A GENERALIZED CAUCHY PROCESS HAVING CUBIC NONLINEARITY

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A generalized Cauchy process with a cubic nonlinear term (a nonlinear friction) is studied under the influence of independent multiplicative and additive Gaussian-white noises. Three methods of parameter estimation (i.e., the maximum likelihood, the moment and the log-amplitude moment) are presented in detail. The effect of nonlinearity-noise interplay associated with the nonlinear friction under the influences of both multiplicative and additive noises are discussed in conjunction with fluctuation-dissipation theorem. The physical significance of nonlinear friction is demonstrated with the use of time series data in economics and fluid turbulence.

Keywords: generalized Cauchy process, cubic nonlinear term, methods of parameter estimation, nonlinearity-noise interplay, physical significance

1 Introduction

Fat-tailed distributions and their related stochastic processes have been discussed extensively in physics, econophysics, biology and others in the past decades (cf.[1-8] and references cited therein). It is known that there are numbers of fat-tailed distributions such as (i) Lévy [1], (ii) stretched exponential [2], (iii) multiplicative log-normal [3,4] and (iv) generalized Cauchy distributions [5,6]. It is an interesting observation that a class of generalized Cauchy distributions can be derived from Tsallis statistics based on a theory of generalized entropy [7,8]. These distributions are utilized to analyze time series data from fluid turbulence, heart beat, stock price and so on. In spite of numerous studies, the role of nonlinearity-noise interplay or nonlinear mechanism are not elucidated completely.

In a previous paper [6], we have studied a class of generalized Cauchy process with two independent/dependent random forces $F_p(t)$ and $F_a(t)$ in the following simple form:

$$\frac{d}{dt}x = -\alpha x + D_p x + xF_p(t) + F_a(t) \ . \tag{1}$$

The maximum likelihood estimators (MLEs) with the SDE (1) are derived with the aid of the method of information geometry [9] in statistics. They are free from the mathematical divergence even in the parameter range where the ordinary moments $\langle x^m \rangle (m = 1, 2, \ldots)$ diverge. Then, availability of the analytical expressions to infer the system parameters have been discussed. The model in eq.(1) is one of the most simple stochastic processes which dose not involve explicit nonlinear terms among the class of generalized Cauchy processes without memory. Various generalizations are possible by introducing a correlation among the noises [6], independent/dependent colored noises [10,11,12], nonlinear terms (nonlinear frictions) [10,11,12] into model (1). In references [10-12], they are discussing a crossover phenomenon (stochastic bifurcation [13], specifically, profile change of the probability density function (pdf) with a double-peaked structure to one with a single-peaked structure) in a double-well potential system under the influence of independent colored noises and/or the cross-correlated colored noises. In their analyses, approximate analytic pdfs are compared with direct Langevin simulations.

In this paper, a model with a cubic nonlinear term (i.e., a nonlinear friction) is studied to see (a) effects of nonlinearity-noise interplay, and to give (b) methods of the parameter estimation for the generalized Cauchy process under the incorporation of a cubic nonlinearity. The paper is organized as follows: Section 2 describes the model equations associated with a generalized Cauchy process (GCP). The analytic expression of the probability density function (pdf) and the profile of the pdf are displayed. Section 3 describes the method of maximum likelihood to infer the parameters of the pdf. Section 4 presents the methods of moment (the ordinary moment and the log-amplitude moment) for estimating parameters. Section 5 discusses (i) the canonical model for the GCP, (ii) the generalized fluctuation-dissipation theorem and (iii) some realistic examples to exhibit the significance of our new model with a nonlinear friction. The final section is devoted to the concluding remarks.

2 Generalized Cauchy Process

2.1 Langevin equation

Let us consider a generalized Cauchy process with a cubic nonlinear term as

$$\frac{d}{dt}x = -\alpha x - \gamma x^3 + D_p x + xF_p(t) + F_a(t) \ , \tag{2}$$

where α and γ are assumed to be positive constants ($\alpha > 0$ and $\gamma > 0$), and the two noises are independent with null mean, $\langle F_p(t) \rangle = \langle F_a(t) \rangle = 0$, and they have the Gaussian-white nature,

$$\langle F_a(t)F_a(t') \rangle = 2D_a\delta(t-t'),\ \langle F_p(t)F_p(t') \rangle = 2D_p\delta(t-t') \text{ and } \langle F_p(t)F_a(t') \rangle = 0 \ . \tag{3}$$

The statistical average of random variable A is denoted by $\langle A \rangle$. The Storatonovich interpretation (SI) is adopted for all Langevin equations throughout the paper. The Langevin equation having the two noise source $F_p(t)$ and $F_a(t)$ in eq.(2) is equivalent to a model with one multiplicative noise $F_m(t)$ in the form:

$$\frac{d}{dt}x = -\alpha x - \gamma x^3 + D_p x + \sqrt{2(D_p x^2 + D_a)} F_m(t) \ , \tag{2'}$$

where $F_m(t)$ is null mean $\langle F_m(t) \rangle = 0$, and it has a Gaussian-white nature, $\langle F_m(t) F_m(t') \rangle = \delta(t - t')$. Equivalence of eq.(2) and eq.(2′) without the nonlinear friction is already shown in our previous paper [6]. This property originate from the fact that Gaussian random processes with null mean are ascribed to the class of same martingale.

2.2 Fokker-Planck equation

It is readily seen that the Fokker-Planck (FP) equation for the generalized Cauchy process in eq.(2) and the one with one multiplicative noise $F_m(t)$ in eq.(2′) reduces to the same FP equation as

$$\frac{\partial}{\partial t} P(x,t) = \frac{\partial}{\partial x}[K(x) P(x,t)] + \frac{\partial^2}{\partial x^2}[D(x) P(x,t)] \ , \tag{4}$$

where the regression term $K(x)$ and the diffusion term $D(x)$ are expressed after Ito-Wong-Zakai noise correction due to the SI by

$$K(x) = (\alpha - D_p)x + \gamma x^3 \ \text{ and } \ D(x) = D_a + D_p x^2 \ . \tag{5}$$

2.3 Probability density function

The formal solution of the probability density function (pdf) $P_s(x)$ at the steady state in eq.(4) is given by

$$P_s(x) = P_0 [D(x)]^{-1} \exp\left[- \int \frac{K(x)}{D(x)} dx \right] \ , \tag{6}$$

where P_0 is the normalization constant. The integration in eq.(6) can be done explicitly for the functions $K(x)$ and $D(x)$ given in eq.(5). The pdf is reduced to a Gaussian-Cauchy hybrid distribution:

$$P_s(x) = \frac{c^{\frac{1}{2}-b}}{\Gamma(\frac{1}{2}, b; ca^2)} \cdot \frac{\exp\big(- cx^2 \big)}{(x^2 + a^2)^b} \ , \tag{7}$$

where $\Gamma(\frac{1}{2}, b; ca^2)$ is a generalized Gamma function defined by

$$\Gamma(z, \lambda; v) \equiv \int_0^\infty \exp(-t) \cdot t^{z-1} \cdot (t+v)^{-\lambda} dt \ (z > 0, \lambda > 0, v > 0). \tag{8}$$

The properties of the generalized Gamma function is described in Appendix. It is suitable to express the moments and the log-amplitude moments which will be shown in sections 4 and 5. Analytical continuity of a generalized Gamma function is studied in detail by Kobayashi [15]. The generalized Gamma function in eq.(8) is proposed by Agarwal and Kalla [16], and Gupta and Ong [17].

The three parameters a, b and c in the pdf in eq.(7) are related to the parameters of the Langevin equation (2) as

$$a^2 \equiv \frac{D_a}{D_p}, \quad b \equiv \frac{\alpha}{2D_p} + \frac{1}{2} - \frac{D_a \gamma}{2D_p^2} \quad \text{and} \quad c = \frac{\gamma}{2D_p} \ . \tag{9}$$

It is shown that the pdf can also be written as

$$P_s(x) = \frac{a^{2b-1}}{B(b-1/2, 1/2; ca^2)} \cdot \frac{\exp\big(-cx^2\big)}{(x^2+a^2)^b} \ , \tag{10}$$

in terms of a generalized Beta function defined by

$$B(\mu,\nu;w) \equiv \int_0^1 \exp\bigg\{-w\bigg(\frac{1-t}{t}\bigg)\bigg\} t^{\mu-1}(1-t)^{\nu-1} dt \ (\mu>0, \nu>0, w>0). \tag{11}$$

In the limit $c \to 0$, the generalized Beta function $B(b-1/2, 1/2; ca^2)$ in eq.(11) reduces to the ordinary Beta function $B(b-1/2, 1/2)$. It seems that the pdf in terms of $B(\mu,\nu;w)$ is a natural extension of the case without the nonlinear term (i.e., c=0). Actually, it is suitable to represent the maximum likelihood estimators in the similar analytical form to the ones without the nonlinear term (i.e., c=0). However, it is not convenient to represent moments since they are expressed by the generalized Beta function with negative parameters ($\mu < 0$). When the generalized Gamma function is adopted, such inconveniences do not arise to represents moments. So theoretical analysis below is given on the basis of eq.(7) with the generalized Gamma function in eq.(8).

2.4 Numerical examples

Now let us consider how the pdf profile changes depending on the values of parameters a, b and c. Figures 1(a)-(d) show the pdf profiles as a function of typical set of parameters b and c with $a = 1$ being fixed. Without the multiplicative noise $F_p(t) = 0$, the pdf is described by $P_s(x) = P_0 \exp(-V(x)/D_a)$ with $V(x) = \frac{\alpha}{2}x^2 + \frac{\gamma}{4}x^4$. From the expression of b in eq.(9), the value of b can be negative when $\alpha D_p + D_p^2 < \gamma D_a$ by virtue of the existence of cubic nonlinear term (γ or $c > 0$). Namely, a stochastic bifurcation (cf. Arnold [10]) (or the noise-induced transition, cf. Hormsenke and Lefever [11]) takes place at $\alpha D_p + D_p^2 = \gamma D_a$ even when $\alpha > 0$ and $\gamma > 0$. At the critical point, the pdf is reduced exactly to the Gaussian type. For the negative values of b, a double-peaked pdf is realized as shown in Fig.1(a).

When the cubic nonlinearity is not accounted, there is a parameter region wherein all the ordinary moments $\langle x^m \rangle$ $(m = 1, 2, \ldots)$ diverge. After accounting the nonlinear term with $\gamma > 0$ or $c > 0$, the ordinary moments do not diverge in the whole parameter region which will be shown in section 4.

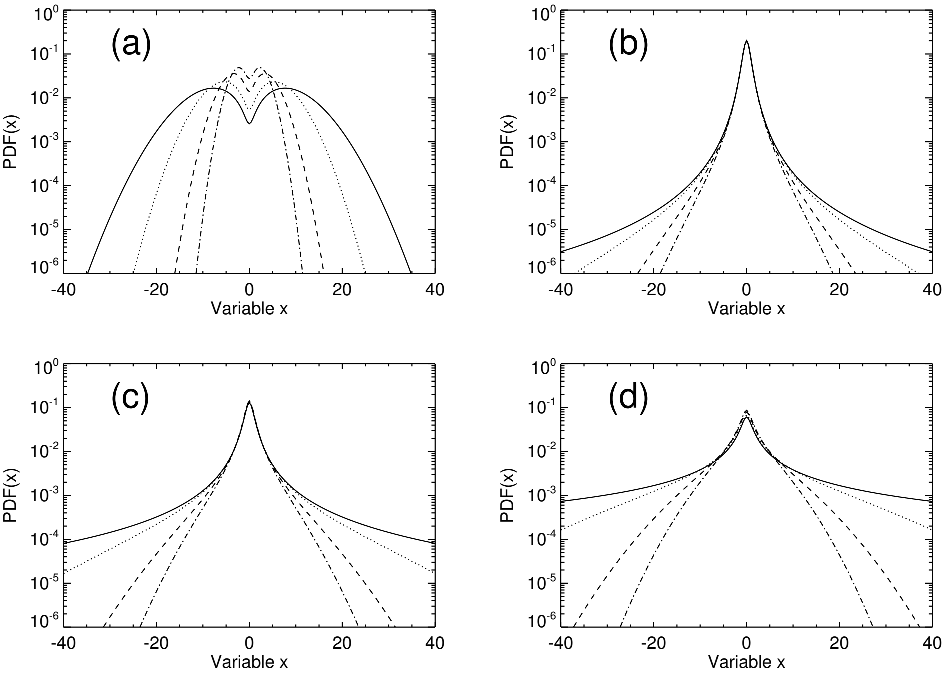

Figure 1: The pdf profile for $a = 1$ as a function of b and c in the cases: (a) $b = -0.6$ (i, SL) $c = 0.01$, (ii, DOL) $c = 0.02$, (iii, DAL) $c = 0.05$, (iv, DDL) $c = 0.1$; (b) $b = 1.5$ (i, SL) $c = 0$, (ii, DOL) $c = 0.001$, (iii, DAL) $c = 0.005$, (iv, DDL) $c = 0.01$; (c) $b = 1.0$ (i, SL) $c = 0$, (ii, DOL) $c = 0.001$, (iii, DAL) $c = 0.005$, (iv, DDL) $c = 0.01$; (d) $b = 0.6$ (i, SL) $c = 0$, (ii, DOL) $c = 0.001$, (iii, DAL) $c = 0.005$, (iv, DDL) $c = 0.01$ (SL=solid line; DOL=dotted line; DAL=dashed line; DDL=dash-dot line). The double-peaked structure arises for the case (a). On the other hand, the single peaked structure appears for the positive values of b in the cases (b), (c) and (d).

3 Maximum Likelihood Method

When a pdf is given in terms of nonlinear functions of $G(x)$ and $F_i(x)$ as $P(x) = \exp(G(x) + \sum_i \theta_i F_i(x) - \Psi(\theta))$, they call $\{\theta_i\}$ as natural (canonical) parameter. Since the pdf in eq.(8) is rewritten in the form, $P_s(x) = \exp(-b\ln(x^2 + a^2) - cx^2 + \ln P_0)$, b and c can be classified into natural parameters and a can be classified into nonnatural one. Namely, $F_1(x) = \ln(x^2 + a^2)$, $F_2(x) = x^2$, $\theta_1 = -b$, $\theta_2 = -c$ and $\Psi(\theta) = \Psi(a, b, c) = -\ln P_0 = (b - \frac{1}{2}) \ln c + \ln\Gamma(\frac{1}{2}, b; ca^2)$[9]. We call the model in eq.(2) as "noncanonical model" since the nonnatural parameter a is involved.

3.1 Maximum likelihood estimator

Consider the maximum likelihood estimators (MLEs) for the pdf in eq.(7) associated with the noncanonical model in eq.(2) or eq.(2′). As is easily expected from the log-likelihood function provided that data set $\{x_i\}$ $(i = 1, 2, \ldots, n)$ is given,

$$L = \ln \prod_{i=1}^{n} P_s(x_i) = n \ln P_0 - b \sum_{i=1}^{n} \ln(x_i^2 + a^2) - c \sum_{i=1}^{n} x_i^2 \ , \tag{12}$$

the nonnatural parameter a and the natural parameters (b and c) are related to the three statistical quantities (i.e., the dual coordinates [9])

$$\eta_1 \equiv \langle \ln(x^2 + a^2) \rangle \ , \eta_2 \equiv \left\langle \frac{1}{x^2 + a^2} \right\rangle \text{ and } \eta_3 \equiv \langle x^2 \rangle. \tag{13}$$

Their analytical expressions in terms of $\Gamma(z, \lambda; v)$ are given by

$$\eta_1 \equiv \langle \ln(x^2 + a^2) \rangle = -\frac{\partial}{\partial b}\Psi(a, b, c) = -\ln c - \psi_{\Gamma,2}\left(\frac{1}{2}, b; ca^2\right) \ , \tag{14}$$

where $\psi_{\Gamma,2}(z, \lambda; v)$ is the generalized digamma function of the second kind (cf. Appendix) defined by

$$\psi_{\Gamma,2}(z, \lambda; v) \equiv \frac{\partial}{\partial \lambda} \ln\Gamma(z, \lambda; v) = \frac{-\int_0^\infty \exp(-t) \cdot t^{z-1} \cdot (t + v)^{-\lambda} \ln(t + v) \cdot dt}{\Gamma(z, \lambda; w)} \ . \tag{15}$$

$$\eta_2 \equiv \left\langle \frac{1}{x^2 + a^2} \right\rangle = -\frac{1}{2ab}\frac{\partial}{\partial a}\Psi(a, b, c) = c \cdot \frac{\Gamma(\frac{1}{2}, b + 1; ca^2)}{\Gamma(\frac{1}{2}, b; ca^2)} \tag{16}$$

and

$$\eta_3 \equiv \langle x^2 \rangle = -\frac{\partial}{\partial c}\Psi(a, b, c) = \frac{1}{c}\frac{\Gamma(\frac{3}{2}, b; ca^2)}{\Gamma(\frac{1}{2}, b; ca^2)} \ . \tag{17}$$

The formula in eq.(17) can be derived from $\Psi(a, b, c)$ with the aid of recurrence relation (A8) in Appendix. Though all the analytical expressions are given in terms of the generalized Gamma function $\Gamma(z, \lambda; v)$ defined in eq.(8), the first η_1 and the second η_2 dual coordinates [9] involve the unnatural parameter a. This is not convenient to infer the parameters (a, b, c). To avoid this inconvenience, we can introduce canonical model wherein no unnatural parameter is involved which will be discussed in section 5. In the limit $c \to 0$, $\eta_1 = 2\ln a + \psi(b) - \psi(b-1/2)$, $\eta_2 = (1-1/2b)/a^2$ and $\eta_3 = a^2/2(b-3/2)$ with $\psi(z)$ being the digamma function. In this limit, (i) the first η_1 and the second η_2 expression are valid for $1/2 < b$, (ii) but the third η_3 expression is valid for $3/2 < b$ since the second moment must be a positive value (when $3/2 < b$, the ordinary second moment diverges: cf. ref. [6]).

3.2 Fisher information

The Fisher information matrix $F(a,b,c)$ in terms of the cross cumulants is given by

$$F(a,b,c) \equiv \begin{pmatrix} \langle [\ln(x^2+a^2)]^2 \rangle_c & \left\langle \frac{\ln(x^2+a^2)}{x^2+a^2} \right\rangle_c & \left\langle x^2 \ln(x^2+a^2) \right\rangle_c \\ * & \left\langle \frac{1}{(x^2+a^2)^2} \right\rangle_c & \left\langle \frac{x^2}{x^2+a^2} \right\rangle_c \\ * & * & \langle x^4 \rangle_c \end{pmatrix} \tag{18}$$

where the matrix elements can be expressed in terms of the generalized Gamma function $\Gamma(z,\lambda;v)$, the digamma function $\psi_{\Gamma,2}(z,\lambda;v)$ and the trigamma function $\psi'_{\Gamma,2}(z,\lambda;v)$ of the second kind (cf. Appendix) as

$$\langle x^4 \rangle_c = \frac{1}{c^2}\left(\frac{\Gamma(\frac{5}{2},b;ca^2)}{\Gamma(\frac{1}{2},b;ca^2)} - \left(\frac{\Gamma(\frac{3}{2},b;ca^2)}{\Gamma(\frac{1}{2},b;ca^2)} \right)^2 \right), \tag{19}$$

$$\left\langle \frac{1}{(x^2+a^2)^2} \right\rangle_c = c^2\left(\frac{\Gamma(\frac{1}{2},b+2;ca^2)}{\Gamma(\frac{1}{2},b;ca^2)} - \left(\frac{\Gamma(\frac{1}{2},b+1;ca^2)}{\Gamma(\frac{1}{2},b;ca^2)} \right)^2 \right), \tag{20}$$

$$\langle [\ln(x^2+a^2)]^2 \rangle_c = \psi'_{\Gamma,2}\left(\frac{1}{2}, b; ca^2 \right), \tag{21}$$

$$\left\langle \frac{\ln(x^2+a^2)}{x^2+a^2} \right\rangle_c = c\frac{\Gamma(\frac{1}{2},b+1;ca^2)}{\Gamma(\frac{1}{2},b;ca^2)} \left\{ \psi_{\Gamma,2}\left(\frac{1}{2},b;ca^2\right) - \psi_{\Gamma,2}\left(\frac{1}{2},b+1;ca^2\right) + \ln c \right\} - c\ln c \tag{22}$$

and

$$\langle x^2 \ln(x^2+a^2) \rangle_c = -\frac{1}{c}\left(\left[\psi_{\Gamma,2}\left(\frac{3}{2},b;ca^2\right) - \psi_{\Gamma,2}\left(\frac{1}{2},b;ca^2\right) \frac{\Gamma(\frac{3}{2},b;ca^2)}{\Gamma(\frac{1}{2},b;ca^2)} \right] \right). \tag{23}$$

Since the matrix $F(a,b,c)$ is a symmetric one, the off-diagonal elements in lower left columns are denoted by $*$ to save space.

It is readily seen that the peak amplitude of the pdf is expressed in terms of $\Gamma(z,\lambda;v)$ as

$$P_s(0) = c^{\frac{1}{2}-b} a^{-2b} \cdot \Gamma\left(\frac{1}{2}, b; ca^2 \right)^{-1}. \tag{24}$$

It can be used to estimate the values of parameters, and to discuss estimation errors in combining with eqs. (14)-(24). However, it is not so convenient since the nonnatural parameter a is included in their statistical quantities except $\langle x^{2m} \rangle$ $(m = 1, 2)$.

4 Methods of Moment

4.1 Ordinary moments

Now let us examine to obtain the expressions of moments in terms of the generalized Gamma function $\Gamma(z, \lambda; v)$ defined in eq.(8):

$$\langle x^{2m} \rangle = \frac{c^{\frac{1}{2}-b}}{\Gamma(\frac{1}{2}, b; ca^2)} \int_{-\infty}^{\infty} \frac{x^{2m} \exp\big(- cx^2 \big)}{(x^2 + a^2)^b} \, dx = c^{-m} \frac{\Gamma(m + \frac{1}{2}, b; ca)}{\Gamma(\frac{1}{2}, b; ca^2)} \ . \tag{25}$$

Since there are three parameters (a, b, c), it is enough to use the second and the fourth moment and the peak value of the pdf $P_s(0)$ in eq.(24) as

$$\langle x^2 \rangle = c^{-1} \frac{\Gamma(\frac{3}{2}, b; ca^2)}{\Gamma(\frac{1}{2}, b; ca^2)} \quad , \quad \langle x^4 \rangle = c^{-2} \frac{\Gamma(\frac{5}{2}, b; ca^2)}{\Gamma(\frac{1}{2}, b; ca^2)} \quad \text{and} \quad P_s(0) = \frac{c^{1/2-b} a^{-2b}}{\Gamma(\frac{1}{2}, b; ca^2)} \ . \tag{26}$$

It should be noted that the ordinary moments do not diverge by virtue of the nonlinear term $cx^3 (c > 0)$.

4.2 Log-amplitude moments

As shown in section 3, the maximum likelihood estimators $\langle \ln(x^2 + a^2) \rangle$ and $\langle \frac{1}{x^2+a^2} \rangle$ involve a nonnatural parameter a. To avoid this inconvenience in estimating the parameters (a, b, c) from time series, one can use the log-amplitude moment $\langle \ln |x| \rangle$ (cf. [28,29]) in place of $\langle \ln(x^2 + a^2) \rangle$. Actually, the log-amplitude moment is expressed as

$$\langle \ln |x| \rangle = \frac{c^{1/2-b}}{\Gamma(\frac{1}{2}, b; ca^2)} \int_{-\infty}^{\infty} \frac{(\ln |x|) \exp\big(- cx^2 \big)}{(x^2 + a^2)^b} \, dx = \frac{1}{2} \left\{ \psi_{\Gamma,1}\left(\frac{1}{2}, b; ca^2 \right) - (\ln c) \right\} , \tag{27}$$

where $\psi_{\Gamma,1}(1/2, b; ca^2)$ is the generalized digamma function of the first kind [cf.,the second kind in eq.(15)] defined by

$$\psi_{\Gamma,1}(z, \lambda; w) \equiv \frac{d}{dz} \ln \Gamma(z, \lambda; w) = \frac{\int_0^\infty \exp(-t) \cdot t^{z-1} \cdot (t + v)^{-\lambda} \ln(t) \cdot dt}{\Gamma(z, \lambda; w)} \ . \tag{28}$$

This new generalized digamma function of the first kind is reduced to the standard digamma function $\psi(z) = (\frac{d}{dz} \ln \Gamma(z))$ as $c \to 0$.

Similarly, the second moment is expressed as

$$\langle (\ln |x|)^2 \rangle = \frac{c^{1/2-b}}{\Gamma(\frac{1}{2}, b; ca^2)} \int_{-\infty}^{\infty} \frac{(\ln |x|)^2 \exp\big(- cx^2 \big)}{(x^2 + a^2)^b} \, dx \tag{29}$$

$$= \frac{1}{4}\left\{\psi'_{\Gamma,1}\left(\frac{1}{2},b;ca^2\right)+\psi_{\Gamma,1}\left(\frac{1}{2},b;ca^2\right)^2-2(\ln c)\psi_{\Gamma,1}\left(\frac{1}{2},b;ca^2\right)+(\ln c)^2\right\}\ , \tag{30}$$

where $\psi'_{\Gamma,1}(z,\lambda;v)$ is the generalized trigamma function of the first kind: $\psi'_{\Gamma,1}(z,\lambda;v)=\frac{d}{dz}\psi_{\Gamma,1}(z,\lambda;v)$. In terms of the cumulant, we have

$$\langle(\ln|x|)^2\rangle_c=\frac{1}{4}\psi'_{\Gamma,1}\left(\frac{1}{2},b;ca^2\right)\ . \tag{31}$$

The formulae $\langle\ln|x|\rangle$ in eq.(27) and $\langle(\ln|x|)^2\rangle_c$ in eq.(31) have simple forms in terms of generalized poly-gamma (digamma and trigamma) functions that are similar to $\langle\ln(x^2+a^2)\rangle$ in eq.(14) and $\langle[\ln(x^2+a^2)]^2\rangle_c$ in eq.(21), respectively. These are corresponding to the mean and the variance of log-amplitude in the system. Since the nonnatural parameter a is not incorporated in the left hand side of the formulae (27) and (31), they are more favorable than those of the MLEs (14) and (21) in section 3 to infer the parameters a,b and c.

5 Discussions

5.1 Canonical model

In this section, it is introduced the canonical model of eq.(2′) which have only canonical parameters. It is convenient (i) to estimate model parameters and (ii) to discuss estimation errors of parameters (cf. section 3.1 and 3.2). Further, one can discuss the fluctuation-dissipation theorem in a state space under the influence of an additive noise after nonlinear transformation (see section 5.2).

When the scale transformation of the variable ($x=aX$) is introduced, the Langevin equation in eq.(2′) under the SI for x reduces to the one for X,

$$\frac{d}{dt}X=-\alpha X-\epsilon X^3+D_pX+\sqrt{2D_p(X^2+1)}F_m(t)\ , \tag{32}$$

where the coefficient of the cubic nonlinear term is scaled by a^2: $\epsilon\equiv a^2\gamma=\frac{D_a}{D_p}\gamma$. The corresponding FP equation for the Langevin equation in eq.(32) becomes

$$\frac{\partial}{\partial t}Q(X,t)=\frac{\partial}{\partial X}[\tilde{K}(X)Q(X,t)]+\frac{\partial^2}{\partial X^2}[\tilde{D}(X)Q(X,t)]\ , \tag{33}$$

where $\tilde{K}(X)=(\alpha-D_p)X+\epsilon X^3$ and $\tilde{D}(X)=D_p(X^2+1)$ due to the SI of eq.(32). The pdf at the steady state $Q_s(X)$ is given by

$$Q_s(X)=\frac{s^{1/2-r}}{\Gamma(\frac{1}{2},r;s)}\frac{\exp\big(-sX^2\big)}{(X^2+1)^r}\ , \tag{34}$$

where the two natural parameters (r,s) are given by $r\equiv\frac{\alpha}{2D_p}+\frac{1}{2}-\frac{\epsilon}{2D_p}$ and $s=\frac{\epsilon}{2D_p}$. Note here that $s\neq c$. The information geometrical potential for the model in eq.(34) is reduced to $\Psi(1,r,s)=(r-\frac{1}{2})\ln c+\ln\Gamma(\frac{1}{2},r;s)$.

Consider the maximum likelihood estimators for the canonical model. As is easily expected from the log-likelihood function, i.e.,

$$L(X) = \ln Q_s(X) = -\left(r - \frac{1}{2}\right)\ln s - \ln\Gamma\left(\frac{1}{2}, r; s\right) - r\ln(X^2+1) - sX^2 \ , \quad (35)$$

the MLE estimators (the dual coordinates) in this case are related to the two statistical quantities $\langle \ln(X^2+1)\rangle$ and $\langle X^2\rangle$. Their analytical expressions are derived in the following form:

$$\tilde{\eta}_1 \equiv \langle \ln(X^2+1)\rangle = -\frac{\partial}{\partial r}\Psi(1,r,s) = -\ln s - \psi_{\Gamma,2}\left(\frac{1}{2}, r; s\right) \quad (36)$$

and

$$\tilde{\eta}_3 \equiv \langle X^2\rangle = -\frac{\partial}{\partial s}\Psi(1,r,s) = \frac{1}{s}\cdot\frac{\Gamma(\frac{3}{2}, r; s)}{\Gamma(\frac{1}{2}, r; s)} \ . \quad (37)$$

All the analytical expressions are given in terms of the generalized Gamma function and the generalized digamma function of the second kind. Also, two dual coordinates $\tilde{\eta}_1$ and $\tilde{\eta}_3$ involve only the natural parameters (r, s). An iterative procedure makes it possible to infer the two parameters (r, s) with the use of the MLEs in eqs.(36) and (37).

The Fisher information (3×3) matrix $F(1, r, s)$ in eq.(18) reduces to the compact (2×2) matrix $G(1, r, s)$ as

$$G(1,r,s) \equiv \begin{pmatrix} \langle [\ln(X^2+1)]^2\rangle_c & \langle X^2\ln(X^2+1)\rangle_c \\ * & \langle X^4\rangle_c \end{pmatrix} \ , \quad (38)$$

where the matrix elements are easily obtained by putting $a = 1$, $b = r$ in eqs.(19), (21) and (23). Since the peak intensity of the pdf becomes $Q_s(0) = s^{\frac{1}{2}-r}\Gamma(\frac{1}{2}, r; s)^{-1}$, the two canonical parameters (r, s) can be estimated by the equation of the peak intensity and the second moment $\frac{\langle X^2\rangle}{Q_s(0)} = s^{r-\frac{3}{2}}\Gamma(\frac{3}{2}, r; s)$, in terms of the generalized Gamma function without using the quantity $\langle\ln(X^2+1)\rangle$ in eq.(36).

5.2 Generalized fluctuation-dissipation theorem

Now let us discuss a generalized fluctuation-dissipation theorem (GFDT) for the Cauchy process in the canonical form with the cubic nonlinear term in eq.(32). After the nonlinear transformation of the state variable $X = \sinh(Y)$ by noticing the noise correction due to the SI, the nonlinear Langevin equation having additive noise is derived as

$$\frac{d}{dt}Y = -(\alpha - D_p - \epsilon)\tanh(Y) - \epsilon\sinh^2(Y)\tanh(Y) + \sqrt{2D_p}F_m(t), \quad (39)$$

$$= -\frac{\partial}{\partial Y}V(Y) + \sqrt{2D_p}F_m(t) \ , \quad (39')$$

where the effective potential $V(Y)$ for the system is

$$V(Y) = (\alpha - \epsilon - D_p)\ln(\cosh(Y)) + \frac{\epsilon}{2}\sinh^2(Y)\ . \tag{40}$$

In deriving the potential in eq.(40), the integral formula $\int \tanh(Y)dY = \ln(\cosh(Y))$ and $\int \sinh^2(Y)\tanh(Y)dY = \sinh^2(Y)/2 - \ln(\cosh(Y))$ are utilized. The shape of the potential is depicted in Fig.2 as a function of ϵ for $\alpha = 3$ and $D_p = 1$. As ϵ increases, potential with slow slope change gradually to potential with steep slope. This is a result of competition between the two types of nonlinear modes, i.e., the single kink $\tanh(Y)$ and the double kink $\sinh(2Y)$ as shown in Fig.3. It is readily see that the probability density function for the model at the steady state is reduced to

$$P_s(Y) = P_0 \exp\left\{ -\frac{V(Y)}{D_p} \right\}\ , \tag{41}$$

where $P_0 = [\int_{-\infty}^{\infty} dY\ \ \exp\{-\frac{V(Y)}{D_p}\}]^{-1}$ is the normalization constant.

A generalized fluctuation-dissipation theorem (GFDT) for the single kink ($\tanh(Y)$) mode and the double kink ($\sinh(2Y)$) mode is expressed as

$$\langle \ln[\cosh(Y)] \rangle = -\frac{1}{2}\ln s - \frac{1}{2}\psi_{\Gamma,2}\left(\frac{1}{2}, r; s\right) \tag{42}$$

and

$$\langle \sinh^2(Y) \rangle = \frac{1}{s}\frac{\Gamma(\frac{3}{2}, r; s)}{\Gamma(\frac{1}{2}, r; s)}\ . \tag{43}$$

As shown in our previous paper [6], eq.(42) and eq(43) with $c = 0$ reduces to the Einstein relation $\langle Y^2 \rangle = D_p/\alpha$ (i.e., $\langle \ln[\cosh(Y)] \rangle \approx \langle \sinh^2(Y) \rangle \approx \langle Y^2 \rangle$) in the limit of small Y.

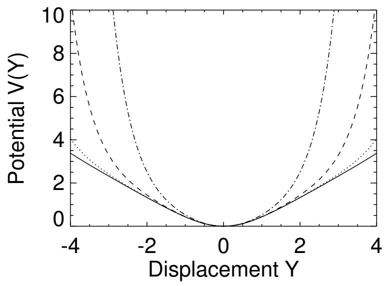

Figure 2: Shape of the potential $V(Y)$ in eq.(40) as a function of parameter ϵ for (a) 0.0001 (solid line), (b) 0.001 (dotted line), (c) 0.01 (dashed line) and (d) 0.1 (dash-dot line) with $\alpha = 3$ and $D_p = 1$.

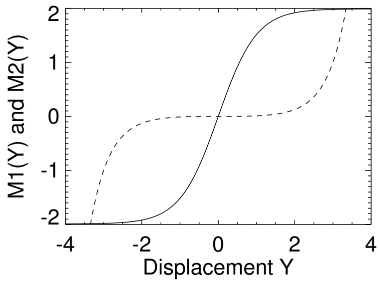

Figure 3: Profile of the two competing nonlinear modes $(\alpha - D_p - \epsilon)\tanh(Y)$ and $\epsilon\sinh^2(Y)\tanh(Y)$ for the parameter $\epsilon = 0.01$ with $\alpha = 3$ and $D_p = 1$.

It is known [24,25,26] for the nonlinear Langevin equation in eq.(39) that the fluctuation-dissipation theorem of the first kind holds:

$$-\frac{d}{dt}C_{YY}(t) = D_p R_Y(t) \ , \tag{44}$$

where $C_{YY}(t)$ is the correlation function of Y, and $R_Y(t)$ is the response function of this system.

The set of equations (42)-(44) (i.e., a generalized FDT or a fluctuation-response relation of "the third kind", since they call eq.(3) as the FDT of the second kind) constitutes the complete piece of information for characterizing the generalized Cauchy process in eq.(2).

5.3 Examples in the real world

Mantegna and Stanley [22,23] reported scaling behaviour in an economic index (the Standard and Poor's 500 with $\Delta t = 1$ min), the central part of the probability density function (pdf) is identified quite well by the Lévy distribution:

$$P_{L\acute{e}vy}(x) = \frac{1}{\pi}\int_0^\infty \exp(-\beta|k|^\alpha)\cos(kx)\ dk \ , \tag{45}$$

with $\alpha = 1.40$ and scale factor $\beta = 0.00375$. It is point out [22,23] that a cut-off effect must be introduced.

The pdf for logarithmic price (index) changes $x(t)$ ($\equiv \log S(t+\Delta t) - \log S(t)$) ($\Delta t = 2$ min) is identified by the pdf given by eq.(7) with $c \neq 0$ with the use of the log-amplitude moments after normalizing the data as $X = \frac{x}{\sqrt{\overline{x^2}}}$ ($\sigma_X^2 = 1$). The good fit is obtained with $a = 0.4$, $b = 1.0$ and $c = 0.035$ in the whole region as shown in the solid line of Fig.4. It seems that the cut-off effect can be achieved by introducing the nonlinear friction (the cubic nonlinear term). The central part of the pdf can be also identified by the generalized Cauchy distribution with $c = 0$ as shown in the dashed line. To check the universality of this nature for economic indexes, a data set for Nikkei 225 ($\Delta t = 1$ min) is examined during the term 1996-2003. It is found that the pdf can be identified also well by the pdf in eq.(7).

In the case of the probability distribution function of acceleration data in turbulence [18,19], the pdf is identified well by

$$P_{turb}(z) = P_0 \exp\left\{ -\frac{z^2}{(1+|\beta z/\sigma|^\gamma)\sigma^2} \right\}, \tag{46}$$

with the parameters $\beta = 0.513$, $\gamma = 1.600$, $\sigma = 0.563$ and the normalization constant $P_0 = 0.733$. This is a hybrid pdf of Gaussian and stretched exponential distribution. The form of the pdf in eq.(46) suggests the effect of nonlinear dissipation is expressed in a different way with the hybrid pdf of Gaussian and Cauchy distribution in eq.(7). Although many theoretical models [18-22] are proposed, the empirical expression shows the best fit among the pdfs proposed by theoretical models in references [18-22]. In this case, a good fit is obtained to the numerical experimental data by the pdf in eq.(7) with parameters $a = 1.00$, $b = 1.85$ and $c = 0.001$.

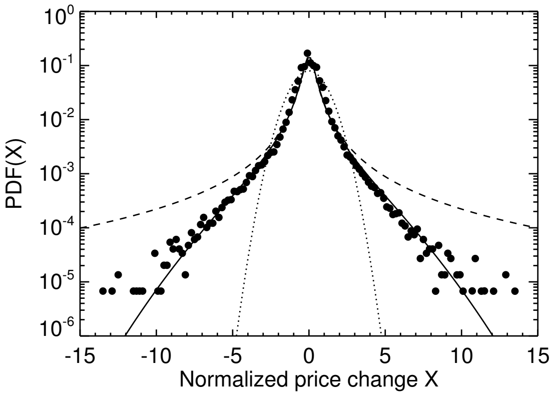

Figure 4: The obtained parameters are used to reconstruct the pdf ($PDF(X)$) of normalized standard and poor data $X(\equiv \frac{x}{\sqrt{x^2}})$ ($\Delta t = 2$ min, 1992-1994, $N = 148000$). The generalized Cauchy distribution with $a = 0.4$, $b = 1.0$ and $c = 0.035$ in the solid line.

6 Concluding Remarks

We have studied the generalized Cauchy process with a cubic nonlinear term in eq.(2) under the influence of Gaussian-white noises. The probability density function takes a Gaussian-Cauchy hybrid distribution. In this case, the normalization constant of the symmetric probability density can be expressed in terms of a generalized Gamma function in eq.(8). Although the generalized Beta function in eq.(10) is suitable to describe the maximum likelihood estimators

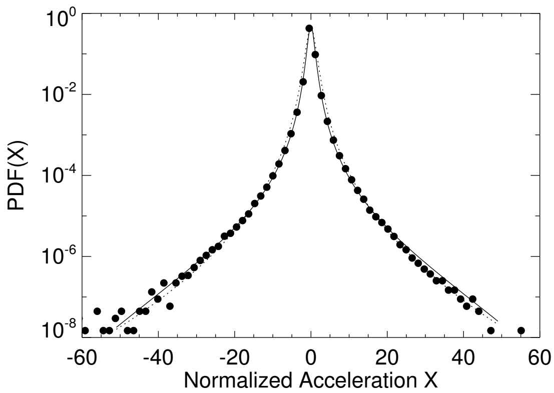

Figure 5: The obtained parameters are used to reconstruct the pdf ($PDF(X)$) of normalized acceleration data ($X(\equiv \frac{x}{\sqrt{x^2}})$) in turbulence due to Mordant et al.[19]. The generalized Cauchy distribution with $a = 1.00$, $b = 1.85$ and $c = 0.001$ in the solid line. The stretched exponential function due to [18,19] is over-plotted in the dotted line.

(MLEs), the generalized Gamma function is more suitable for describing moments and log-amplitude moments. When one tries to represent the MLEs, the moments and the log-amlitude moments, it is necessary to introduce two types of digamma functions (the first kind and the second kind) as shown in eqs.(15) and (28). In any case, the analytical expressions to infer the parameters are obtained.

It is known that the maximum likelihood estimator (MLE) is recommended by virtue of the minimum estimation error. In the case of the model with noncanonical parameter a, (i.e., (x^2+a^2)), it is not convenient to use the method of the MLE. In the present model, it is convenient to use the moment methods in eqs.(26), (27) and (31) due to simplicity and due to reducing estimation error.

We have applied the method to the present generalized Cauchy process with a cubic nonlinear term to identify an economic index (the Standard and Poor's 500) and acceleration data of fluid turbulence. The cut-off effect of fat-tail in the pdf due to the nonlinear friction can be incorporated successfully to get a good fit to the two kinds of experimental data. The model with the method of parameter estimation with eqs.(26),(27) and (31) might be useful to investigate various time series data with the symmetric density (7) in real world complex systems.

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Appendix

A. Generalized Gamma function and its relation to the hyper geometric function $U(z,u;v)$

Let us define $I_0(a,b,c)$ by

$$I_0(a,b,c) = 2\int_0^\infty \frac{\exp(-cx^2)}{(x^2+a^2)^b} \cdot dx \ . \tag{A1}$$

Changing from the variable x to the new one y with $y = cx^2$ one gets

$$I_0(a,b,c) = c^{b-1/2}\int_0^\infty \exp(-y)y^{-1/2}(y+ca^2)^{-b}dy \ . \tag{A2}$$

So, the pdf is expressed by

$$P_s(x) = \frac{c^{1/2-b}}{\Gamma(\frac{1}{2},b;ca^2)} \cdot \frac{\exp(-cx^2)}{(x^2+a^2)^b} \ , \tag{A3}$$

where $\Gamma(z,\lambda;v)$ is a generalized Gamma function defined by

$$\Gamma(z,\lambda;v) \equiv \int_0^\infty \exp(-t)\cdot t^{z-1}\cdot(t+v)^{-\lambda}\cdot dt \ . \tag{A4}$$

In the limit $\lambda \to 0$, eq.(A4) reduces to the Gamma function:

$$\lim_{\lambda\to 0}\Gamma(z,\lambda;v) = \Gamma(z). \tag{A5}$$

The generalized Gamma function in eq.(A4) is expressed in terms of the confluent hypergeometric function $U(z,\beta;v)$ of the second kind as

$$\Gamma(z,\lambda;v) = \Gamma(z)(v)^{z-\lambda}U(z,1+z-\lambda;v) \ , \tag{A6}$$

where

$$U(z,u;v) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{\exp(-vt)t^{z-1}}{(1+t)^{z+1-u}}dt \ . \tag{A7}$$

B. Properties of $\Gamma(z,\lambda;v)$

Here describes some mathematical formula on differentiations and recurrence relation. It is easy to prove them.
(i) recursion relation

$$z\Gamma(z,\lambda;v) = \Gamma(z+1,\lambda;v) + \lambda\Gamma(z+1,\lambda+1;v) \ . \tag{A8}$$

(ii) a generalized digamma function of the first kind which is taken derivative with respect to z

$$\psi_{\Gamma,1}(z,\lambda;v) = \frac{\partial}{\partial z} \ln \Gamma(z,\lambda;v) = \frac{\Gamma_1'(z,\lambda;v)}{\Gamma(z,\lambda;v)} \ . \tag{A9}$$

(iii) $\Gamma_1'(z,\lambda;v)$ is evaluated by

$$\Gamma_1'(z,\lambda;v) = \int_0^\infty \frac{t^{z-1}\exp(-t)\ln(t)}{(t+v)^\lambda} dt \ . \tag{A10}$$

(iv) the 2nd derivative of the generalized Gamma function of the first kind can be defined by

$$\Gamma_1^{(2)}(z,\lambda;v) = \int_0^\infty t^{z-1}\exp(-t)(t+v)^{-\lambda}(\ln t)^2 dt \ . \tag{A11}$$

and the generalized trigamma function of the first kind is defined by

$$\psi_{\Gamma,1}'(z,\lambda;v) = \frac{d^2}{dz^2} \ln \Gamma(z,\lambda;v) = \frac{\Gamma_1^{(2)}(z,\lambda;v)}{\Gamma(z,\lambda;v)} - \left(\frac{\Gamma_1'(z,\lambda;v)}{\Gamma(z,\lambda;v)}\right)^2 \ . \tag{A12}$$

(v) a generalized digamma function of the second kind which is taken derivative with respect to λ

$$\psi_{\Gamma,2}(z,\lambda;v) = \frac{\partial}{\partial \lambda} \ln \Gamma(z,\lambda;v) = \frac{\Gamma_2'(z,\lambda;v)}{\Gamma(z,\lambda;v)} \ . \tag{A13}$$

(vi) $\Gamma_2'(z,\lambda;v)$ is evaluated by

$$\Gamma_2'(z,\lambda;v) = -\int_0^\infty \frac{t^{z-1}\exp(-t)\ln(t+v)}{(t+v)^\lambda} dt \ . \tag{A14}$$

(vii) the 2nd derivative of the generalized Gamma function of the second kind can be defined by

$$\Gamma_2^{(2)}(z,\lambda;v) = (-1)^2 \int_0^\infty t^{z-1}\exp(-t)(t+v)^{-\lambda}(\ln(t+v))^2 dt \ . \tag{A15}$$

The generalized trigamma function of the second kind is defined by

$$\psi_{\Gamma,2}'(z,\lambda;v) = \frac{d^2}{d\lambda^2} \ln \Gamma(z,\lambda;v) = \frac{\Gamma_2^{(2)}(z,\lambda;v)}{\Gamma(z,\lambda;v)} - \left(\frac{\Gamma_2'(z,\lambda;v)}{\Gamma(z,\lambda;v)}\right)^2 \ . \tag{A16}$$

C. Some special cases of $\Gamma(z,\lambda;v)$

$$\Gamma(1/2,1;v) = \pi v^{-1/2}\exp(v)\ \mathrm{erfc}(v^{1/2}) \ , \tag{A17}$$

$$\Gamma(3/2,1;v) = \pi^{1/2} - \pi v^{1/2}\exp(v)\ \mathrm{erfc}(v^{1/2}) \tag{A18}$$

and

$$\Gamma(5/2,1;v) = (1/2-v)\sqrt{\pi} + \pi\exp(v)v^{3/2}\ \mathrm{erfc}(v^{1/2}) \ , \tag{A19}$$

where erfc(z) is the complementary error function defined by

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-t^2) dt \ . \tag{A20}$$

Also, it is worth in the special cases

$$\Gamma(1, 1; v) = E_i(1, v) \exp(v) \tag{A21}$$

and

$$\Gamma(2, 1; v) = 1 - v \exp(v) E_i(1, v) \ , \tag{A22}$$

where $E_i(a, z)$ is a generalized exponential integral defined by

$$E_i(a, z) = \int_1^\infty \frac{\exp(-zt)}{t^a} dt \ . \tag{A23}$$

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