet) \\ &\cong \mathrm{Tor}_{k-n}^R(X, \mathrm{Z}'^{-n}(M \otimes_A^\bullet Q^\bullet)) \\ &= 0\end{aligned}$$

for $k > m + n$. □

Definition 1.11 ([8])**.** A family of idempotents $\{e_\lambda\}_{\lambda\in\Lambda}$ in A is said to be orthogonal if $e_\lambda e_\mu = 0$ unless $\lambda = \mu$. An idempotent $e \in A$ is said to be local if $eAe \cong \mathrm{End}_A(eA)$ is local. A ring A is said to be semiperfect if $1 = \Sigma_{i=1}^n e_i$ in A with the e_i orthogonal local idempotents.

Throughout the rest of this section, R is a commutative noetherian ring and A is a noetherian R-algebra.

Lemma 1.12. *Assume that R is a complete local ring. Then every noetherian R-algebra A is semiperfect.*

We denote by $\mathrm{Spec}(R)$ the set of prime ideals of R. For each $\mathfrak{p} \in \mathrm{Spec}(R)$ we denote by $(-)_\mathfrak{p}$ the localization at $\mathfrak{p}$ and for each $M \in$ Mod-R we denote by $\mathrm{Supp}_R(M)$ the set of $\mathfrak{p} \in \mathrm{Spec}(R)$ with $M_\mathfrak{p} \neq 0$.

Lemma 1.13. *For any $\mathfrak{p}, \mathfrak{q} \in \mathrm{Supp}_R(A)$ with $\mathfrak{p} \neq \mathfrak{q}$ we have*

$$\mathrm{add}(\mathrm{Hom}_R(A, E_R(R/\mathfrak{p}))) \ \cap \ \mathrm{add}(\mathrm{Hom}_R(A, E_R(R/\mathfrak{q}))) = \{0\}$$

in Mod-A.

Lemma 1.14. *Assume that R is a local ring with the maximal ideal $\mathfrak{m}$. Then $\mathrm{Hom}_R(A, E_R(R/\mathfrak{m})) \in \mathrm{Mod}\text{-}A$ is artinian.*

Remark 1.15. For any module $M \in \mathrm{mod}\text{-}R$ with $R_\mathfrak{p}$ Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(M)$, the following are equivalent:

(1) M is maximal Cohen-Macaulay.

(2) $\mathrm{Ext}^i_R(M, R) = 0$ for $i \neq 0$.

(3) M has Gorenstein dimension zero.

2 Auslander-Gorenstein algebras

Throughout this section, R is a commutative noetherian ring with a minimal injective resolution $R \to I^\bullet$ and A is a noetherian R-algebra such that $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$ and $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq 0$. Set $\Omega = \mathrm{Hom}_R(A, R)$.

In this section, assuming R being a complete Gorenstein local ring, we will provide a necessary and sufficient condition for A to be an Auslander-Gorenstein ring (see Definition 3.2 below). We refer to [9] for commutative Gorenstein rings.

Remark 2.1. The following hold:

(1) A has Gorenstein dimension zero as an R-module, i.e., $A \xrightarrow{\sim} \mathrm{Hom}_R(\Omega, R)$ and $\mathrm{Ext}^i_R(\Omega, R) = 0$ for $i \neq 0$.

(2) $A \xrightarrow{\sim} \mathrm{End}_A(\Omega)$ and $A \xrightarrow{\sim} \mathrm{End}_{A^{\mathrm{op}}}(\Omega)^{\mathrm{op}}$ canonically.

(3) $\mathrm{Ext}^i_A(\Omega, \Omega) = \mathrm{Ext}^i_{A^{\mathrm{op}}}(\Omega, \Omega) = 0$ for $i \neq 0$.

(4) If proj dim $\Omega_A < \infty$ and proj dim ${}_A\Omega < \infty$, then Ω_A is a tilting module with proj dim ${}_A\Omega =$ proj dim Ω_A.

Proof. (1) For any $\mathfrak{p} \in \mathrm{Supp}_R(A)$, since $\mathrm{Ext}^i_{R_\mathfrak{p}}(A_\mathfrak{p}, R_\mathfrak{p}) \cong \mathrm{Ext}^i_R(A, R)_\mathfrak{p} = 0$ for $i \neq 0$, by Lemma 1.8(1) $A_\mathfrak{p} \in \mathrm{mod}\text{-}R_\mathfrak{p}$ has Gorenstein dimension zero, so that $A \in \mathrm{mod}\text{-}R$ has Gorenstein dimension zero.

(2) and (3) We have an injective resolution $\Omega \to \mathrm{Hom}^\bullet_R(A, I^\bullet)$ in Mod-A, so that for any $i \geq 0$ we have

$$\begin{aligned}\mathrm{Ext}^i_A(\Omega, \Omega) &\cong \mathrm{H}^i(\mathrm{Hom}^\bullet_A(\Omega, \mathrm{Hom}^\bullet_R(A, I^\bullet))) \\ &\cong \mathrm{H}^i(\mathrm{Hom}^\bullet_R(\Omega, I^\bullet)) \\ &\cong \mathrm{Ext}^i_R(\Omega, R).\end{aligned}$$

Similarly, $\mathrm{Ext}^i_{A^{\mathrm{op}}}(\Omega, \Omega) \cong \mathrm{Ext}^i_R(\Omega, R)$ for all $i \geq 0$.

(4) According to (2), (3) above, the first assertion follows by [20, Proposition 1.6]. The last assertion follows by Lemma 1.3(2). □

Lemma 2.2. *The following are equivalent:*

(1) proj dim $\Omega_A \le 1$.

(2) proj dim ${}_A\Omega \le 1$.

(3) Ω_A *is a tilting module with* proj dim ${}_A\Omega =$ proj dim $\Omega_A \le 1$.

Proof. Obviously, (3) $\Rightarrow$ (1) and (2).

(2) $\Rightarrow$ (1). Let $M \in$ mod-A. We claim that $\mathrm{Ext}^2_A(\Omega, M) = 0$. It suffices to show that $\mathrm{Ext}^2_A(\Omega, M)_{\mathfrak{p}} \cong \mathrm{Ext}^2_{A_{\mathfrak{p}}}(\Omega_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$. We have $\mathrm{Ext}^i_{R_{\mathfrak{p}}}(A_{\mathfrak{p}}, R_{\mathfrak{p}}) \cong \mathrm{Ext}^i_R(A, R)_{\mathfrak{p}} = 0$ for $i \ne 0$, $\Omega_{\mathfrak{p}} \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(A_{\mathfrak{p}}, R_{\mathfrak{p}})$ and proj dim ${}_{A_{\mathfrak{p}}}\Omega_{\mathfrak{p}} \le 1$ for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$, so that we may assume that R is a Gorenstein local ring with the maximal ideal $\mathfrak{m}$. Denote by $(\hat{-})$ the $\mathfrak{m}$-adic completion. Since $\hat{R}$ is faithfully flat over R, it suffices to show that $\mathrm{Ext}^2_A(\Omega, M) \otimes_R \hat{R} \cong \mathrm{Ext}^2_{\hat{A}}(\hat{\Omega}, \hat{M}) = 0$. Since $\mathrm{Ext}^i_{\hat{R}}(\hat{A}, \hat{R}) \cong \mathrm{Ext}^i_R(A, R) \otimes_R \hat{R} = 0$ for $i \ne 0$, $\hat{\Omega} \cong \mathrm{Hom}_{\hat{R}}(\hat{A}, \hat{R})$ and proj dim ${}_{\hat{A}}\hat{\Omega} \le 1$, we may assume that R is complete. Then by Bongartz's Lemma (see [12, Section 2]) there exists $T \in$ mod-A^{op} with $\Omega \oplus T$ a tilting module, so that by [18, Proposition 4.9] ${}_A\Omega$ is a tilting module. Thus by Remark 2.1(2) and Lemma 1.3(2) Ω_A is a tilting module with proj dim ${}_A\Omega =$ proj dim $\Omega_A \le 1$.

(1) $\Rightarrow$ (2). By symmetry.

(2) $\Rightarrow$ (3). Since (2) $\Rightarrow$ (1), the assertion follows by Remark 2.1(4). $\square$

Lemma 2.3. *Assume that R is a Gorenstein local ring. Then we have*

$$\text{inj dim } {}_AA = \text{proj dim } \Omega_A + \dim R.$$

Proof. It follows by Lemma 1.13 that we have a minimal injective resolution $\Omega \to \mathrm{Hom}_R(A, I^\bullet)$ in Mod-A. We claim first that inj dim ${}_AA < \infty$ implies proj dim $\Omega_A < \infty$. Assume that inj dim ${}_AA < \infty$. Then by Lemma 1.6 we have flat dim $\mathrm{Hom}_R(A, I^i)_A \le$ inj dim ${}_AA < \infty$ for all $i \ge 0$. It follows that proj dim $\Omega_A =$ flat dim $\Omega_A < \infty$.

Next, assume that proj dim $\Omega_A = m < \infty$. Setting $d = \dim R$, we claim that inj dim ${}_AA = m + d$. Take a projective resolution $P^\bullet \to \Omega$ in mod-A. Then by Remark 2.1(1) we have a right resolution $A \to \mathrm{Hom}^\bullet_R(P^\bullet, R)$ in mod-A^{op}. Also, we have an injective resolution $\mathrm{Hom}_R(P^{-j}, R) \to \mathrm{Hom}^\bullet_R(P^{-j}, I^\bullet)$ in Mod-A^{op} for each $0 \le j \le m$. It follows by Lemma 1.1 that we have an injective resolution $A \to \mathrm{Hom}^\bullet_R(P^\bullet, I^\bullet)$ in Mod-A^{op}, so that inj dim ${}_AA \le m + d$. To prove the equality, it suffices to show that

$$\partial = (\partial' \ \partial'') : \mathrm{Hom}_R(P^{-m}, I^{d-1}) \oplus \mathrm{Hom}_R(P^{-m+1}, I^d) \to \mathrm{Hom}_R(P^{-m}, I^d)$$

is not a split epimorphism. By Lemma 1.13 ∂ is a split epimorphism if and only if so is ∂''. Suppose to the contrary that ∂'' is a split epimorphism. In case R is complete, we have a commutative diagram

$$\begin{tikzcd}
P^{-m} \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_R(\mathrm{Hom}_R(P^{-m}, I^d), I^d) \arrow[d, "\mathrm{Hom}_R(\partial'', I^d)"] \\
P^{-m+1} \arrow[r, "\sim"] & \mathrm{Hom}_R(\mathrm{Hom}_R(P^{-m+1}, I^d), I^d)
\end{tikzcd}$$

and hence $P^{-m} \to P^{-m+1}$ is a split monomorphism, a contradiction. Next, let $\mathfrak{m}$ be the maximal ideal of R and denote by $(\hat{-})$ the $\mathfrak{m}$-adic completion. Then $\mathrm{Ext}^m_{\hat{A}}(\hat{\Omega}, \hat{A}) \cong \mathrm{Ext}^m_A(\Omega, A) \otimes_R \hat{R} \neq 0$, so that proj dim $\hat{\Omega}_{\hat{A}} =$ proj dim Ω_A. Thus inj dim ${}_{\hat{A}}\hat{A} = m + d$ and by Lemma 1.14 there exists a simple module $S \in$ mod-$\hat{A}^{\mathrm{op}}$ with $\mathrm{Ext}^{m+d}_{\hat{A}^{\mathrm{op}}}(S, \hat{A}) \neq 0$. Note that S is an $\hat{R}/\mathfrak{m}\hat{R}$-module. Since $\hat{R}/\mathfrak{m}\hat{R} \cong R/\mathfrak{m}R$, S has finite length as an R-module, so that $S \cong \hat{S}$ and hence $\mathrm{Ext}^{m+d}_{A^{\mathrm{op}}}(S, A) \otimes_R \hat{R} \cong \mathrm{Ext}^{m+d}_{\hat{A}^{\mathrm{op}}}(S, \hat{A}) \neq 0$. Thus $\mathrm{Ext}^{m+d}_{A^{\mathrm{op}}}(S, A) \neq 0$ and inj dim ${}_A A = m + d$. □

Theorem 2.4. *Assume that R is a Gorenstein local ring. Then the following are equivalent:*

(1) inj dim $A_A \leq$ dim $R + 1$.

(2) inj dim ${}_A A \leq$ dim $R + 1$.

Proof. (2) $\Rightarrow$ (1). By Lemma 2.3 proj dim $\Omega_A \leq 1$, so that by Lemma 2.2 proj dim ${}_A\Omega \leq 1$. Thus applying Lemma 2.3 to A^{op} we have inj dim ${}_A A \leq$ dim $R + 1$.

(1) $\Rightarrow$ (2). By symmetry. □

Every ring B derived equivalent to A is a noetherian R-algebra ([21, Proposition 9.4]), but it may happen that $\mathrm{Ext}^i_R(B, R) \neq 0$ for some $i \geq 1$ (see [1, Examle 4.7]).

Corollary 2.5. *Assume that R is a Gorenstein local ring. Let B be a ring derived equivalent to A and with $\mathrm{Ext}^i_R(B, R) = 0$ for $i \neq 0$. If* inj dim $B_B \leq$ dim $R + 1$, *then* inj dim ${}_A A =$ inj dim $A_A < \infty$.

Proof. By [17, Proposition 1.7(2)] inj dim $A_A < \infty$. Next, since by Theorem 2.4 inj dim ${}_B B < \infty$, and since by [21, Proposition 9.1] A^{op} and B^{op} are derived equivalent, again by [17, Proposition 1.7(2)] inj dim ${}_A A < \infty$. The assertion now follows by [23, Lemma A]. □

Lemma 2.6. *For any module $T \in$ mod-A with $\mathrm{Ext}^i_A(T, T) = \mathrm{Ext}^i_R(T, R) = 0$ for $i \neq 0$, setting $B = \mathrm{End}_A(T)$, we have $\mathrm{Ext}^i_R(B, R) = 0$ for $i \neq 0$.*

Proof. Localizing at each $\mathfrak{p} \in \mathrm{Supp}_R(A)$, we may assume that R is a Gorenstein local ring with $d =$ dim R. Take a projective resolution $P^\bullet \to T$ in mod-A and apply $\mathrm{Hom}_A(-, T)$. Then we have a right resolution $B \to T^\bullet$ in mod-B^{op} with $T^i \in \mathrm{add}(T)$ for all $i \geq 0$, so that $\mathrm{Ext}^i_R(B, R) \cong \mathrm{Ext}^{i+d}_R(\mathrm{Z}^d(T^\bullet), R) = 0$ for $i \geq 1$. □

Proposition 2.7. *Let $0 \to K \xrightarrow{g} P \xrightarrow{f} \Omega \to 0$ be an exact sequence in* mod-A. *Set $T = P \oplus K$ and $B = \mathrm{End}_A(T)$. Assume that $P \in \mathrm{add}(\Omega) \cap \mathcal{P}_A$. Then the following hold:*

(1) A and B are derived equivalent.

(2) $\mathrm{Ext}^i_R(B,R)=0$ *for* $i\neq 0$.

Proof. (1) Since $P\in\mathcal{P}_A$, $\mathrm{Hom}_A(P,f)$ is surjective. Also, since $P\in\mathrm{add}(\Omega)$, by Remark 2.1(3) $\mathrm{Hom}_A(g,P)$ is surjective. It follows by [3, Lemma 1.1] that $\mathrm{End}_A(P\oplus\Omega)$ and B are derived equivalent. Finally, $P\in\mathrm{add}(\Omega)$ implies that $\mathrm{End}_A(P\oplus\Omega)$ is Morita equivalent to $\mathrm{End}_A(\Omega)\cong A$.

(2) We claim first that $\mathrm{Ext}^i_A(T,T)=0$ for $i\neq 0$. We have $\mathrm{Ext}^i_A(P,T)=0$ for $i\neq 0$. Applying $\mathrm{Hom}_A(-,P)$, we have $\mathrm{Ext}^i_A(K,P)=0$ for $i\neq 0$. Also, applying $\mathrm{Hom}_A(\Omega,-)$, we have $\mathrm{Ext}^i_A(\Omega,K)=0$ for $i\geq 2$. Thus applying $\mathrm{Hom}_A(-,K)$ we have $\mathrm{Ext}^i_A(K,K)=0$ for $i\neq 0$. Next, applying $\mathrm{Hom}_R(-,R)$, we have $\mathrm{Ext}^i_R(K,R)=0$ for $i\neq 0$, so that $\mathrm{Ext}^i_R(T,R)=0$ for $i\neq 0$. Thus the assertion follows by Lemma 2.6. □

In the proposition above, $T\in\text{mod-}A$ is not a tilting module in general. Also, if proj dim $\Omega_A\leq 1$, then by Lemmas 2.2, 1.3(1) $T\in\text{mod-}A$ is a projective generator, so that B is Morita equivalent to A.

Throughout the rest of this section, we assume that R is a Gorenstein local ring with $d=\dim R$ and that proj dim ${}_A\Omega=$ proj dim $\Omega_A=m<\infty$. Then by Lemma 2.3 inj dim ${}_AA=$ inj dim $A_A=m+d$. Take a projective resolution $P^\bullet\to\Omega$ in mod-A^{op} and set $Q^\bullet=\mathrm{Hom}_R(P^\bullet,R)$. Then we have a right resolution $0\to A\to Q^0\to\cdots\to Q^m\to 0$ in mod-A with $Q^i=\mathrm{Hom}_R(P^{-i},R)\in\mathrm{add}(\Omega)$ for all $i\geq 0$.

Remark 2.8. The following hold:

(1) Every $Q^i\in\text{mod-}R$ is a reflexive module with $\mathrm{Ext}^j_R(\mathrm{Hom}_R(Q^i,R),R)=0$ for $j\neq 0$.

(2) $\oplus_{i\geq 0}\mathrm{Hom}_R(Q^i,R)\in\text{mod-}A^{\mathrm{op}}$ is a projective generator.

(3) proj dim $Q^i<\infty$ in mod-A for all $i\geq 0$.

Proof. Obviously, (3) holds. By Remark 2.1(1) Ω_A has Gorenstein dimension zero, so that (1) holds. Also, by Remark 2.1(4) ${}_A\Omega$ is a tilting module, so that by Lemma 1.3(1) $\oplus_{i\geq 0}P^{-i}\in\text{Mod-}A^{\mathrm{op}}$ is a projective generator. Thus, since $\mathrm{Hom}_R(Q^i,R)\cong P^{-i}$ for all $i\geq 0$, (2) holds. □

In the following, we assume further that R is complete and that $P^\bullet\to\Omega$ is a minimal projective resolution (cf. Lemma 1.12). Let $A\to E^\bullet$ be a minimal injective resolution in Mod-A. Then, since by Lemma 1.1 we have an injective resolution $A\to\mathrm{Hom}^\bullet_R(P^\bullet,I^\bullet)$ in Mod-A, we have

$$\mathrm{Hom}^\bullet_R(P^\bullet,I^\bullet)\cong E^\bullet\oplus(\oplus_{n\geq 0}C(\mathrm{id}_{Z_n})[-n-1]),$$

where $C(\mathrm{id}_{Z_n})$ is the mapping cone of the identity mapping of Z_n which is a direct summand of $\mathrm{Hom}^n_R(P^\bullet,I^\bullet)=\oplus_{i+j=n}\mathrm{Hom}_R(P^{-i},I^j)$.

In the next proposition, the implication (1) ⇒ (2) holds true without the completeness of R.

Proposition 2.9. *The following are equivalent:*

(1) proj dim $Q^i \le i$ *in* mod-A *for all* $i \ge 0$.

(2) flat dim $E^n \le n$ *in* Mod-A *for all* $n \ge 0$.

Proof. (1) $\Rightarrow$ (2). By Lemmas 1.5 and 1.10.

(2) $\Rightarrow$ (1). For any $0 \le i \le m$ and any indecomposable direct summand P of P^{-i}, we claim that $I = \mathrm{Hom}_R(P, I^d) \in \mathrm{add}(E^{d+i})$. Suppose to the contrary that $I \notin \mathrm{add}(E^{d+i})$. Then either $C(\mathrm{id}_I)[-d-i-1] \in \mathrm{add}(\oplus_{n\ge 0} C(\mathrm{id}_{Z_n})[-n-1])$ or $C(\mathrm{id}_I)[-d-i] \in \mathrm{add}(\oplus_{n\ge 0} C(\mathrm{id}_{Z_n})[-n-1])$. Thus by Lemma 1.13 either $C(\mathrm{id}_I)[-i-1] \in \mathrm{add}(\mathrm{Hom}_R^\bullet(P^\bullet, I^d))$ or $C(\mathrm{id}_I)[-i] \in \mathrm{add}(\mathrm{Hom}_R^\bullet(P^\bullet, I^d))$, so that either $C(\mathrm{id}_P)[i+1] \in \mathrm{add}(P^\bullet)$ or $C(\mathrm{id}_P)[i] \in \mathrm{add}(P^\bullet)$, which contradicts to the minimality of $P^\bullet$. Thus for any $i \ge 0$ we have $\mathrm{Hom}_R(P^{-i}, I^d) \in \mathrm{add}(E^{d+i})$ and flat dim $\mathrm{Hom}_R(P^{-i}, I^d)_A \le d + i$. Since by Lemma 1.5 we have $\mathrm{Hom}_R(P^{-i}, I^d) \cong I^d \otimes_R Q^i$, it suffices to show that flat dim $I^d \otimes_R Q^i = d +$ flat dim Q^i in Mod-A. Set $r =$ flat dim Q^i and $J = \mathrm{rad}(A)$, the Jacobson radical of A. By Lemma 1.10 we have flat dim $I^d \otimes_R Q^i \le d + r$. Take minimal projective resolutions $Q'^\bullet \to Q^i$ in mod-A and $P'^\bullet \to A/J$ in mod-A^{op}. We have $\mathrm{Tor}_r^A(Q^i, A/J) \cong Q'^{-r} \otimes_A A/J \ne 0$. Also, we have an exact sequence

$$0 \to \mathrm{Tor}_r^A(Q^i, A/J) \to \mathrm{Z}'^{-r}(Q^i \otimes_A P'^\bullet) \to \mathrm{B}^{-r+1}(Q^i \otimes_A P'^\bullet) \to 0$$

and hence, applying $\mathrm{Hom}_R(-, R)$, we have an epimorphism

$$\mathrm{Ext}_R^d(\mathrm{Z}'^{-r}(Q^i \otimes_A P'^\bullet), R) \to \mathrm{Ext}_R^d(\mathrm{Tor}_r^A(Q^i, A/J), R).$$

Since $\mathrm{Tor}_r^A(Q^i, A/J)$ is semisimple as an R-module, $\mathrm{Ext}_R^d(\mathrm{Tor}_r^A(Q^i, A/J), R) \cong \mathrm{Tor}_r^A(Q^i, A/J) \ne 0$. Note that we have a left resolution in mod-R

$$\cdots \to Q^i \otimes_A P'^{-r-1} \to Q^i \otimes_A P'^{-r} \to \mathrm{Z}'^{-r}(Q^i \otimes_A P'^\bullet) \to 0.$$

Since by Remark 2.1(1) $\mathrm{Tor}_k^R(I^d, \Omega) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^k(\Omega, R), I^d) = 0$ for $k \ne 0$, for any $j \ge r$ we have $\mathrm{Tor}_k^R(I^d, Q^i \otimes_A P'^{-j}) = 0$ for $k \ne 0$ and hence

$$\begin{aligned}
\mathrm{Tor}_{d+r}^A(I^d \otimes_R Q^i, A/J) &\cong \mathrm{H}^{-d-r}(I^d \otimes_R^\bullet Q^i \otimes_A^\bullet P'^\bullet) \\
&\cong \mathrm{Tor}_d^R(I^d, \mathrm{Z}'^{-r}(Q^i \otimes_A P'^\bullet)) \\
&\cong \mathrm{Hom}_R(\mathrm{Ext}_R^d(\mathrm{Z}'^{-r}(Q^i \otimes_A P'^\bullet), R), I^d) \\
&\ne 0,
\end{aligned}$$

so that flat dim $I^d \otimes_R Q^i = d + r$. $\square$

In the proposition above, the condition (2) is left-right symmetric (see Proposition 3.1 below) and hence so is the condition (1).

3 Auslander-Gorenstein resolution

In this section, formulating Remark 2.8 and Proposition 2.9, we will introduce the notion of Auslander-Gorenstein resolution and show that a noetherian ring is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over another Auslander-Gorenstein ring.

We start by recalling the Auslander condition. In the following, Λ stands for an arbitrary noetherian ring.

Proposition 3.1 (Auslander). *For any $n \geq 0$ the following are equivalent:*

(1) In a minimal injective resolution $\Lambda \to I^\bullet$ in Mod-Λ*,* flat dim $I^i \leq i$ *for all $0 \leq i \leq n$.*

(2) In a minimal injective resolution $\Lambda \to J^\bullet$ in Mod-Λ^{op}*,* flat dim $J^i \leq i$ *for all $0 \leq i \leq n$.*

(3) For any $1 \leq i \leq n+1$, any $M \in$ mod-Λ *and any submodule X of* $\mathrm{Ext}^i_\Lambda(M, \Lambda) \in$ mod-Λ^{op} *we have* $\mathrm{Ext}^j_{\Lambda^{\mathrm{op}}}(X, \Lambda) = 0$ *for all $0 \leq j < i$.*

(4) For any $1 \leq i \leq n+1$, any $X \in$ mod-Λ^{op} *and any submodule M of* $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(X, \Lambda) \in$ mod-Λ *we have* $\mathrm{Ext}^j_\Lambda(M, \Lambda) = 0$ *for all $0 \leq j < i$.*

Proof. See e.g. [15, Theorem 3.7]. □

Definition 3.2 ([11]). We say that Λ satisfies the Auslander condition if it satisfies the equivalent conditions in Proposition 3.1 for all $n \geq 0$, and that Λ is an Auslander-Gorenstein ring if inj dim ${}_\Lambda\Lambda =$ inj dim $\Lambda_\Lambda < \infty$ and if it satisfies the Auslander condition.

Definition 3.3. We denote by $\mathcal{G}_\Lambda$ the full subcategory of mod-Λ consisting of reflexive modules $M \in$ mod-Λ with $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_\Lambda(M, \Lambda), \Lambda) = 0$ for $i \neq 0$.

Throughout the rest of this section, R and A are noetherian rings. We do not require the existence of a ring homomorphism $R \to A$. Also, even if we have a ring homomorphism $R \to A$ with R commutative, the image of which may fail to be contained in the center of A (cf. [2]).

Definition 3.4. A right resolution $0 \to A \to Q^0 \to \cdots \to Q^m \to 0$ in Mod-A is said to be a Gorenstein resolution of A over R if the following conditions are satisfied:

(1) Every Q^i is an R-A-bimodule.

(2) $Q^i \in \mathcal{G}_{R^{\mathrm{op}}}$ in Mod-R^{op} for all $i \geq 0$.

(3) $\oplus_{i \geq 0} \mathrm{Hom}_{R^{\mathrm{op}}}(Q^i, R) \in$ Mod-A^{op} is faithfully flat.

(4) flat dim $Q^i < \infty$ in Mod-A for all $i \geq 0$.

Definition 3.5. A Gorenstein resolution $0 \to A \to Q^0 \to \cdots \to Q^m \to 0$ of A over R is said to be an Auslander-Gorenstein resolution if the following stronger condition is satisfied:

(4)′ flat dim $Q^i \leq i$ in Mod-A for all $i \geq 0$.

Theorem 3.6. *Assume that A admits a Gorenstein resolution*

$$0 \to A \to Q^0 \to \cdots \to Q^m \to 0$$

over R and that inj dim ${}_R R =$ inj dim $R_R = d < \infty$*. Then the following hold:*

(1) For an injective resolution $R \to I^\bullet$ in Mod-R *we have an injective resolution $A \to E^\bullet$ in* Mod-A *such that*

$$E^n = \bigoplus_{i+j=n} I^j \otimes_R Q^i$$

for all $n \geq 0$. In particular, inj dim ${}_A A =$ inj dim $A_A \leq m + d$ *and*

$$\text{flat dim } E^n \leq \sup\{\text{flat dim } I^j + \text{flat dim } Q^i \mid i + j = n\}$$

for all $n \geq 0$.

(2) If R is an Auslander-Gorenstein ring, and if $A \to Q^\bullet$ is an Auslander-Gorenstein resolution, then A is an Auslander-Gorenstein ring.

Proof. (1) For each $0 \leq i \leq m$, since $Q^i \in \mathcal{G}_{R^{\mathrm{op}}}$ and $\mathrm{Hom}_{R^{\mathrm{op}}}(Q^i, R) \in \text{Mod-}A^{\mathrm{op}}$ is flat, and since by Lemma 1.5 $\mathrm{Hom}^\bullet_R(\mathrm{Hom}_{R^{\mathrm{op}}}(Q^i, R), I^\bullet) \cong I^\bullet \otimes^\bullet_R Q^i$ as complexes over Mod-A, we have an injective resolution $Q^i \to I^\bullet \otimes^\bullet_R Q^i$ in Mod-A. Thus by Lemma 1.1 we have an injective resolution $A \to E^\bullet$ with $E^n = \oplus_{i+j=n} I^j \otimes_R Q^i$ for all $n \geq 0$. In particular, inj dim $A_A \leq m + d$. Also, by Lemma 1.10 flat dim $E^n \leq \sup\{\text{flat dim } I^j + \text{flat dim } Q^i \mid i + j\} < \infty$. It only remains to see that inj dim ${}_A A =$ inj dim A_A. By Lemma 1.6, it suffices to show that $\oplus_{n \geq 0} E^n \in \text{Mod-}A$ is an injective cogenerator. Let $M \in \text{Mod-}A$ with $\mathrm{Hom}_A(M, I^j \otimes_R Q^i) = 0$ for all i, j. Note that for any i, j we have

$$\begin{aligned}
\mathrm{Hom}_R(M \otimes_A \mathrm{Hom}_{R^{\mathrm{op}}}(Q^i, R), I^j) &\cong \mathrm{Hom}_A(M, \mathrm{Hom}_R(\mathrm{Hom}_{R^{\mathrm{op}}}(Q^i, R), I^j)) \\
&\cong \mathrm{Hom}_A(M, I^j \otimes_R Q^i) \\
&= 0
\end{aligned}$$

and that by Corollary 1.9 $\oplus_{j \geq 0} I^j \in \text{Mod-}R$ is an injective cogenerator. Thus $M \otimes_A \mathrm{Hom}_{R^{\mathrm{op}}}(Q^i, R) = 0$ for all i and hence, since $\oplus_{i \geq 0} \mathrm{Hom}_{R^{\mathrm{op}}}(Q^i, R)$ is faithfully flat, we have $M = 0$.

(2) We have flat dim $I^j +$ flat dim $Q^i \leq i + j$ for all i, j. □

In case $m = 0$, a Gorenstein resolution of A over R is just an R-A-bimodule Q such that $Q \cong A$ in Mod-A, $Q \in \mathcal{G}_{R^{\mathrm{op}}}$ in Mod-R^{op} and $\mathrm{Hom}_{R^{\mathrm{op}}}(Q, R) \in \text{Mod-}A^{\mathrm{op}}$ is faithfully flat. In particular, if A is a Frobenius extension of R in the sense of [2], then both A itself and $\mathrm{Hom}_R(A, R)$ are Gorenstein resolutions of A over R, where $A \cong \mathrm{Hom}_R(A, R)$ in Mod-A but $A \not\cong \mathrm{Hom}_R(A, R)$ as R-A bimodules in general.

4 Examples

In this section, we will provide several examples of Auslander-Gorenstein resolution.

Example 4.1. Let R be a commutative noetherian ring and A a noetherian R-algebra such that $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$ and $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq 0$. Set $\Omega = \mathrm{Hom}_R(A, R)$ and assume that Ω admits a projective resolution $0 \to P^{-1} \to P^0 \to \Omega \to 0$ in mod-A^{op} with $P^0 \in \mathrm{add}(\Omega)$. Then applying $\mathrm{Hom}_R(-, R)$ we have a right resolution $0 \to A \to Q^0 \to Q^1 \to 0$ in mod-A with $Q^0 \in \mathrm{add}(\Omega)$, where $Q^i = \mathrm{Hom}_R(P^{-i}, R)$ for $0 \leq i \leq 1$, which must be an Auslander-Gorenstein resolution of A over R because by Lemmas 2.2 and 1.3(1) $P^0 \oplus P^{-1} \in$ mod-A^{op} is a projective generator.

Example 4.2 (cf. [6])**.** Let R be a complete Gorenstein local ring of dimension one and Λ a noetherian R-algebra with $\mathrm{Ext}^i_R(\Lambda, R) = 0$ for $i \neq 0$. Denote by $\mathcal{L}_\Lambda$ the full subcategory of mod-Λ consisting of modules X with $\mathrm{Ext}^i_R(X, R) = 0$ for $i \neq 0$. It should be noted that $\mathcal{L}_\Lambda$ is closed under submodules. Assume that $\mathcal{L}_\Lambda = \mathrm{add}(M)$ with $M \in$ mod-Λ non-projective and set $A = \mathrm{End}_\Lambda(M)$. Since A is a subalgebra of $\mathrm{End}_R(M)$, and since $\mathrm{End}_R(M)$ is embedded in a finite direct sum of copies of M, we have $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq 0$.

We claim first that gl dim $A = 2$. Set $F = \mathrm{Hom}_\Lambda(M, -) : \mathcal{L}_\Lambda \xrightarrow{\sim} \mathcal{P}_A$ and $S_X = FX/\mathrm{rad}(FX)$ for each indecomposable $X \in \mathcal{L}_\Lambda$. If $X \in \mathcal{P}_\Lambda$, then we have an exact sequence

$$0 \to F(\mathrm{rad}(X)) \to FX \to S_X \to 0,$$

so that proj dim $S_X \leq 1$. Assume that $X \notin \mathcal{P}_\Lambda$. There exists $f : Y \to X$ in $\mathcal{L}_\Lambda$ such that $FY \xrightarrow{Ff} FX \to S_X \to 0$ is a minimal projective presentation. Thus, setting $Z = \mathrm{Ker}\ f \in \mathcal{L}_\Lambda$, we have an exact sequence

$$0 \to FZ \xrightarrow{Fg} FY \xrightarrow{Ff} FX \to S_X \to 0$$

and proj dim $S_X \leq 2$. Since $\mathrm{Hom}_A(F\Lambda, S_X) = 0$, $\mathrm{Hom}_A(F\Lambda, Ff)$ is surjective and so is $\mathrm{Hom}_\Lambda(\Lambda, f)$. Thus f is an epimorphism. If proj dim $S_X \leq 1$, then Fg is a split monomorphism and so is g, so that f is a split epimorphism and so is Ff, a contradiction.

Next, set $D = \mathrm{Hom}_R(-, R)$ and $\Omega = DA$. It then follows by Lemmas 2.2 and 2.3 that Ω_A is a tilting module with proj dim $\Omega_A =$ proj dim ${}_A\Omega = 1$. Take a minimal projective presentation $P^{-1} \to P^0 \to DM \to 0$ in mod-Λ^{op}. Applying $F \circ D$, we have an exact sequence in mod-A

$$0 \to A \to F(DP^{-1}) \xrightarrow{f} F(DP^0).$$

We have $D(M \otimes_\Lambda P^{-1}) \cong F(DP^{-1})$ and hence $DF(DP^{-1}) \cong M \otimes_\Lambda P^{-1} \cong \mathrm{Hom}_\Lambda(\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P^{-1}, \Lambda), M) \in \mathcal{P}_{A^{\mathrm{op}}}$. Thus, setting $Q^0 = F(DP^{-1})$ and $Q^1 = \mathrm{Im}\ f$, we have an Auslander-Gorenstein resolution of A over R.

Throughout the rest of this section, R stands for an arbitrary noetherian ring. We refer to [5, Chapter II] for the way to construct an extension ring A of R by a quiver with relations.

Example 4.3. Let $n \geq 2$ be an integer and $A = \mathrm{T}_n(R)$, the ring of $n \times n$ upper triangular matrices over R. Namely, A is a free right R-module with a basis $\mathfrak{B} = \{e_{ij} \mid 1 \leq i \leq j \leq n\}$ and the multiplication in A is defined subject to the following axioms: (A1) $e_{ij}e_{kl} = 0$ unless $j = k$ and $e_{ij}e_{jk} = e_{ik}$ for all $i \leq j \leq k$; and (A2) $xv = vx$ for all $x \in R$ and $v \in \mathfrak{B}$. Set $e_i = e_{ii}$ for all i. Then A is a noetherian ring with $1 = \Sigma_{i=1}^{n} e_i$, where the e_i are orthogonal idempotents. We consider R as a subring of A via the injective ring homomorphism $\varphi : R \to A, x \mapsto x1$. Denote by $\mathfrak{B}^* = \{e_{ij}^* \mid 1 \leq i \leq j \leq n\}$ the dual basis of $\mathfrak{B}$ for the left R-module $\mathrm{Hom}_R(A, R)$, i.e., we have $a = \Sigma_{v \in \mathfrak{B}} v v^*(a)$ for all $a \in A$. It is not difficult to check the following:

(1) $e_1A \xrightarrow{\sim} \mathrm{Hom}_R(Ae_n, R), a \mapsto e_{1n}^* a$ as R-A-bimodules.

(2) For each $2 \leq i \leq n$, setting $f : e_1A \to \mathrm{Hom}_{R^{\mathrm{op}}}(Ae_{i-1}, R), a \mapsto e_{1,i-1}^* a$ and $g : e_iA \to e_1A, a \mapsto e_{1i}a$, we have an exact sequence of R-A-bimodules

$$0 \to e_iA \xrightarrow{g} e_1A \xrightarrow{f} \mathrm{Hom}_R(Ae_{i-1}, R) \to 0.$$

(3) $\mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Hom}_R(Ae_i, R), R) \cong Ae_i$ as A-R-bimodules for all $1 \leq i \leq n$.

Consequently, we have an exact sequence of R-A-bimodules

$$0 \to A \to \overset{n}{\oplus} e_1A \to \overset{n}{\underset{i=2}{\oplus}} \mathrm{Hom}_R(Ae_{i-1}, R) \to 0,$$

which is an Auslander-Gorenstein resolution of A over R.

Example 4.4. Define a ring A by a quiver

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "\alpha"] & \overset{2}{\bullet} \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\gamma"] & \overset{3}{\bullet} \arrow[l, shift left, "\delta"] \end{tikzcd}$$

with relations $\alpha\beta = 0$, $\alpha\gamma = 0$, $\delta\gamma = 0$, $\delta\beta = 0$ and $\beta\alpha - \gamma\delta = 0$ over R. Namely, A is a free left R-module with a basis $\mathfrak{B} = \{e_1, e_2, e_3, \alpha, \beta, \gamma, \delta, w\}$ and the multiplication in A is defined subject to the following axioms: (A1) $e_ie_j = 0$ unless $i = j$ and $e_ie_i = e_i$ for all i; (A2) $\alpha = e_1\alpha e_2$, $\beta = e_2\beta e_1$, $\gamma = e_2\gamma e_3$ and $\delta = e_3\delta e_2$; (A3) $\alpha\beta = \alpha\gamma = \delta\beta = \delta\gamma = 0$ and $w = \beta\alpha = \gamma\delta$; and (A4) $xv = vx$ for all $x \in R$ and $v \in \mathfrak{B}$. It is not difficult to see that A is a noetherian ring with $1 = \Sigma_{i=1}^{3} e_i$, where the e_i are orthogonal idempotents. We consider R as a subring of A via the injective ring homomorphism $\varphi : R \to A, x \mapsto x1$. Set $\Omega = \mathrm{Hom}_{R^{\mathrm{op}}}(A, R)$ and denote by $\mathfrak{B}^* = \{e_1^*, e_2^*, e_3^*, \alpha^*, \beta^*, \gamma^*, \delta^*, w^*\}$ the dual basis of $\mathfrak{B}$ for the right R-module Ω, i.e., we have $a = \Sigma_{v \in \mathfrak{B}} v^*(a)v$ for all $a \in A$. We have $\Omega \cong \oplus_{i=1}^{3} \mathrm{Hom}_{R^{\mathrm{op}}}(e_iA, R)$ as A-R-bimodules and the following hold:

(1) $Ae_2 \xrightarrow{\sim} \mathrm{Hom}_{R^{\mathrm{op}}}(e_2A, R), a \mapsto aw^*$ as A-R-bimodules.

(2) Set $f : Ae_2 \to \mathrm{Hom}_{R^{\mathrm{op}}}(e_1A, R), a \mapsto a\alpha^*$ and $g : Ae_3 \to Ae_2, a \mapsto a\delta$. Then we have an exact sequence of A-R-bimodules

$$0 \to Ae_3 \xrightarrow{g} Ae_2 \xrightarrow{f} \mathrm{Hom}_{R^{\mathrm{op}}}(e_1A, R) \to 0.$$

(3) Set $f' : Ae_2 \to \mathrm{Hom}_{R^{\mathrm{op}}}(e_3A, R), a \mapsto a\delta^*$ and $g' : Ae_1 \to Ae_2, a \mapsto a\alpha$. Then we have an exact sequence of A-R-bimodules

$$0 \to Ae_1 \xrightarrow{g'} Ae_2 \xrightarrow{f'} \mathrm{Hom}_{R^{\mathrm{op}}}(e_3A, R) \to 0.$$

(4) $e_iA \cong \mathrm{Hom}_R(\mathrm{Hom}_{R^{\mathrm{op}}}(e_iA, R), R)$ as R-A-bimodules for all $1 \le i \le 3$.

Consequently, we have an exact sequence of A-R-bimodules

$$0 \to Ae_1 \oplus Ae_3 \to \overset{3}{\oplus} Ae_2 \to \Omega \to 0$$

and applying $\mathrm{Hom}_R(-, R)$ we have an exact sequence of R-A-bimodules

$$0 \to A \to \overset{3}{\oplus} e_2A \to \mathrm{Hom}_R(Ae_1, R) \oplus \mathrm{Hom}_R(Ae_3, R) \to 0,$$

which is an Auslander-Gorenstein resolution of A over R.

Example 4.5. Define a ring A by a quiver

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, "\alpha"] & \overset{2}{\bullet} \arrow[r, "\beta"] & \overset{3}{\bullet} \arrow[ll, bend left, "\gamma"] \end{tikzcd}$$

with a relation $\gamma\alpha = 0$. Namely, A is a free left R-module with a basis $\mathfrak{B} = \{e_1, e_2, e_3, \alpha, \beta, \gamma, v_{13}, v_{21}, w\}$ and the multiplication in A is defined by the following axioms: (A1) $e_ie_j = 0$ unless $i = j$ and $e_ie_i = e_i$ for all i; (A2) $\alpha = e_1\alpha e_2$, $\beta = e_2\beta e_1$ and $\gamma = e_2\gamma e_3$; (A3) $\gamma\alpha = 0$, $\alpha\beta = v_{13}$, $\beta\gamma = v_{21}$ and $w = \alpha\beta\gamma$; and (A4) $xv = vx$ for all $x \in R$ and $v \in \mathfrak{B}$. It is not difficult to see that A is a noetherian ring with $1 = \Sigma_{i=1}^3 e_i$, where the e_i are orthogonal idempotents. We consider R as a subring of A via the injective ring homomorphism $\varphi : R \to A, x \mapsto x1$. Let $\Omega = \mathrm{Hom}_{R^{\mathrm{op}}}(A, R)$ and denote by $\mathfrak{B}^* = \{e_1^*, e_2^*, e_3^*, \alpha^*, \beta^*, \gamma^*, v_{13}^*, v_{21}^*, w^*\}$ the dual basis of $\mathfrak{B}$ for the right R-module Ω, i.e., we have $a = \Sigma_{v \in \mathfrak{B}} v^*(a)v$ for all $a \in A$. We have $\Omega \cong \oplus_{i=1}^3 \mathrm{Hom}_{R^{\mathrm{op}}}(e_iA, R)$ as A-R-bimodules and the following hold:

(1) $Ae_1 \xrightarrow{\sim} \mathrm{Hom}_{R^{\mathrm{op}}}(e_1A, R), a \mapsto aw^*$ as A-R-bimodules.

(2) Set $f : Ae_1 \to \mathrm{Hom}_{R^{\mathrm{op}}}(e_2A, R), a \mapsto av_{21}^*$, $g : Ae_1 \to Ae_1, a \mapsto aw$ and $h : Ae_3 \to Ae_1, a \mapsto a\gamma$. Then we have an exact sequence of A-R-bimodules

$$0 \to Ae_3 \xrightarrow{h} Ae_1 \xrightarrow{g} Ae_1 \xrightarrow{f} \mathrm{Hom}_{R^{\mathrm{op}}}(e_2A, R) \to 0.$$

(3) Set $f' : Ae_1 \to \mathrm{Hom}_{R^{\mathrm{op}}}(e_3A, R), a \mapsto a\gamma^*$ and $g' : Ae_2 \to Ae_1, a \mapsto av_{21}$. Then we have an exact sequence of A-R-bimodules

$$0 \to Ae_2 \xrightarrow{g'} Ae_1 \xrightarrow{f'} \mathrm{Hom}_{R^{\mathrm{op}}}(e_3A, R) \to 0.$$

(4) $e_iA \xrightarrow{\sim} \mathrm{Hom}_R(\mathrm{Hom}_{R^{\mathrm{op}}}(e_iA, R), R)$ as R-A-bimodules for all $1 \leq i \leq 3$.

Consequently, we have an exact sequence of A-R-bimodules

$$0 \to Ae_3 \to Ae_1 \oplus Ae_2 \to \overset{3}{\oplus} Ae_1 \to \Omega \to 0$$

and applying $\mathrm{Hom}_R(-, R)$ we have an exact sequence of R-A-bimodules

$$0 \to A \to \overset{3}{\oplus} e_1A \to e_1A \oplus \mathrm{Hom}_R(Ae_2, R) \to \mathrm{Hom}_R(Ae_3, R) \to 0,$$

which is an Auslander-Gorenstein resolution of A over R.

Example 4.6. Let $n \geq 3$ be an integer and define a ring A by a quiver

$$\begin{array}{ccccccc} 1 & & 2 & & & & n \\ \bullet & \xrightarrow{\alpha_1} & \bullet & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-1}} & \bullet \end{array}$$

with relations $\alpha_i\alpha_{i+1} = 0$ for $1 \leq i < n-1$. Namely, A is a free left R-module with a basis $\mathfrak{B} = \{e_1, e_2, \cdots, e_n, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$ and the multiplication in A is defined by the following axioms: (A1) $e_ie_j = 0$ unless $i = j$ and $e_ie_i = e_i$ for all i; (A2) $\alpha_i = e_i\alpha_ie_{i+1}$ for all i; (A3) $\alpha_i\alpha_{i+1} = 0$ for all i; and (A4) $xv = vx$ for all $x \in R$ and $v \in \mathfrak{B}$. It is not difficult to see that A is a noetherian ring with $1 = \Sigma_{i=1}^n e_i$, where the e_i are orthogonal idempotents. We consider R as a subring of A via the injective ring homomorphism $\varphi : R \to A, x \mapsto x1$. Set $\Omega = \mathrm{Hom}_{R^{\mathrm{op}}}(A, R)$ and denote by $\mathfrak{B}^* = \{e_1^*, e_2^*, \cdots, e_n^*, \alpha_1^*, \alpha_2^*, \cdots, \alpha_{n-1}^*\}$ the dual basis of $\mathfrak{B}$ for the right R-module Ω, i.e., we have $a = \Sigma_{v \in \mathfrak{B}} v^*(a)v$ for all $a \in A$. We have $\Omega \cong \oplus_{i=1}^n \mathrm{Hom}_{R^{\mathrm{op}}}(e_iA, R)$ as A-R-bimodules and the following hold:

(1) $Ae_{i+1} \xrightarrow{\sim} \mathrm{Hom}_{R^{\mathrm{op}}}(e_iA, R), a \mapsto a\alpha_i^*$ as A-R-bimodules for all $1 \leq i < n$.

(2) Set $f : Ae_n \to \mathrm{Hom}_{R^{\mathrm{op}}}(e_nA, R), a \mapsto ae_n^*$ and $g_i : Ae_i \to Ae_{i+1}, a \mapsto a\alpha_i$ for $1 \leq i < n$. Then we have an exact sequence of A-R-bimodules

$$0 \to Ae_1 \xrightarrow{g_1} Ae_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} Ae_n \xrightarrow{f} \mathrm{Hom}_{R^{\mathrm{op}}}(e_nA, R) \to 0.$$

(3) $e_iA \cong \mathrm{Hom}_R(\mathrm{Hom}_{R^{\mathrm{op}}}(e_iA, R), R)$ as R-A-bimodules for all $1 \leq i \leq n$.

Consequently, we have an exact sequence of A-R-bimodules

$$0 \to Ae_1 \to Ae_2 \to \cdots \to (\overset{n}{\underset{i=2}{\oplus}} Ae_i) \oplus Ae_n \to \Omega \to 0$$

and applying $\mathrm{Hom}_R(-, R)$ we have an exact sequence of R-A-bimodules

$$0 \to A \to (\overset{n-1}{\underset{i=1}{\oplus}} e_iA) \oplus e_{n-1}A \to e_{n-2}A \to \cdots \to e_1A \to \mathrm{Hom}_R(Ae_1, R) \to 0,$$

which is an Auslander-Gorenstein resolution of A over R.

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