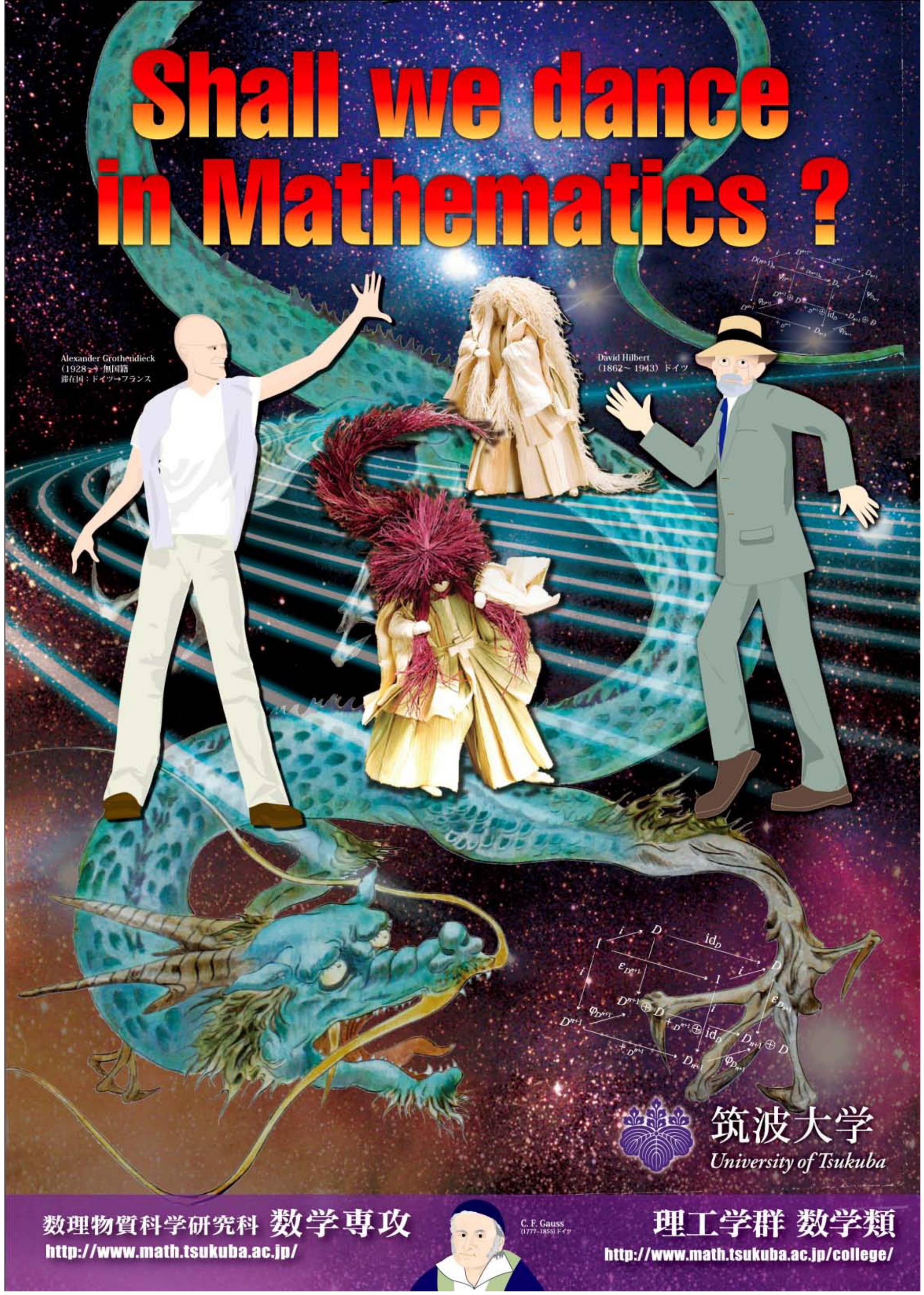
Shall we dance
in Mathematics ?
Alexander Grothendieck
（1928～）無国籍
滞在国：ドイツ→フランス
David Hilbert
（1862～ 1943）ドイツ
筑波大学
University of Tsukuba
数理物質科学研究科 数学専攻
http://www.math.tsukuba.ac.jp/
C. F. Gauss
(1777-1855) ドイツ
理工学群 数学類
http://www.math.tsukuba.ac.jp/college/

Differential Geometry of Microlinear Frölicher Spaces

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Abstract:

Three distinct approaches to jet bundles are presented

1 Nilpotent Infinitesimals and Weil Algebras

$$D = D_1 = \{d \in \mathbb{R} \mid d^2 = 0\} \approx \mathbb{R}[X]/(X^2)$$

$$D_n = \{d \in \mathbb{R} \mid d^{n+1} = 0\} \approx \mathbb{R}[X]/(X^{n+1})$$

$$D(m)_n = \{(d_1, ..., d_m) \in \mathbb{R}^m | d_{i_1} ... d_{i_{n+1}} = 0\} \approx \mathbb{R}[X_1, ..., X_m] / \left(X_{i_1} ... X_{i_{n+1}}\right)$$

$(i_1, ..., i_{n+1}$ shall range over natural numbers between 1 and m including both ends)

$$\begin{array}{c} \text{The category} \\ \text{of Weil algebras} \end{array} \underset{\mathcal{W}}{\overset{\mathcal{D}}{\rightleftarrows}} \begin{array}{c} \text{The category} \\ \text{of infinitesimal objects} \end{array}$$

$$\mathcal{D}_{\mathbb{R}[X]/(X^{n+1})} = D_n$$

$$\mathcal{W}_{D_n} = \mathbb{R}[X]/(X^{n+1})$$

Weil, André : Théorie des points proches sur les variétés différentiables, Colloques Internationaux du Centre National de la Reserche Scientifique, Strassbourg, pp.111-117, 1953.

2 Synthetic Differential Geometry

Our Real World of Mathematics	$\rightleftarrows$	Virtual World of Mathematics

<u>the construction of Grothendieck topos</u>

MacLane and Moerdijk : Sheaves in Geometry and Logic
Universitext, Springer-Verlag, 1992

Real World	**Virtual World**
classical logic	intuitionistic logic
$D = \{0\}$	$D \neq \{0\}$
manifolds	microlinear spaces
Many non-smooth entities !	Everything is smooth !
$\vdots$	$\vdots$

Anders Kock : Synthetic Differential Geometry 2nd ed.
London Mathematical Society Lecture Note Series **333**
Cambridge University Press 2006

2.1 Differential Calculus

The Kock-Lawvere Axiom

$$(\forall f : D \to \mathbb{R})\,(\exists! a \in \mathbb{R})\,(\forall d \in D)\,(f(d) = f(0) + ad)$$

More abstractly,

The canonical homomorphism $\mathbb{R}[X]/(X^2) \to \mathbb{R}^D$ of $\mathbb{R}$-algebras is an isomorphism.

Generally,
The Generalized Kock-Lawvere Axiom

The canonical homomorphism $W \to \mathbb{R}^{\mathcal{D}_W}$ of $\mathbb{R}$-algebras is an isomorphism for any Weil algebra W.

2.2 Microlinear Spaces

Definition 1 *A space M is microlinear iff $M^{\mathcal{D}_{\mathbb{L}}}$ is a limit diagram for any finite limit diagram $\mathbb{L}$ in the category of Weil algebras.*

Proposition 2 *The space $\mathbb{R}$ is microlinear.*

Proof. By the generalized Kock-Lawvere axiom. ■

Proposition 3 *Given a microlinear space M and $x \in M$, the space*

$$\mathbf{T}_x M = \left(M^D\right)_x = \{t : D \to M \mid t(0) = x\}$$

is naturally an $\mathbb{R}$-module.

2.3 Vector Fields

Exponential Laws

$$\left(M^D\right)^M = M^{D\times M} = \left(M^M\right)^D$$

1. The first viewpoint as a section of the tangent bundle:

$$X : M \to M^D \text{ with } X_x(0) = x \text{ for any } x \in M.$$

2. The second viewpoint as an infinitesimal flow:

$$X : D \times M \to M \text{ with } X(0, x) = x \text{ for any } x \in M.$$

3. The third viewpoint as an infinitesimal transformation:

$$X : D \to M^M \text{ with } X_0 = \mathrm{id}_M.$$

Theorem 4 *The totality of vector fields on M forms a Lie algebra.*

Addition

$$(X+Y)_d = X_d \circ Y_d = Y_d \circ X_d$$

for any $d \in D$.

Scalar Multiplication

$$(\alpha X)_d = X_{\alpha d}$$

for any $d \in D$ and any $\alpha \in \mathbb{R}$.

Lie Bracket

$$[X, Y]_{d_1 d_2} = Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1}$$

for any $d_1, d_2 \in D$.

3 Our Framework

Our General Framework

$$\pi : E \to M$$

with microlinear spaces E, M.

4 The First Approach to Jet Bundles

$$\tilde{\mathbf{J}}^n(\pi) \quad \text{non-holonomic jet bundle}$$
$$\hat{\mathbf{J}}^n(\pi)\text{...semi-holonomic jet bundle}$$
$$\mathbf{J}^n(\pi)\text{...holonomic jet bundle}$$

Definition 5 *We let*

$$\bar{\mathbf{J}}^0(\pi) = \hat{\mathbf{J}}^0(\pi) = \mathbf{J}^0(\pi) = E$$
$$\bar{\pi}_{0,0} = \hat{\pi}_{0,0} = \pi_{0,0} = \mathrm{id}_E$$
$$\bar{\pi}_0 = \hat{\pi}_0 = \pi_0 = \pi$$

$\bar{\mathbf{J}}^1(\pi) = \hat{\mathbf{J}}^1(\pi) = \mathbf{J}^1(\pi)$ *consists of* $\nabla_x : \left(M^D\right)_{\pi(x)} \to \left(E^D\right)_x$ $(x \in E)$ *abiding by the following two conditions:*

1.
$$\pi \circ (\nabla_x(t)) = t$$
for any $t \in \left(M^D\right)_{\pi(x)}$.

2.
$$\nabla_x(\alpha t) = \alpha(\nabla_x(t))$$
for any $t \in \left(M^D\right)_{\pi(x)}$ *and any* $\alpha \in \mathbb{R}$.

We define $\bar{\pi}_{1,0} = \hat{\pi}_{1,0} = \pi_{1,0} : \bar{\mathbf{J}}^1(\pi) = \hat{\mathbf{J}}^1(\pi) = \mathbf{J}^1(\pi) \to E$ *to be the assignment of* x *to each* $\nabla_x \in \bar{\mathbf{J}}^1(\pi) = \hat{\mathbf{J}}^1(\pi) = \mathbf{J}^1(\pi)$. *We define* $\bar{\pi}_1 = \hat{\pi}_1 = \pi_1 : \bar{\mathbf{J}}^1(\pi) = \hat{\mathbf{J}}^1(\pi) = \mathbf{J}^1(\pi) \to M$ *to be* $\pi \circ \pi_{1,0}$.

We proceed by induction on n. We are going to define $\bar{\mathbf{J}}^{n+1}(\pi)$, $\hat{\mathbf{J}}^{n+1}(\pi)$ and $\mathbf{J}^{n+1}(\pi)$ together with mappings $\bar{\pi}_{n+1,n} : \bar{\mathbf{J}}^{n+1}(\pi) \to \bar{\mathbf{J}}^n(\pi)$, $\hat{\pi}_{n+1,n} : \hat{\mathbf{J}}^{n+1}(\pi) \to \hat{\mathbf{J}}^n(\pi)$ and $\pi_{n+1,n} : \mathbf{J}^{n+1}(\pi) \to \mathbf{J}^n(\pi)$ by induction on $n \geq 1$. We let $\bar{\pi}_{k+1} = \bar{\pi}_k \circ \bar{\pi}_{k+1,k}$, $\hat{\pi}_{k+1} = \hat{\pi}_k \circ \hat{\pi}_{k+1,k}$ and $\pi_{k+1} = \pi_k \circ \pi_{k+1,k}$.

Definition 6 *We define* $\bar{\mathbf{J}}^{n+1}(\pi)$ *to be* $\mathbf{J}^1(\bar{\pi}_k)$ *with* $\bar{\pi}_{k+1,k} = (\bar{\pi}_k)_{1,0}$.

Definition 7 *We define* $\hat{\mathbf{J}}^{n+1}(\pi)$ *to be the subspace of* $\mathbf{J}^1(\hat{\pi}_n)$ *consisting of* ∇_x*'s with* $x = \nabla_y \in \hat{\mathbf{J}}^n(\pi)$ *subject to the following condition:The diagram*

$$\begin{array}{ccc} & D & \\ \nabla_x(t) \swarrow & & \searrow \nabla_y(t) \\ \hat{\mathbf{J}}^n(\pi) & \underset{\hat{\pi}_{n,n-1}}{\longrightarrow} & \hat{\mathbf{J}}^{n-1}(\pi) \end{array}$$

commutes for any $t \in \left(M^D\right)_{\hat{\pi}_n(x)}$.

Definition 8 *We define* $\mathbf{J}^{n+1}(\pi)$ *to be the subspace of* $\mathbf{J}^1(\pi_n)$ *consisting of* ∇_x*'s with* $x = \nabla_y \in \mathbf{J}^n(\pi)$ *subject to the following two conditions:*

1. *The diagram*

$$\begin{array}{ccc} & D & \\ \nabla_x(t) \swarrow & & \searrow \nabla_y(t) \\ \mathbf{J}^n(\pi) & \underset{\pi_{n,n-1}}{\longrightarrow} & \mathbf{J}^{n-1}(\pi) \end{array}$$

   commutes for any $t \in \left(M^D\right)_{\pi_n(x)}$.

2. *Given* $\gamma \in \left(M^{D^2}\right)_{\pi_n(x)}$ *with* $d_1, d_2 \in D$*, we have*

$$\nabla_z(\gamma(d_1,\cdot))(d_2) = \nabla_w(\gamma(\cdot,d_2))(d_1)$$

   with

$$z = \nabla_y(\gamma(\cdot,\mathbf{0}))(d_1)$$
$$w = \nabla_y(\gamma(\mathbf{0},\cdot))(d_2)$$

Nishimura, Hirokazu : Holonomicity in sythetic differential geometry of jet bundles, Beiträge zur Algebra und Geometrie, 44 (2003), 471-481.

5 The Second Approach to Jet Bundles

We proceed by induction on n.

Definition 9 *Let* $\mathbb{J}^{D}(\pi) = \mathbf{J}^1(\pi)$

Definition 10 $\mathbb{J}^{D^{n+1}}$ *consists of mappings* $\nabla_x : \left(M^{D^{n+1}}\right)_{\pi(x)} \to \left(E^{D^{n+1}}\right)_x$ *(*$x \in E$*) subject to the following conditions:*

1. *Given* $\gamma \in \left(M^{D^{n+1}}\right)_{\pi(x)}$, *we have*

$$\pi \circ (\nabla_x(\gamma)) = \gamma$$

2.
$$\nabla_x(\alpha \underset{i}{\cdot} \gamma) = \alpha \underset{i}{\cdot} \nabla_x(\gamma) \qquad (1 \leq i \leq n+1)$$

with

$$(\alpha \underset{i}{\cdot} \gamma)(d_1, ..., d_{n+1}) = \gamma(d_1, ..., \alpha d_i, ..., d_{n+1})$$

3.
$$\nabla_x(\gamma^\sigma) = (\nabla_x(\gamma))^\sigma \ \ (\sigma \in \mathbf{S}_{n+1})$$

with

$$\gamma^\sigma(d_1, ..., d_{n+1}) = \gamma\left(d_{\sigma(1)}, ..., d_{\sigma(n+1)}\right)$$

4. *We have* $\pi_{n+1,n}(\nabla_x) \in \mathbb{J}^{D^n}$ *with*

$$\begin{aligned}&\nabla_x\left((d_1, ..., d_{n+1}) \in D^{n+1} \longmapsto \gamma(d_1, ..., d_n)\right) \\ &= (d_1, ..., d_{n+1}) \in D^{n+1} \longmapsto \left((\pi_{n+1,n}(\nabla_x))(\gamma)\right)(d_1, ..., d_n)\end{aligned}$$

for any $\gamma \in \left(M^{D^n}\right)_{\pi(x)}$.

5. *Given* $\gamma \in \left(M^{D^n}\right)_{\pi(x)}$, *we have*

$$\begin{aligned}\nabla_x((d_1, ..., d_{n+1}) \in D^{n+1} &\longmapsto \gamma(d_1, ..., d_{n-1}, d_n d_{n+1})) \\ &= (d_1, ..., d_{n+1}) \in D^{n+1} \longmapsto \\ &((\pi_{n+1,n}(\nabla_x))(\gamma))(d_1, ..., d_{n-1}, d_n d_{n+1})\end{aligned}$$

Nishimura, Hirokazu : Higher-order preconnections in synthetic differential geometry of jet bundles, Beiträge zur Algebra und Geometrie, **45** (2004), 677-696.

6 The Third Approach to Jet Bundles

We proceed by induction on n.

Definition 11 *$\mathbb{J}^{D_{n+1}}$ consists of mappings $\nabla_x : \left(M^{D_{n+1}}\right)_{\pi(x)} \rightarrow \left(E^{D_{n+1}}\right)_x$ $(x \in E)$ subject to the following conditions:*

1.
$$\pi \circ (\nabla_x(\gamma)) = \gamma$$

2.
$$\nabla_x(\alpha\gamma) = \alpha\nabla_x(\gamma)$$
with
$$(\alpha\gamma)(d) = \gamma(\alpha d)$$

3. *We have $\pi_{n+1,n}(\nabla_x) \in \mathbb{J}^{D_n}$ with*
$$\begin{aligned}&\nabla_x (d \in D_{n+1} \longmapsto \gamma(dd')) \\ &= d \in D_{n+1} \longmapsto ((\pi_{n+1,n}(\nabla_x))(\gamma))(dd')\end{aligned}$$
for any $\gamma \in \left(M^{D^n}\right)_{\pi(x)}$ and any $d' \in D_n$.

4. *The other technical condition*

Nishimura, Hirokazu : Synthetic differential geometry of higher-order total differentials, Cahiers de Topologie et Géométrie Différentielle Catégoriques, 47 (2006), 129-154 and 207-232.

7 From the First Approach to the Second Approach

Definition 12 *Mappings* $\varphi_0 : \mathbf{J}^0(\pi) \to \mathbb{J}^1(\pi)$ *and* $\mathbf{J}^1(\pi) \to \mathbb{J}^D(\pi)$ *shall be the identity mappings. We are going to define* $\varphi_n : \mathbf{J}^n(\pi) \to \mathbb{J}^{D^n}(\pi)$ *for any natural number* n *by induction on* n. *Given* $\nabla_x \in \mathbf{J}^{n+1}(\pi)$, *we define* $\varphi_{n+1}(\nabla_x) \in \mathbb{J}^{D^{n+1}}(\pi)$ *to be*

$$\begin{aligned}&\varphi_{n+1}(\nabla_x)(\gamma)(d_1, ..., d_{n+1})\\&= \varphi_n(\nabla_x(\gamma(0, ..., 0, \cdot))(d_{n+1}))(\gamma(\cdot, ..., \cdot, d_{n+1}))(d_1, ..., d_n)\end{aligned}$$

8 From the Second Approach to the Third Approach

Definition 13 *Mappings* $\varphi_0 : \mathbf{J}^0(\pi) \to \mathbb{J}^1(\pi)$ *and* $\mathbf{J}^1(\pi) \to \mathbb{J}^D(\pi)$ *shall be the identity mappings. We are going to define* $\psi_n : \mathbb{J}^{D^n}(\pi) \to \mathbb{J}^{D_n}(\pi)$ *for any natural number* n *by induction on* n. *Given* $\nabla_x \in \mathbb{J}^{D^{n+1}}(\pi)$, *we have*

$$\begin{aligned}&\nabla_x((d_1, ..., d_{n+1}) \in D^{n+1} \longmapsto \gamma(d_1 + ... + d_{n+1}))\\&\qquad = (d_1, ..., d_{n+1}) \in D^{n+1} \longmapsto \varphi_{n+1}(\nabla_x)(d_1 + ... + d_{n+1})\end{aligned}$$

9 The Equivalence of the Three Approaches with Coordinates

9.1 The Conventional Framework

$$E = \mathbb{R}^{p+q}$$
$$M = \mathbb{R}^{p}$$

with π being the canonical projection.

9.2 The Conventional Description

Definition 14 *We denote by $\mathcal{J}^n(\pi)$ the totality of*

$$\gamma \in E^{D(p)_n}$$

such that $\pi \circ \gamma$ is constant.

Proposition 15 *Each $\nabla \in \mathcal{J}^n(\pi)$ can be identified uniquely with a sequence*

$$\left(x^1, ..., x^p, u^1, ..., u^q, u^j_{i_1}, u^j_{i_1,i_2}, ..., u^j_{i_1,i_2,...,i_n}\right)_{1 \leq i_1 \leq i_2 \leq ... \leq i_n \leq p}$$

of real numbers at length

$$p + q + qp + ... + q_{p+n-1}C_n$$

in the sense that

$$\begin{aligned}(d_1, ..., d_p) \in D(p)_n \mapsto & \left(x^1, ..., x^p, u^1, ..., u^q\right) + \sum_{1 \leq i_1 \leq p} \left(0, ...0, u^1_{i_1}, ..., u^q_{i_1}\right) d_{i_1} \\ & + \sum_{1 \leq i_1 \leq i_2 \leq p} \left(0, ...0, u^1_{i_1,i_2}, ..., u^q_{i_1,i_2}\right) d_{i_1} d_{i_2} + ... \\ & + \sum_{1 \leq i_1 \leq i_2 \leq ... \leq i_n \leq p} \left(0, ...0, u^1_{i_1,i_2,...,i_n}, ..., u^q_{i_1,i_2,...,i_n}\right) d_{i_1} d_{i_2} ... d_{i_n} \\ & \in \mathbb{R}^{p+q}\end{aligned}$$

9.3 The First Approach with Coordinates

Definition 16 *1. We define* $\theta_{\mathbf{J}^1(\pi)}^{\mathcal{J}^1(\pi)}:\mathcal{J}^1(\pi)\rightarrow\mathbf{J}^1(\pi)$ *to be*

$$\begin{aligned}&\theta_{\mathbf{J}^1(\pi)}^{\mathcal{J}^1(\pi)}\left(x^1,...,x^p,u^1,...,u^q,u_i^j\right)\\&=\left[d\in D\mapsto\left(x^1,...,x^p\right)+\left(y^1,...,y^p\right)d\in\mathbb{R}^p\right]\in\left(M^D\right)_{(x^1,...,x^p)}\mapsto\\&\left[d\in D\mapsto\left(x^1,...,x^p,u^1,...,u^q\right)+\left(y^1,...,y^p,\sum_{i=1}^p u_i^j y^i\right)d\in\mathbb{R}^{p+q}\right]\\&\in\left(E^D\right)_{(x^1,...,x^p,u^1,...,u^q)}\end{aligned}$$

2. *Taking a step forward, we define* $\theta_{\mathbf{J}^2(\pi)}^{\mathcal{J}^2(\pi)}:\mathcal{J}^2(\pi)\rightarrow\mathbf{J}^2(\pi)$ *to be*

$$\begin{aligned}&\theta_{\mathbf{J}^2(\pi)}^{\mathcal{J}^2(\pi)}\left(x^i,u^j,u_{i_1}^j,u_{i_1,i_2}^j\right)\\&=\left[d\in D\mapsto\left(x^i\right)+\left(y^i\right)d\in\mathbb{R}^p\right]\in\left(M^D\right)_{(x^i)}\mapsto\\&\left[d\in D\mapsto\theta_{\mathbf{J}^1(\pi)}^{\mathcal{J}^1(\pi)}\left(\left(x^i,u^j,u_{i_1}^j\right)+\left(y^i,\sum_{i_1=1}^p u_{i_1}^j y^{i_1},\sum_{i_2=1}^p u_{i_1,i_2}^j y^{i_2}\right)d\right)\in\mathbf{J}^1(\pi)\right]\\&\in\left(\mathbf{J}^1(\pi)^D\right)_{\theta_{\mathbf{J}^1(\pi)}^{\mathcal{J}^1(\pi)}\left(x^i,u^j,u_{i_1}^j\right)}\end{aligned}$$

3. *Generally, proceeding by induction on* n*, we define* $\theta_{\mathbf{J}^{n+1}(\pi)}^{\mathcal{J}^{n+1}(\pi)}:\mathcal{J}^{n+1}(\pi)\rightarrow$ $\mathbf{J}^{n+1}(\pi)$ *to be*

$$\begin{aligned}&\theta_{\mathbf{J}^{n+1}(\pi)}^{\mathcal{J}^{n+1}(\pi)}\left(x^i,u^j,u_{i_1}^j,u_{i_1,i_2}^j,...,u_{i_1,i_2,...,i_{n+1}}^j\right)\\&=\left[d\in D\mapsto\left(x^i\right)+\left(y^i\right)d\in\mathbb{R}^p\right]\in\left(M^D\right)_{(x^i)}\mapsto\\&\left[\theta_{\mathbf{J}^n(\pi)}^{\mathcal{J}^n(\pi)}\left(\begin{array}{c}d\in D\mapsto\\\left(x^i,u^j,u_{i_1}^j,...,u_{i_1,i_2,...,i_n}^j\right)+\\\left(y^i,\sum_{i_1=1}^p u_{i_1}^j y^{i_1},...,\sum_{i_{n+1}=1}^p u_{i_1,i_2,...,i_{n+1}}^j y^{i_{n+1}}\right)d\end{array}\right)\in\mathbf{J}^n(\pi)\right]\\&\in\left(\mathbf{J}^n(\pi)^D\right)_{\theta_{\mathbf{J}^n(\pi)}^{\mathcal{J}^n(\pi)}\left(x^i,u^j,u_{i_1}^j,...,u_{i_1,i_2,...,i_n}^j\right)}\end{aligned}$$

Theorem 17 *The mapping* $\theta_{\mathbf{J}^n(\pi)}^{\mathcal{J}^n(\pi)}:\mathcal{J}^n(\pi)\rightarrow\mathbf{J}^n(\pi)$ *is bijective.*

9.4 The Second Approach with Coordinates

Definition 18 *We define mappings* $\theta_{\mathbb{J}^{D^n}(\pi)}^{\mathcal{J}^n(\pi)} : \mathcal{J}^n(\pi) \to \mathbb{J}^{D^n}(\pi)$ *as* $\varphi_n \circ \theta_{\mathbf{J}^n(\pi)}^{\mathcal{J}^n(\pi)}$.

Theorem 19 *The mapping* $\theta_{\mathbb{J}^{D^n}(\pi)}^{\mathcal{J}^n(\pi)}$ *is bijective and is of the following description:*

$$\theta_{\mathbb{J}^{D^n}(\pi)}^{\mathcal{J}^n(\pi)}(x^i, u^j, u^j_{i_1}, u^j_{i_1,i_2}, \ldots, u^j_{i_1,i_2,\ldots,i_n})$$
$$= \left[(\delta_1, \ldots, \delta_n) \in \mathbb{R}^n \longmapsto (x^i) + \sum_{r=1}^{n} \sum_{1 \le k_1 < \ldots < k_r \le n} \delta_{k_1} \ldots \delta_{k_r} (y^i_{k_1,\ldots,k_r}) \in \mathbb{R}^p \right]$$
$$\in (M \otimes \mathcal{W}_{D^n})_{(x^i)} \mapsto$$
$$\left[\begin{array}{c} (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n \longmapsto (x^i, u^j) + \\ \sum_{r=1}^{n} \sum_{1 \le k_1 < \ldots < k_r \le n} \delta_{k_1} \ldots \delta_{k_r} (y^i_{k_1,\ldots,k_r}, \sum \sum_{i_1=1}^{p} \cdots \sum_{i_s=1}^{p} y^{i_1}_{\mathbf{J}_1} \ldots y^{i_s}_{\mathbf{J}_s} u^j_{i_1,\ldots,i_s}) \in \mathbb{R}^{p+q} \end{array} \right]$$
$$\in (E \otimes \mathcal{W}_{D^n})_{(x^i, u^j)}$$

where the completely undecorated $\sum$ *is taken over all partitions of the set* $\{k_1, \ldots, k_r\}$ *into nonempty subsets* $\{\mathbf{J}_1, \ldots, \mathbf{J}_s\}$, *and if* $\mathbf{J} = \{k_1, \ldots, k_t\}$ *is a set of natural numbers with* $k_1 < \ldots < k_t$, *then* $y^i_{\mathbf{J}}$ *denotes* $y^i_{k_1,\cdots,k_t}$.

9.5 The Third Approach with Coordinates

Definition 20 *We define mappings* $\theta_{\mathbb{J}^{D_n}(\pi)}^{\mathcal{J}^n(\pi)} : \mathcal{J}^n(\pi) \to \mathbb{J}_{D_n}(\pi)$ *as* $\psi_n \circ \theta_{\mathbb{J}^{D^n}(\pi)}^{\mathcal{J}^n(\pi)}$.

Theorem 21 *The mapping* $\theta_{\mathbb{J}^{D_n}(\pi)}^{\mathcal{J}^n(\pi)} : \mathcal{J}^n(\pi) \to \mathbb{J}_{D_n}(\pi)$ *is bijective and is of the following description:*

$$\theta_{\mathbb{J}^{D_n}(\pi)}^{\mathcal{J}^n(\pi)}(x^i, u^j, u^j_{i_1}, u^j_{i_1,i_2}, ..., u^j_{i_1,i_2,...,i_n})$$
$$= \left[\delta \in \mathbb{R} \longmapsto (x^i) + \sum_{k=1}^{n} \frac{\delta^k}{k!}(y^i_k) \in \mathbb{R}^p\right] \in (M \otimes \mathcal{W}_{D_n})_{(x^i)} \mapsto$$
$$\left[\delta \in \mathbb{R} \longmapsto (x^i, u^j) + \sum_{k=1}^{n} \frac{\delta^k}{k!} \sum \sum_{i_1=1}^{p} ... \sum_{i_r=1}^{p} \left(y^i_k, u^j_{i_1,...,i_r} y^{i_1}_{k_1} ... y^{i_r}_{k_r}\right) \in \mathbb{R}^{p+q}\right]$$
$$\in (E \otimes \mathcal{W}_{D_n})_{(x^i,u^j)}$$

where the undecorated $\sum$ *is taken over all partitions of the positive integer* k *into positive integers* $k_1, ..., k_r$ *(so that* $k = k_1 + ... + k_r$*) with* $1 \leq k_1 \leq ... \leq k_r \leq n$.

10 Frölicher Spaces

Theorem 22 *The category of Frölicher spaces and smooth mappings is cartesian closed.*

Theorem 23 *The category of convenient vector spaces and smooth linear mappings is cartesian closed.*

However

Remark 24 *The category of smooth manifolds modelled after convenient vector spaces as local models and smooth mappings is by no means cartesian closed !!!*

Alfred Frölicher and Anreas Kriegl : Linear Spaces and Differentiation Theory
John Wiley and Sons 1988

11 Externalized Microlinearity

Weil prolongation

$$X \otimes W \quad (X\text{:Frölicher space, } W\text{:Weil algebra})$$

standing intuitively for

$$X^{\mathcal{D}_W}$$

Weil exponentiability

Definition 25 *A Frölicher space X is called Weil exponentiable if*

$$(X \otimes (W_1 \otimes_\infty W_2))^Y = (X \otimes W_1)^Y \otimes W_2$$

holds naturally for any Frölicher space Y and any Weil algebras W_1 and W_2.

If $Y = 1$,

$$X \otimes (W_1 \otimes_\infty W_2) = (X \otimes W_1) \otimes W_2$$

If $W_1 = \mathbb{R}$,

$$(X \otimes W_2)^Y = X^Y \otimes W_2$$

Nishimura, Hirokazu : A muchlarger class of Frölicher spaces than that of convenient vector spaces may embed into the Cahiers topos, Far East Journal of Mathematical Sciences, **35**(2009), 211-223.

Microlinearity

Definition 26 *A Frölicher space X is called microlinear providing that any finite limit diagram $\mathbb{L}$ in the category of Weil algebras yields a limit diagram $X \otimes \mathbb{L}$ in the category of Frölicher spaces.*

Theorem 27 *Weil exponentiable and microlinear Frölicher spaces, together with smooth mappings among them, form a cartesian closed category*

Nishimura, Hirokazu : Microlinearity in Frölicher spaces -beyond the regnant philosophy of manifolds-, International Journal of Pure and Applied Mathematics, **60** (2010), 15-24.

Thank you for your attention!