Radiation by a heavy quark in $\mathcal{N} = 4$ SYM at strong coupling

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Abstract: Using the AdS/CFT correspondence in the supergravity approximation, we compute the energy density radiated by a heavy quark undergoing some arbitrary motion in the vacuum of the strongly coupled $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. We find that this energy is fully generated via backreaction from the near–boundary endpoint of the dual string attached to the heavy quark. Because of that, the energy distribution shows the same space–time localization as the classical radiation that would be produced by the heavy quark at weak coupling. We believe that this and some other unnatural features of our result (like its anisotropy and the presence of regions with negative energy density) are artifacts of the supergravity approximation, which will be corrected after including string fluctuations. For the case where the quark trajectory is bounded, we also compute the radiated power, by integrating the energy density over the surface of a sphere at infinity. For sufficiently large times, we find agreement with a previous calculation by Mikhailov [hep-th/0305196].

Contents

1 Introduction

One of the basic problems that one can think of in the context of any gauge theory, and in particular within a strongly–coupled, conformal, field theory as described by the AdS/CFT correspondence [1–3], is that of the radiation by a moving, classical, charged particle. By 'classical' we mean a particle which is heavy enough to be treated as pointlike and assumed to follow a well–identified, classical, trajectory (say, under the action of an external force). And by 'radiation' we mean the emission of quanta of the underlying gauge theory which escape at infinity, thus generating energy loss. For asymptotically weak coupling, these quanta need to be strictly on–shell and hence propagate at the speed of light (for the radiation in the vacuum). But in general, the emitted quanta can be also off–shell, in which case they are subjected to further evolution (e.g., time–like quanta can decay). In particular, when the coupling is strong, we expect such off–shell effects to be very important and generate a very different space–time pattern for the radiated energy as compared to weak coupling.

Consider, for instance, the radiation produced by a heavy quark subjected to a kick, *i.e.* an external force which is localized in space and time. In a classical calculation, which is the same as the weak coupling limit of the corresponding field theory, the radiation will propagate away from the quark as a spherical shell which is radially expanding at the speed of light ($r = t$), with a small width Δr determined by the duration of the original perturbation. This is quite different from the picture we would expect at strong coupling [4]. There, the radiation should typically proceed via the emission of a few virtual quanta, which will then decay into other quanta, thus eventually generating a system of partons with a wide distribution in virtualities. Since time–like quanta propagate slower than the speed of light, the energy taken away by those quanta should exhibit a rather broad distribution in r at $r \lesssim t$. Since moreover the various quanta can be randomly emitted along any direction (in the quark rest frame), this picture also implies that the energy distribution should be isotropic (up to a Lorentz boost). This last prediction has been checked via an explicit calculation within AdS/CFT of the angular distribution of the energy produced by the decay of a time–like wavepacket at strong coupling [5].

In view of the above, it appeared as a surprise when other AdS/CFT calculations [6, 7], which have also investigated the radial distribution, found that there is no broadening (at least, within the limits of the respective calculations): the energy radiated within the vacuum of the $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory at strong coupling appears to be as sharply localized in r as the corresponding classical result. This was first noticed in Ref. [6] for the example of the synchrotron radiation produced by a heavy quark in uniform rotation and then extended in Ref. [7] to other situations, including the two problems alluded to above — a heavy quark perturbed by a kick and the decay of a time–like 'photon'. As pointed out in [7] (but already visible at the level of the calculations in [6]), this lack of radial broadening is to be attributed to the fact that, within the supergravity approximation used in these calculations, the whole contribution to radiation is generated via backreaction from points near the Minkowski boundary of AdS_5.

At this level, it is useful to recall that the supergravity approximation is the classical limit of the dual string theory, which neglects string loops and string fluctuations, and is generally accepted to faithfully describe the strong 't Hooft coupling limit $\lambda = g^2 N_c \to \infty$ with fixed $g \ll 1$ of the $\mathcal{N} = 4$ SYM theory [1–3]. (g is the Yang–Mills coupling and N_c the number of colors.) Furthermore, the 'backreaction' refers to the AdS/CFT calculation of the energy density in the gauge theory, which involves the response of the AdS_5 metric to the 5D stress tensor of the bulk object dual to the physical excitation on the boundary. (For instance, this bulk object is a Nambu–Goto string in the case of a heavy quark in the fundamental representation of $SU(N_c)$, and a supergravity vector field wave–packet in the case of a time–like photon.)

A priori, the calculation of the backreaction involves an integral over all the points within the support of the bulk stress tensor, say along the string in the case of a heavy quark. Similar calculations at *finite temperature* [8–11] have shown that, in that context, all the points along the string provide non–trivial contributions to the energy density on the boundary, which therefore shows broadening: points of the string which lie further and further away from the boundary provide contributions which are more and more spread in space–time. This is in the spirit of the *ultraviolet/infrared correspondence* [12, 13], which associates increasing distance from the boundary of AdS_5 with increasing virtuality in the original gauge theory. However, in the case of the radiation in the *vacuum* (say, as produced by an accelerated quark), the calculations in Refs. [6, 7] show that the integral expressing the backreaction reduces to a boundary contribution from the string endpoint at the Minkowski boundary. Thus, there is effectively no virtuality involved in this calculation, which 'explains' why the final result shows no spreading. But this 'explanation' leaves us with a physical paradox, namely why should radiation in a quantum field theory at infinitely strong coupling involve only on–shell (light–like) modes, without any trace of virtual quantum fluctuations.

In Ref. [7] we have also proposed a possible solution to this puzzle, by identifying a class of stringy corrections which are *not* suppressed in the strong coupling limit and which when included in the calculation of the backreaction — in an admittedly heuristic way, by lack of a proper treatment of string fluctuations in AdS_5 — seem to provide energy broadening, in conformity with the UV/IR correspondence. It would be of course very interesting to make further progress with the understanding of string corrections in AdS_5, which is an outstanding open problem. This is however not the purpose of this paper. Rather, here we shall be more modest and extend the results in Ref. [7] in a different direction: we shall provide an exact result for the energy density radiated in the supergravity approximation by heavy quark undergoing some arbitrary motion in the vacuum of the $\mathcal{N}=4$ SYM theory.

In spite of our own criticism of the supergravity approximation for the type of problems at hand, we believe that the present results are nevertheless interesting for several reasons. First, a precise knowledge of the classical, supergravity, result is a first and mandatory step in any effort aiming at including string corrections. Second, by itself, this classical calculation is rather non–trivial, as it requires an exact, analytic, solution to the problem of the backreaction. Previously, such analytic solutions have been given only for particular cases — the most non–trivial one being the calculation of the synchrotron radiation in Ref. [6]. Our general results below will allow us to simply recover such previous results and extend them to an arbitrary motion. Third, we shall explicitly verify that, also in the general case, the whole contribution to the backreaction is still coming from the string endpoint near the boundary; hence, in this approximation, the radiation propagates at the speed of light, like in a classical field theory. Fourth, our results exhibit some other surprising features (besides the lack of radial broadening), which in our opinion reflect the limitations of the supergravity approximation: the energy density appears to be anisotropic and also negative in some regions of space–time. The anisotropy is unnatural at strong coupling, for reasons explained before; it is even more so in the context of the $\mathcal{N}=4$ SYM theory, where, as we shall see, already the corresponding *classical* result is fully isotropic[1] (up to boost effects). As for the negativity of the energy density, which was already noticed (for the example of uniform rotation) in [6], this is in principle acceptable in a quantum field theory, but we find it very unnatural in the context of radiation, for reasons to be discussed in Sect. 8.

Our analysis of the backreaction, that we now outline, will build upon previous constructions in the literature. An essential ingredient in that sense is the exact solution, due to Mikhailov [15], for the string profile corresponding to an arbitrary motion of the heavy quark. This solution is not fully explicit — it still depends upon a 'retardation time', determined as the solution of a generally transcendental equation (see Sect. 2 for details) —, but this is not less explicit than the usual textbook treatment of radiation in classical electrodynamics [14], where the results are written as a function of the 'retardation time' (the time of emission, related to the time and point of measurement by the condition of propagation at the speed of light). By studying the energy carried by the accelerated string, Mikhailov has also deduced a formula for the radiated power, which appears to be similar to Liénard formula in classical electrodynamics. His results have been extended in Refs. [16, 17], where the total energy of the moving quark (proper energy plus radiation) has been inferred via a world–sheet analysis.

[1] To better appreciate the non–trivial character of this property, one should recall that it does not hold for radiation in classical electrodynamics [14]. See also the discussion in Sect. 7.

Furthermore, in computing the backreaction, we shall use the general formulæ established in Ref. [6] which express the energy density on the boundary as a convolution between the bulk stress tensor of the string and the bulk–to–boundary propagator. Using Mikhailov's solution for the string profile, we shall express the bulk stress tensor in terms of the quark motion on the boundary (in Sect. 3), and this will allow us to explicitly perform the integrals yielding the backreaction (in Sect. 4). We shall thus find that, due to remarkable cancelations between contributions arising from various components of the stress tensor, the only terms which are left in the final result for the energy density are boundary contributions from the string endpoint at the heavy quark. Then, in Sect. 5, we shall extract the *radiative* energy density, defined as the component of the total energy showing the slowest decay ($\sim 1/R^2$) at large distances. This is the main result of our paper. By integrating this result over the surface of a sphere at infinity (an operation which is well defined when the quark trajectory is confined to a bounded region in space), we shall also compute the radiated power (still in Sect. 5). We shall thus find the term originally obtained by Mikhailov [15] and also a second term, which is however subleading at large times[2]. In the remaining part of the paper, we shall further discuss our results, compare them to some known limits in the literature (in Sect. 6) and also to the corresponding classical results (that we shall derive in the context of the $\mathcal{N} = 4$ SYM theory in Sect. 7). In our final discussion in Sect. 8, we shall emphasize some peculiar features of these results, which point towards limitations of the supergravity approximation.

2 The string profile

We consider a heavy quark in the fundamental representation of the gauge group SU(N_c) which undergoes some arbitrary motion, with trajectory $\boldsymbol{r} = \boldsymbol{r}_q(t)$, within the vacuum of the $\mathcal{N} = 4$ SYM theory at strong coupling. (We assume the quark to be arbitrarily heavy, so that the notion of classical trajectory makes indeed sense for it.) The quark can either be in isolation, or it can be a part of a quark–antiquark pair. The dual supergravity description of the quark dynamics is a Nambu–Goto string hanging in AdS$_5$, with one endpoint attached to a D7–brane. The string can be either finite, with the other endpoint attached to the same D7–brane (the case of a quark–antiquark pair), or it can extend all the way to the center of AdS$_5$ (the case of a single quark).

We shall parameterize the AdS$_5$ space–time using Poincaré coordinates, with metric

$$\mathrm{d}s^2 \,\equiv\, G_{MN}\,\mathrm{d}x^M\mathrm{d}x^N \,=\, \frac{L^2}{z^2}\left(-\mathrm{d}t^2 + \mathrm{d}\boldsymbol{r}^2 + \mathrm{d}z^2\right). \tag{2.1}$$

Here, $x^M = (x^\mu, z)$ where $x^\mu = (t, \boldsymbol{r})$ are the Minkowski coordinates in the physical space–time and z (with $0 \le z < \infty$) is the fifth dimension, also known as the 'radial coordinate in AdS$_5$' (not to be confused with the physical radius $r = |\boldsymbol{r}|$). In these coordinates, the Minkowski boundary lies at $z = 0$, the center of AdS$_5$ is at $z \to \infty$, and the D7–brane ends at a distance $z_m = \sqrt{\lambda}/(2\pi m_q)$ from the boundary, with m_q the quark mass. In what follows we shall choose m_q to be large enough for z_m to be much smaller than any other interesting space–time scale.

[2]Interestingly, this second term is similar to a piece of the total energy of the accelerated string which in Refs. [16, 17] has been interpreted as a part of the quark proper (or kinetic) energy. See the discussion in Sect. 6.

The string dynamics is encoded in the Nambu–Goto action

$$S = -T_0 \int \mathrm{d}\tau\, \mathrm{d}\sigma \sqrt{-g}\,, \qquad g_{ab} \,=\, G_{MN}\partial_a X^M \partial_b X^N\,, \tag{2.2}$$

where $T_0 = \sqrt{\lambda}/2\pi L^2$ is the string tension, τ and σ are the two coordinates on the world–sheet, $X^M(\tau,\sigma)$ are the string coordinates in AdS$_5$, and g_{ab}, with $a,\, b = \tau,\, \sigma$, is the induced metric on the string world–sheet.

Choosing $\tau = t$ and $\sigma = z$ as the two coordinates parametrizing the world-sheet we can write the embedding functions and its derivatives as

$$X^M = (t, \boldsymbol{r}_s, z), \qquad \dot{X}^M = (1, \dot{\boldsymbol{r}}_s, 0), \qquad X^{M\prime} = (0, \boldsymbol{r}_s', 1), \tag{2.3}$$

where a dot or a prime on $\boldsymbol{r}_s$ denote a derivative with respect to t or z respectively. The individual components and the determinant of the induced metric read

$$\begin{aligned} g_{\tau\tau} &= \dot{X}\cdot\dot{X} = |G_{00}|\,(-1+\dot{\boldsymbol{r}}_s^2), \qquad & g_{\sigma\sigma} &= X'\cdot X' = |G_{00}|\,(1+\boldsymbol{r}_s'^2),\\ g_{\tau\sigma} &= \dot{X}\cdot X' = |G_{00}|\,\dot{\boldsymbol{r}}_s\cdot\boldsymbol{r}_s', & \sqrt{-g} &= |G_{00}|\,\sqrt{1-\dot{\boldsymbol{r}}_s^2+\boldsymbol{r}_s'^2-(\dot{\boldsymbol{r}}_s\times\boldsymbol{r}_s')^2}, \end{aligned} \tag{2.4}$$

with $|G_{00}| = L^2/z^2$. The condition that the action (2.2) be stationary under small variations $\boldsymbol{r}_s \to \boldsymbol{r}_s + \delta\boldsymbol{r}_s(t,z)$ yields the string equations of motion

$$\frac{\partial}{\partial t}\,\frac{(1+\boldsymbol{r}_s'^2)\dot{\boldsymbol{r}}_s-(\dot{\boldsymbol{r}}_s\cdot\boldsymbol{r}_s')\boldsymbol{r}_s'}{\sqrt{1-\dot{\boldsymbol{r}}_s^2+\boldsymbol{r}_s'^2-(\dot{\boldsymbol{r}}_s\times\boldsymbol{r}_s')^2}} - \frac{1}{|G_{00}|}\,\frac{\partial}{\partial z}\,\frac{|G_{00}|[(1-\dot{\boldsymbol{r}}_s^2)\boldsymbol{r}_s'+(\dot{\boldsymbol{r}}_s\cdot\boldsymbol{r}_s')\dot{\boldsymbol{r}}_s]}{\sqrt{1-\dot{\boldsymbol{r}}_s^2+\boldsymbol{r}_s'^2-(\dot{\boldsymbol{r}}_s\times\boldsymbol{r}_s')^2}} = 0\,. \tag{2.5}$$

As shown by Mikhailov [15], the general solution $\boldsymbol{r}_s(t,z)$ to the above equation is implicitly determined by

$$t \,=\, t_q + \gamma_q z, \qquad \boldsymbol{r}_s \,=\, \boldsymbol{r}_q + \boldsymbol{v}_q(t-t_q)\,, \tag{2.6}$$

where $\boldsymbol{r}_q$, $\boldsymbol{v}_q$ and γ_q are evaluated at t_q, with $\boldsymbol{v}_q \equiv \dot{\boldsymbol{r}}_q$ the quark velocity and $\gamma_q \equiv 1/(1-v_q^2)^{1/2}$. These two equations should be understood as follows: by solving the first equation (2.6), one obtains t_q as a function of t and z, which is then inserted into the second equation (2.6) to obtain the function $\boldsymbol{r}_s(t,z)$. The solutions thus obtained must be restricted to $z \geq z_m$. By assumption, z_m is arbitrarily small, but the limit $z_m \to 0$ can be taken only after performing the 'ultraviolet renormalization', *i.e.*, after absorbing a would–be divergent contribution in that limit in the definition of the quark mass. To clarify the physical interpretation of eqs. (2.6), notice that they imply

$$(\boldsymbol{r}_s-\boldsymbol{r}_q)^2 + z^2 = (t-t_q)^2 \qquad \text{and} \qquad t_q(t,z=0) = t\,, \qquad \boldsymbol{r}_s(t,z=0) = \boldsymbol{r}_q(t)\,, \tag{2.7}$$

and that the velocity of light in AdS$_5$ is equal to one within the present conventions. Hence eq. (2.7) can be interpreted as follows: a light signal emitted at time t_q at the point with $\boldsymbol{r} = \boldsymbol{r}_q$ and $z = 0$ reaches the string (at the point with coordinates $\boldsymbol{r}_s$ and z) at the later time t. Thus, eqs. (2.6) show how the string gets built within the bulk via radiation from the quark on the boundary.

We shall often need the derivatives of t_q w.r.t. t and z, which read (below, $\boldsymbol{a}_q \equiv \dot{\boldsymbol{v}}_q$)

$$\frac{\partial t_q}{\partial t} = \frac{1}{1+z\gamma_q^3\,\boldsymbol{v}_q\cdot\boldsymbol{a}_q}\,, \qquad \frac{\partial t_q}{\partial z} = -\frac{\gamma_q}{1+z\gamma_q^3\,\boldsymbol{v}_q\cdot\boldsymbol{a}_q}\,. \tag{2.8}$$

Using these formulæ together with eqs. (2.6) it is straightforward to express the derivatives of $\boldsymbol{r}_s$ in terms of the boundary motion:

$$\dot{\boldsymbol{r}}_s = \boldsymbol{v}_q + \frac{\gamma_q z \boldsymbol{a}_q}{1 + z\gamma_q^3\, \boldsymbol{v}_q \cdot \boldsymbol{a}_q}, \quad \boldsymbol{r}'_s = -\frac{\gamma_q^2 z \boldsymbol{a}_q}{1 + z\gamma_q^3\, \boldsymbol{v}_q \cdot \boldsymbol{a}_q}, \tag{2.9}$$

The following identity is also useful:

$$\sqrt{1 - \dot{\boldsymbol{r}}_s^2 + \boldsymbol{r}_s'^2 - (\dot{\boldsymbol{r}}_s \times \boldsymbol{r}'_s)^2} = \frac{1}{\gamma_q} \frac{\partial t_q}{\partial t}\,. \tag{2.10}$$

3 5D bulk stress tensor and backreaction

The calculation of the space–time distribution of the energy produced by the heavy quark requires to solve the 'backreaction problem', that is, to compute the perturbation δG_{MN} of the AdS$_5$ metric associated with the string. For the problem at hand, this perturbation is relatively small, of $\mathcal{O}(1/N_c^2)$, and can be computed by solving the linearized Einstein equation with t_{MN} (the string stress tensor) as a source. The expectation value $\langle T_{\mu\nu}\rangle$ of the physical stress tensor in the boundary gauge theory is then obtained from the near–boundary ($z \to 0$) behaviour of $\delta G_{\mu\nu}$. Thus, it becomes apparent that we first need to compute the 5D bulk tensor and since our source is a string it will be proportional to $\delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_s)$. More precisely

$$t^{MN}(t, \boldsymbol{r}, z) \;=\; -\frac{T_0}{\sqrt{-G}}\sqrt{-g}\, g^{ab}\, \partial_a X^M\, \partial_b X^N\, \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_s) \equiv \tilde{t}_{MN}\, \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}_s), \tag{3.1}$$

where it is not hard to see that $\tilde{t}_{MN}$ is given by

$$\tilde{t}^{MN} = \frac{T_0}{\sqrt{-g}\sqrt{-G}} \left[g_{\sigma\sigma} \dot{X}^M \dot{X}^N + g_{\tau\tau} X^{M\prime} X^{N\prime} - g_{\tau\sigma} \left(\dot{X}^M X^{N\prime} + X^{M\prime} \dot{X}^N \right) \right]. \tag{3.2}$$

Now we calculate the components of the above and at the same time lower the indices. The metric is diagonal and we do this by multiplying each component by $\pm|G_{00}|^2$, where we use the minus sign only when one of the two indices is equal to 0. After substitution of the common coefficient (notice that there is a third $|G_{00}|$ factor coming from g_{ab}, eq. (2.4))

$$\frac{T_0 |G_{00}|^3}{\sqrt{-g}\sqrt{-G}} = \frac{\sqrt{\lambda}}{2\pi} \frac{z}{L^3} \frac{1}{\sqrt{1 - \dot{\boldsymbol{r}}_s^2 + \boldsymbol{r}_s'^2 - (\dot{\boldsymbol{r}}_s \times \boldsymbol{r}'_s)^2}} = \frac{\sqrt{\lambda}}{2\pi} \frac{z\gamma_q}{L^3} \frac{\partial t}{\partial t_q}, \tag{3.3}$$

we have

$$\tilde{t}_{00} = \frac{\sqrt{\lambda}}{2\pi} \frac{z\gamma_q}{L^3} \frac{\partial t}{\partial t_q} (1 + \boldsymbol{r}_s'^2), \tag{3.4}$$

$$\tilde{t}_{0i} = \frac{\sqrt{\lambda}}{2\pi} \frac{z\gamma_q}{L^3} \frac{\partial t}{\partial t_q} [-(1 + \boldsymbol{r}_s'^2)\dot{x}_s^i + (\dot{\boldsymbol{r}}_s \cdot \boldsymbol{r}'_s) x_s'^i], \tag{3.5}$$

$$\tilde{t}_{05} = \frac{\sqrt{\lambda}}{2\pi} \frac{z\gamma_q}{L^3} \frac{\partial t}{\partial t_q} \dot{\boldsymbol{r}}_s \cdot \boldsymbol{r}'_s, \tag{3.6}$$

$$\tilde{t}_{ij} = \frac{\sqrt{\lambda}}{2\pi} \frac{z\gamma_q}{L^3} \frac{\partial t}{\partial t_q} [(1 + \boldsymbol{r}_s'^2)\dot{x}_s^i \dot{x}_s^j + (-1 + \dot{\boldsymbol{r}}_s^2) x_s'^i x_s'^j - (\dot{\boldsymbol{r}}_s \cdot \boldsymbol{r}'_s)(\dot{x}_s^i x_s'^j + \dot{x}_s^j x_s'^i)], \tag{3.7}$$

$$\tilde{t}_{i5} = \frac{\sqrt{\lambda}}{2\pi} \frac{z\gamma_q}{L^3} \frac{\partial t}{\partial t_q} [(-1 + \dot{\boldsymbol{r}}_s^2) x_s'^i - (\dot{\boldsymbol{r}}_s \cdot \boldsymbol{r}'_s)\dot{x}_s^i], \tag{3.8}$$

$$\tilde{t}_{55} = \frac{\sqrt{\lambda}}{2\pi} \frac{z\gamma_q}{L^3} \frac{\partial t}{\partial t_q} (-1 + \dot{\boldsymbol{r}}_s^2). \tag{3.9}$$

Expressing the components in terms of the boundary motion according to eq. (2.9), we finally deduce

$$\tilde{t}_{00} = \frac{\sqrt{\lambda}}{2\pi}\frac{z\gamma_q}{L^3}\frac{\partial t_q}{\partial t}\left\{1+2z\gamma_q^3\,\boldsymbol{v}_q\cdot\boldsymbol{a}_q+z^2\gamma_q^6[\boldsymbol{a}_q^2-(\boldsymbol{v}_q\times\boldsymbol{a}_q)^2]\right\}, \tag{3.10}$$

$$\tilde{t}_{0i} = \frac{\sqrt{\lambda}}{2\pi}\frac{z\gamma_q}{L^3}\frac{\partial t_q}{\partial t}\left\{-z\gamma_q a_q^i-(1+2z\gamma_q^3\boldsymbol{v}_q\cdot\boldsymbol{a}_q)v_q^i-z^2\gamma_q^6[\boldsymbol{a}_q^2-(\boldsymbol{v}_q\times\boldsymbol{a}_q)^2]v_q^i\right\}, \tag{3.11}$$

$$\tilde{t}_{05} = \frac{\sqrt{\lambda}}{2\pi}\frac{z\gamma_q}{L^3}\frac{\partial t_q}{\partial t}\left\{-z\gamma_q^2\,\boldsymbol{v}_q\cdot\boldsymbol{a}_q-z^2\gamma_q^5[\boldsymbol{a}_q^2-(\boldsymbol{v}_q\times\boldsymbol{a}_q)^2]\right\}, \tag{3.12}$$

$$\begin{aligned}\tilde{t}_{ij} = \frac{\sqrt{\lambda}}{2\pi}\frac{z\gamma_q}{L^3}\frac{\partial t_q}{\partial t}\big\{&z\gamma_q(v_q^i a_q^j+v_q^j a_q^i)+(1+2z\gamma_q^3\,\boldsymbol{v}_q\cdot\boldsymbol{a}_q)v_q^i v_q^j\\&+z^2\gamma_q^6[\boldsymbol{a}_q^2-(\boldsymbol{v}_q\times\boldsymbol{a}_q)^2]v_q^i v_q^j\big\},\end{aligned} \tag{3.13}$$

$$\tilde{t}_{i5} = \frac{\sqrt{\lambda}}{2\pi}\frac{z\gamma_q}{L^3}\frac{\partial t_q}{\partial t}\left\{za_q^i+z\gamma_q^2\boldsymbol{v}_q\cdot\boldsymbol{a}_q v_q^i+z^2\gamma_q^5[\boldsymbol{a}_q^2-(\boldsymbol{v}_q\times\boldsymbol{a}_q)^2]v_q^i\right\}, \tag{3.14}$$

$$\tilde{t}_{55} = \frac{\sqrt{\lambda}}{2\pi}\frac{z\gamma_q}{L^3}\frac{\partial t_q}{\partial t}\left\{-\frac{1}{\gamma_q^2}+z^2\gamma_q^4[\boldsymbol{a}_q^2-(\boldsymbol{v}_q\times\boldsymbol{a}_q)^2]\right\}. \tag{3.15}$$

4 Energy density in the gauge theory

The details of this analysis relating the energy density $\mathcal{E}\equiv\langle T_{00}\rangle$ on the boundary to the string stress tensor in the bulk can be found in Ref. [6] from which we shall simply borrow the final formulæ. Namely, one has

$$\mathcal{E}(t,\boldsymbol{r})=\mathcal{E}_{\rm A}(t,\boldsymbol{r})+\mathcal{E}_{\rm B}(t,\boldsymbol{r}), \tag{4.1}$$

where the two contributions are

$$\mathcal{E}_{\rm A}=\frac{2L^3}{\pi}\int\frac{{\rm d}^4\acute{r}\,{\rm d}z}{z^2}\,\Theta(t-\acute{t})\delta''(\mathcal{W})\big[z(2t_{00}-t_{55})-(t-\acute{t})t_{05}+(x-\acute{x})^i t_{i5}\big], \tag{4.2}$$

$$\mathcal{E}_{\rm B}=\frac{2L^3}{3\pi}\int\frac{{\rm d}^4\acute{r}\,{\rm d}z}{z}\,\Theta(t-\acute{t})\delta'''(\mathcal{W})\big[|\boldsymbol{r}-\acute{\boldsymbol{r}}|^2(2t_{00}-2t_{55}+t_{ii})-3(x-\acute{x})^i(x-\acute{x})^j t_{ij}\big]. \tag{4.3}$$

The argument of t_{MN} is $(\acute{t},\acute{\boldsymbol{r}},z)$ and the quantity

$$\mathcal{W}\equiv-(t-\acute{t})^2+(\boldsymbol{r}-\acute{\boldsymbol{r}})^2+z^2 \tag{4.4}$$

is proportional to the 5D invariant distance between the source point in the bulk and the measurement point on the boundary. Eqs. (4.2)–(4.3) are essentially convolutions of t_{MN} with the graviton bulk–to–boundary propagator.

The integration over ${\rm d}^3\acute{\boldsymbol{r}}$ is trivially done using the δ-function of the string stress tensor given in eq. (3.1) and then by using eqs. (3.10)–(3.15) we can express the integrands in eqs. (4.2) and (4.3) in terms of the boundary motion. Since $\boldsymbol{r}_q,\boldsymbol{v}_q$ and $\boldsymbol{a}_q$ are evaluated at t_q, the calculation simplifies if we change variable from $\acute{t}$ to t_q. Then for the $\mathcal{E}_{\rm A}$ term we have

$$\mathcal{E}_{\rm A}=\frac{\sqrt{\lambda}}{\pi^2}\int{\rm d}t_q\,{\rm d}z\,\delta''(\mathcal{W}_q+2\gamma_q\Xi z)[A_0(t_q)+zA_1(t_q)+z^2A_2(t_q)], \tag{4.5}$$

with the definitions

$$\mathcal{W}_q\equiv-(t-t_q)^2+|\boldsymbol{r}-\boldsymbol{r}_q|^2,\qquad \Xi\equiv(t-t_q)-\boldsymbol{v}_q\cdot(\boldsymbol{r}-\boldsymbol{r}_q)=\frac{1}{2}\frac{{\rm d}\mathcal{W}_q}{{\rm d}t_q}, \tag{4.6}$$

and where the coefficients of the polynomial in z in the square bracket of eq. (4.5) are

$$A_0 = 3\gamma_q - \gamma_q v_q^2 + \gamma_q^3 \boldsymbol{v}_q \cdot \boldsymbol{a}_q [t - t_q + \boldsymbol{v}_q \cdot (\boldsymbol{r} - \boldsymbol{r}_q)] + \gamma_q \, \boldsymbol{a}_q \cdot (\boldsymbol{r} - \boldsymbol{r}_q), \tag{4.7}$$

$$A_1 = 2\gamma_q^4 \boldsymbol{v}_q \cdot \boldsymbol{a}_q + \gamma_q^6 [\boldsymbol{a}_q^2 - (\boldsymbol{v}_q \times \boldsymbol{a}_q)^2][t - t_q + \boldsymbol{v}_q \cdot (\boldsymbol{r} - \boldsymbol{r}_q)], \tag{4.8}$$

$$A_2 = 0. \tag{4.9}$$

Note that, given the z–dependencies of the tensor components $\tilde{t}_{MN}$ in eqs. (3.10)–(3.15) and the other factors of z in the integrand of eq. (4.2), one would *a priori* expect the integrand of eq. (4.5) to contain a polynomial of second order in z. However, in reality this polynomial is *linear* since, as shown in eq. (4.9) above, the coefficient A_2 of the quadratic term is identically zero, due to rather non–trivial cancelations. For example, for the terms proportional to $z^2[\boldsymbol{a}_q^2 - (\boldsymbol{v}_q \times \boldsymbol{a}_q)^2]$, there are four different contributions yielding a total coefficient $2\gamma_q^2 - 1 - \gamma_q^2 - \gamma_q^2 v_q^2$, which indeed vanishes. This has important consequences to which we shall shortly return.

After replacing $\acute{t} \to t_q$ as the integration variable, the argument of the δ–function has become linear in z and thus it is easier to perform first the corresponding integration. The two derivatives in δ'' can be taken w.r.t. $\mathcal{W}_q$ and pulled outside the z–integration. Then the δ–function sets

$$z = -\frac{\mathcal{W}_q}{2\gamma_q \Xi} \,. \tag{4.10}$$

Since causality requires $\Xi > 0$ and z takes only non–negative values, it is clear that the above result is non–zero only for $\mathcal{W}_q \leq 0$. Therefore,

$$\mathcal{E}_{\rm A} = \frac{\sqrt{\lambda}}{\pi^2} \int {\rm d}t_q \left\{ -\frac{A_1}{4\gamma_q^2 \Xi^2} \frac{\partial^2}{\partial \mathcal{W}_q^2} [\Theta(-\mathcal{W}_q)\mathcal{W}_q] + \frac{A_0}{2\gamma_q \Xi} \frac{\partial^2}{\partial \mathcal{W}_q^2} \Theta(-\mathcal{W}_q) \right\}. \tag{4.11}$$

For the first term it is straightforward to compute the two derivatives. In the second term, we first take one derivative, then rewrite the second one as $\partial/\partial\mathcal{W}_q = (2\Xi)^{-1}\partial/\partial_{t_q}$, and finally integrate by parts to obtain

$$\mathcal{E}_{\rm A} = \frac{\sqrt{\lambda}}{4\pi^2} \int {\rm d}t_q \, \delta(\mathcal{W}_q) \left(\frac{A_1}{\gamma_q^2 \Xi^2} + \frac{\partial}{\partial t_q} \frac{A_0}{\gamma_q \Xi^2} \right). \tag{4.12}$$

Let $t_r = t_r(t, \boldsymbol{r})$ denote the value of t_q for which $\mathcal{W}_q(t_q) = 0$, that is

$$t - t_r = |\boldsymbol{r} - \boldsymbol{r}_q(t_r)|. \tag{4.13}$$

Writing $\delta(\mathcal{W}_q) = \delta(t_q - t_r)/2\Xi$ we finally arrive at

$$\mathcal{E}_{\rm A} = \frac{\sqrt{\lambda}}{8\pi^2} \frac{A_1}{\gamma_q^2 \Xi^3} + \frac{\sqrt{\lambda}}{8\pi^2} \frac{1}{\Xi} \frac{\partial}{\partial t_r} \frac{A_0}{\gamma_q \Xi^2} \,. \tag{4.14}$$

One should be cautious to treat t_r as a symbolic variable: only after the derivative is performed, one can replace t_r by its actual dependence on t and $\boldsymbol{r}$ via the solution to eq. (4.13). Also, in the previous manipulations we have been a little imprecise about the integration limits in t_q and z after the change of variables and the associated boundary terms. The most interesting case for us here will be a situation where the motion keeps going for ever, meaning $-\infty < t' < t$. Then eq. (2.6) implies $-\infty < t_q < t$ and $0 < z < \infty$, and one can easily check that the integration by parts generates no boundary terms. Indeed, at the upper limit $t = t_q$ and the constraint eq. (4.13) can be satisfied only when $\boldsymbol{r} = \boldsymbol{r}_q(t)$, a situation that we shall not consider. Also, at the lower limit $t_q \to -\infty$, eq. (4.13) cannot be satisfied for any finite $\boldsymbol{r}$. Other situations, where the integration domain for t' is finite, need to be considered case by case.

To summarize, the above integral over t_q has support only at $t_q = t_r$, where $\mathcal{W}_q = 0$, cf. eq. (4.12). Via eq. (4.10), this implies that the integration over z receives contributions from the endpoint at $z = 0$ alone. These special properties are the consequence of the above mentioned cancelation of the terms proportional to z^2 in the integrand of eq. (4.5). In turn, they imply that the final result (4.14) has *the same causal structure as the corresponding classical result*, that is, as the energy density produced by a source with trajectory $\boldsymbol{r}_q(t_q)$ in a classical field theory. Indeed, the condition $\mathcal{W}_q = 0$ is recognized as the classical retardation condition for the propagation of a signal at the speed of light. In particular, $t_r(t, \boldsymbol{r})$ coincides with the classical 'retarded time' — the time t_q at which a light signal must be emitted by the source located at $\boldsymbol{r}_q$ in order to be received at the point $\boldsymbol{r}$ at some latter time t. Furthermore, the space–time pattern of the energy in eq. (4.14) must be the same as in the corresponding classical problem, since this is entirely fixed by the trajectory of the source together with the condition that the signal propagates at the speed of light. In particular, when focusing on the radiation part we expect no quantum broadening to emerge.

For the calculation of $\mathcal{E}_{\rm B}$ we proceed similarly. Expressing the integrand in terms of the boundary motion, and after some tedious but straightforward algebra we arrive at

$$\mathcal{E}_{\rm B} = \frac{\sqrt{\lambda}}{\pi^2} \int {\rm d}t_q\, {\rm d}z\, \delta'''(\mathcal{W}_q + 2\gamma_q \Xi z)[B_0(t_q) + zB_1(t_q) + z^2 B_2(t_q) + z^3 B_3(t_q) + z^4 B_4(t_q)], \tag{4.15}$$

where the coefficients are given by

$$B_0 = \frac{4}{3\gamma_q}(\boldsymbol{r} - \boldsymbol{r}_q)^2 + \gamma_q[(\boldsymbol{r} - \boldsymbol{r}_q) \times \boldsymbol{v}_q]^2, \tag{4.16}$$

$$B_1 = -\frac{8}{3}\boldsymbol{v}_q \cdot (\boldsymbol{r} - \boldsymbol{r}_q) + 2\gamma_q^2[(\boldsymbol{r} - \boldsymbol{r}_q) \times \boldsymbol{v}_q] \cdot \{(\boldsymbol{r} - \boldsymbol{r}_q) \times \,[\gamma_q^2(\boldsymbol{v}_q \cdot \boldsymbol{a}_q)\boldsymbol{v}_q + \boldsymbol{a}_q]\}\,, \tag{4.17}$$

$$B_2 = \frac{4}{3}\gamma_q v_q^2 - 2\gamma_q^3[(\boldsymbol{r} - \boldsymbol{r}_q) \times \boldsymbol{v}_q] \cdot (\boldsymbol{v}_q \times \boldsymbol{a}_q) + \gamma_q^7[\boldsymbol{a}_q^2 - (\boldsymbol{v}_q \times \boldsymbol{a}_q)^2][(\boldsymbol{r} - \boldsymbol{r}_q) \times \boldsymbol{v}_q]^2, \tag{4.18}$$

$$B_3 = B_4 = 0. \tag{4.19}$$

A priori, the integrand can involve a quartic polynomial in z, but in reality this polynomial is just quadratic, since the terms proportional to z^3 and z^4 have exactly canceled among various contributions. Thus the integration has similar properties to the one for $\mathcal{E}_{\rm A}$, since we now have three derivatives to take. Once again, the integrations over z and t_q are fixed by eq. (4.10) and respectively the condition $\mathcal{W}_q = 0$, which together imply $t_q = t_r$ and $z = 0$. One finally obtains

$$\mathcal{E}_{\rm B} = -\frac{\sqrt{\lambda}}{8\pi^2}\frac{B_2}{\gamma_q^3 \Xi^4} - \frac{\sqrt{\lambda}}{16\pi^2}\frac{1}{\Xi}\frac{\partial}{\partial t_r}\frac{B_1}{\gamma_q^2 \Xi^3} - \frac{\sqrt{\lambda}}{16\pi^2}\frac{1}{\Xi}\frac{\partial}{\partial t_r}\left(\frac{1}{\Xi}\frac{\partial}{\partial t_r}\frac{B_0}{\gamma_q \Xi^2}\right). \tag{4.20}$$

The same discussion as for eq. (4.14) applies to potential boundary terms. Eqs. (4.14) and (4.20) are our final results for the total energy density produced by the heavy quark. It is interesting to notice that B_0 and B_1 are related as $B_1 = \frac{\partial}{\partial t_q}(\gamma_q B_0)$, so the last two terms in eq. (4.20) can be combined to yield a somewhat simpler expression for $\mathcal{E}_{\rm B}$:

$$\mathcal{E}_{\rm B} = -\frac{\sqrt{\lambda}}{8\pi^2}\,\frac{B_2}{\gamma_q^3\Xi^4} - \frac{\sqrt{\lambda}}{8\pi^2}\,\frac{1}{\Xi}\,\frac{\partial}{\partial t_r}\left(\frac{1}{\gamma_q\Xi^2}\,\frac{\partial}{\partial t_r}\,\frac{B_0}{\Xi}\right). \tag{4.21}$$

In the next section, we shall use these results to extract the energy radiated by the heavy quark. But before that, let us perform a first, non–trivial check of these formulæ by using them to recover the known result for the Coulomb energy of a heavy quark which moves at constant velocity (and thus it does not radiate). Assuming uniform linear motion with velocity v along the x axis, we have $\boldsymbol{a}_q = 0$ and then the expressions for the coefficients A_i and B_i simplify considerably. After simple manipulations, we deduce

$$\mathcal{E}_{\rm A} = \frac{\sqrt{\lambda}}{4\pi^2}\,\frac{\gamma^2(3-v^2)}{[\boldsymbol{x}_\perp^2 + \gamma^2(x-vt)^2]^2}\,. \tag{4.22}$$

and respectively

$$\mathcal{E}_{\rm B} = -\frac{\sqrt{\lambda}\gamma^2}{6\pi^2}\,\frac{(4-2v^2)[\boldsymbol{x}_\perp^2 + \gamma^2(x-vt)^2] + v^2\gamma^2(x-vt)^2}{[\boldsymbol{x}_\perp^2 + \gamma^2(x-vt)^2]^3}. \tag{4.23}$$

which add together to the expected result [8–10] :

$$\mathcal{E} = \frac{\sqrt{\lambda}\gamma^2}{12\pi^2}\,\frac{(1+v^2)\boldsymbol{x}_\perp^2 + (x-vt)^2}{[\boldsymbol{x}_\perp^2 + \gamma^2(x-vt)^2]^3}. \tag{4.24}$$

Note that all the three terms in eq. (4.20) for $\mathcal{E}_{\rm B}$ contribute to eq. (4.23), while eq. (4.22) receives contributions only from the second, derivative, term in eq. (4.14).

5 Radiated energy and power

From now on we shall focus on the part of the energy density which is radiated. Following the standard definition in the literature, we shall identify the radiation as the part of the energy density which falls like $1/R^2$ (with $\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{r}_q(t_r)$) at large distances from the source. For this definition to be meaningful, we shall consider only observation points $\boldsymbol{r}$ which are sufficiently far away from the position $\boldsymbol{r}_q(t)$ of the quark at the observation time t for the dominant contribution of the energy density at $\boldsymbol{r}$ to be falling like $1/R^2$. (Indeed, if $\boldsymbol{r}$ is relatively close to $\boldsymbol{r}_q(t)$, then the retardation condition (4.13) allows for solutions t_r with $t_r \simeq t$ and $\boldsymbol{r}_q(t_r) \simeq \boldsymbol{r}$, and then the energy density at $\boldsymbol{r}$ is dominated by the near–field of the heavy quark, *i.e.* by its Coulomb energy, and not by radiation.)

To make the power counting with respect to $1/R$ more transparent, it is useful to notice that, for $t_q = t_r$ obeying eq. (4.13), one has $\Xi = R(1 - \boldsymbol{n}\cdot\boldsymbol{v}_q)$, where we have defined $\boldsymbol{n}$ as the unit vector along $\boldsymbol{R}$. Then, by inspection of the expressions in the previous section, one can check that, first, the pieces showing the slowest decay at large distances in Eqs. (4.14) and (4.20) are those which behave like $1/R^2$, as expected, and, second, in order to isolate these pieces, it is enough to keep the terms in the coefficients A_i which are proportional to R or $t - t_q$ and the terms in the coefficients B_i which are proportional to R^2. By doing that, one eventually finds that radiative contributions $\propto 1/R^2$ to the energy density come from all the terms in Eqs. (4.14) and (4.20) which are proportional to either the square of the acceleration, or to its time derivative (also known as the 'jerk'). Thus, the radiation vanishes in the absence of acceleration, as expected. However, unlike what would happen in a classical theory, or in a weakly coupled theory at leading order, where the radiation involves *only* terms proportional to the square of the acceleration (see e.g. the discussion in Sec. 7 below), in the present calculation at strong coupling we also find contributions proportional to the jerk $\dot{\boldsymbol{a}}_q$.

In what follows, we shall exhibit all the radiative contributions to the energy density in Eqs. (4.14) and (4.20). In turns out that, in view of the subsequent physical discussion and also of the comparison with the respective classical results in Sec. 7, it is meaningful to separate between two types of such contributions: (i) those generated by terms in the string stress tensor which are by themselves proportional to the square of the acceleration, and (ii) those coming from the terms in Eqs. (4.14) and (4.20) which involve derivatives w.r.t. t_r.

(i) Contributions proportional to the acceleration squared, more precisely to the structure $\boldsymbol{a}_q^2 - (\boldsymbol{v}_q \times \boldsymbol{a}_q)^2$, and which are originating from the components (3.10)–(3.15) of $\tilde{t}_{MN}$, are visible in eq. (4.8) for A_1 and in eq. (4.18) for B_2. They contribute to the energy density via the terms without derivatives in Eqs. (4.14) and (4.20), and yield

$$\mathcal{E}_{\rm A}^{(1)} = \frac{\sqrt{\lambda}}{8\pi^2}\,\frac{\gamma_q^4[\boldsymbol{a}_q^2 - (\boldsymbol{v}_q \times \boldsymbol{a}_q)^2]}{R^2}\,\frac{1+\boldsymbol{n}\cdot\boldsymbol{v}_q}{(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^3}\,, \tag{5.1}$$

$$\mathcal{E}_{\rm B}^{(1)} = -\frac{\sqrt{\lambda}}{8\pi^2}\,\frac{\gamma_q^4[\boldsymbol{a}_q^2 - (\boldsymbol{v}_q \times \boldsymbol{a}_q)^2]}{R^2}\,\frac{(\boldsymbol{n}\times\boldsymbol{v}_q)^2}{(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^4}\,. \tag{5.2}$$

These two contributions combine to give the following, relatively simple, expression

$$\mathcal{E}_{\rm rad}^{(1)}(t,\boldsymbol{r}) \;=\; \frac{\sqrt{\lambda}}{8\pi^2}\,\frac{\gamma_q^2[\boldsymbol{a}_q^2 - (\boldsymbol{v}_q \times \boldsymbol{a}_q)^2]}{(\boldsymbol{r}-\boldsymbol{r}_q)^2(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^4}\,, \tag{5.3}$$

where it is understood that all quantities related to the motion of the quark ($\boldsymbol{r}_q$, $\boldsymbol{v}_q$, and $\boldsymbol{a}_q$) are evaluated at $t_q = t_r(t,\boldsymbol{r})$. Note that the other terms in A_1 and B_2 do not generate contributions of order $1/R^2$ to the energy density.

(ii) The remaining terms of order $1/R^2$ arise from the 3rd and 4th term of A_0, from B_0, and from the 2nd term of B_1. *A priori*, that is, within the coefficients A_i and B_i, these terms depend only upon the quark velocity $\boldsymbol{v}_q$ and are at most linear in the acceleration $\boldsymbol{a}_q$, but after taking the derivatives w.r.t. t_r in eqs. (4.14) and (4.20), they generate contributions proportional to the square of the acceleration, or to its derivative. Defining $\xi = 1-\boldsymbol{n}\cdot\boldsymbol{v}_q$, we find

$$\mathcal{E}_{\rm A}^{(2)} = \frac{\sqrt{\lambda}}{8\pi^2 R^2\xi}\,\frac{\partial}{\partial t_r}\left[\frac{\boldsymbol{n}\cdot\boldsymbol{a}_q}{\xi^2} + \frac{\gamma_q^2\boldsymbol{v}_q\cdot\boldsymbol{a}_q(2-\xi)}{\xi^2}\right] \tag{5.4}$$

$$\mathcal{E}_{\rm B}^{(2)} = -\frac{\sqrt{\lambda}}{8\pi^2 R^2\xi}\,\frac{\partial}{\partial t_r}\left[-\frac{\boldsymbol{n}\cdot\boldsymbol{a}_q(1-\xi)}{\xi^3} + \frac{\gamma_q^2\boldsymbol{v}_q\cdot\boldsymbol{a}_q(2-\xi)}{\xi^2} + \frac{1}{\xi}\,\frac{\partial}{\partial t_r}\left(\frac{1}{6\gamma_q^2\xi^2}+\frac{1}{\xi}\right)\right], \tag{5.5}$$

where we have neglected derivatives acting on $\boldsymbol{R}$ or $t-t_q$ since they generate terms which fall faster than $1/R^2$. Performing the derivative on $1/\xi$ of the last term in Eq. (5.5) (we do not differentiate the unit vector $\boldsymbol{n}$ since this would lead again to terms falling faster than $1/R^2$; that is, we use $\partial_{t_r}\xi \simeq -\boldsymbol{n}\cdot\boldsymbol{a}_q$), we see that $\mathcal{E}_{\rm B}^{(2)}$ becomes

$$\mathcal{E}_{\rm B}^{(2)} = -\frac{\sqrt{\lambda}}{8\pi^2 R^2\xi}\,\frac{\partial}{\partial t_r}\left[\frac{\boldsymbol{n}\cdot\boldsymbol{a}_q}{\xi^2} + \frac{\gamma_q^2\boldsymbol{v}_q\cdot\boldsymbol{a}_q(2-\xi)}{\xi^2} + \frac{1}{6\xi}\,\frac{\partial}{\partial t_r}\,\frac{1}{\gamma_q^2\xi^2}\right]. \tag{5.6}$$

Thus, adding the two contributions from eqs. (5.4) and (5.6) we are left only with the last term in the last equation, which, after preforming the first derivative w.r.t. t_r and returning to the original variables, is finally rewritten as

$$\mathcal{E}^{(2)}_{\rm rad}(t,\boldsymbol{r}) \,=\, \frac{\sqrt{\lambda}}{24\pi^2}\,\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_q|^2(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)}\,\frac{\partial}{\partial t_r}\left[\frac{\boldsymbol{v}_q\cdot\boldsymbol{a}_q}{(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^3}-\frac{\boldsymbol{n}\cdot\boldsymbol{a}_q}{\gamma_q^2(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^4}\right]. \tag{5.7}$$

If one also performs the remaining derivative w.r.t. t_r, one finds that all the ensuing terms are proportional to either the square or the derivative of the acceleration, as anticipated.

The radiated energy density $\mathcal{E}_{\rm rad}$ given by the sum of eqs. (5.3) and (5.7) is the main result of this paper. The (relatively high) powers of $1-\boldsymbol{n}\cdot\boldsymbol{v}_q$ visible in the denominators of these expressions have a kinematical origin: they express the angular collimation of the radiation due to the Lorentz boost, which was to be expected, independently of the value of the coupling. In the ultrarelativistic limit $v_q\simeq 1$ or $\gamma_q\gg 1$, one can write (with α denoting the angle between the vectors $\boldsymbol{v}_q$ and $\boldsymbol{n}$): $1-v_q\cos\alpha\simeq(1/2)(1/\gamma_q^2+\alpha^2)$. This makes it clear that the radiation is emitted within a small angle $\alpha\sim 1/\gamma_q$ around the direction of the quark velocity $\boldsymbol{v}_q(t_r)$, so like for the corresponding classical problem [14].

Using the above results for $\mathcal{E}_{\rm rad}=\mathcal{E}^{(1)}_{\rm rad}+\mathcal{E}^{(2)}_{\rm rad}$, we shall now compute the radiated power. By integrating the energy conservation law $\partial_t\langle T^{00}\rangle+\partial_i\langle T^{0i}\rangle=0$ over the whole space and using Gauss' theorem together with $\langle T^{0i}\rangle\approx n^i\langle T^{00}\rangle$ for the dominant respective contributions, proportional to $1/R^2$, at large distances, one finds (recall the notation $\langle T^{00}\rangle=\mathcal{E}$)

$$-\,\frac{{\rm d}E}{{\rm d}t}\,=\,\lim_{r\to\infty}\,r^2\int{\rm d}\Omega\;\mathcal{E}(t,\boldsymbol{r})\,, \tag{5.8}$$

where the left hand side represents the energy radiated per unit of *observation* time. In practice, it is more convenient to define the power $P_{\rm rad}$ as the energy radiated per unit of *emission* time t_r. Then by using the above formula together with ${\rm d}t/{\rm d}t_r=\xi$, one sees that the power radiated per unit solid angle reads (below, $R\to\infty$)

$$\frac{{\rm d}P_{\rm rad}}{{\rm d}\Omega}=\frac{{\rm d}t}{{\rm d}t_r}\,R^2\mathcal{E}_{\rm rad}\;\;\Rightarrow\;\;P_{\rm rad}=\int{\rm d}\Omega\,(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)\,R^2\mathcal{E}_{\rm rad}. \tag{5.9}$$

Note an important, implicit, assumption in the above argument: we have made the hypothesis that, at all the points on a sphere at infinity ($r\to\infty$), we have $R\equiv|\boldsymbol{r}-\boldsymbol{r}_q(t_r)|\simeq r$ (which in turn implies that only the far–zone contributions $\propto 1/R^2$ to the energy density and flux have to be retained). This is correct provided the motion of the quark is *bounded*, such that its trajectory $\boldsymbol{r}_q(t_r)$ does not cross the sphere at infinity. More precisely, it is enough that this condition be satisfied during the *acceleration phase* of its motion, since this is the only phase which creates radiation. Thus, our subsequent results for $P_{\rm rad}$ are only valid provided there exists some fixed, but arbitrary, distance r_0 such that $\boldsymbol{r}_q(t_q)\le r_0$ for any t_q within the acceleration phase. We shall later make some comments on the case of unbounded motion.

Some standard and useful integrals to perform the angular integrations in eq. (5.9) are listed in Appendix A. For the first contribution coming from $\mathcal{E}^{(1)}_{\rm rad}$ we find

$$P^{(1)}_{\rm rad}=\frac{\sqrt{\lambda}}{2\pi}\,\gamma_q^6\,[\boldsymbol{a}_q^2-(\boldsymbol{v}_q\times\boldsymbol{a}_q)^2]\,, \tag{5.10}$$

which is the result inferred in [15] from a world–sheet analysis (*i.e.*, without an explicit calculation of the backreaction, but merely via a calculation of the energy flux down the string).

Remarkably, this expression has the same structure as the respective classical result, that is, the Liénard formula in classical electrodynamics [14], that we shall extend to the case of the $\mathcal{N}=4$ SYM theory in Sec. 7. For the second contribution from $\mathcal{E}^{(2)}_{\rm rad}$ we obtain

$$P^{(2)}_{\rm rad} = -\frac{\sqrt{\lambda}}{18\pi}\,\frac{\partial}{\partial t_r}\,\gamma_q^4 \boldsymbol{v}_q\cdot\boldsymbol{a}_q. \tag{5.11}$$

The fact that this term is a total derivative w.r.t. the emission time t_r rises some puzzles for its interpretation as a contribution to the radiated energy (see the discussion at the end of Sect. 6). It is therefore interesting to notice that a term with a similar structure has been interpreted in Ref. [16, 17] as a contribution to the *proper* energy of the quark (and not to its radiation). We shall return to this point in the next section.

Eqs. (5.10) and (5.11) are our final results for the radiated power. It is important to keep in mind that these results have been derived here for the case of a bounded quark motion. Because of that, in evaluating these formulæ one can use the approximation $t_r \simeq t - r$ for most (but not all) purposes.

6 Applications

Here we shall apply the general results derived in the previous section for the radiated energy density and the power to specific quark motions.

(i) Uniform rotation: We shall first recover the results for uniform circular motion originally obtained in [6]. Using spherical coordinates $\boldsymbol{r}=(r,\theta,\phi)$ and parametrizing the boundary motion as

$$\boldsymbol{r}_q(t_r) = (R_0, \pi/2, \omega t_r), \tag{6.1}$$

our expressions in eq. (5.3) and eq. (5.7) lead to the following two contributions to the density of the radiated energy

$$\mathcal{E}^{(1)}_{\rm rad} = \frac{\sqrt{\lambda}}{8\pi^2}\,\frac{a^2}{r^2\xi^4}\,, \tag{6.2}$$

$$\mathcal{E}^{(2)}_{\rm rad} = \frac{\sqrt{\lambda}\,\omega^2}{24\pi^2 r^2}\,\frac{4-7\xi-4v^2\sin^2\theta+3\xi^2}{\gamma^2\xi^6}, \tag{6.3}$$

with $v=\omega R_0$, $a=\omega^2 R_0$, and where, according to our earlier definition above eq. (5.4), we have

$$\xi = 1 - v\sin\theta\sin(\phi-\omega t_r). \tag{6.4}$$

Note that the contribution in eq. (6.3) is fully arising from the last term, proportional to $\boldsymbol{n}\cdot\boldsymbol{a}_q$, in eq. (5.7), since $\boldsymbol{v}_q\cdot\boldsymbol{a}_q = 0$ for the problem at hand.

Adding the two pieces in eqs. (6.2) and (6.3) we find

$$\mathcal{E}_{\rm rad} = \frac{\sqrt{\lambda}\omega^2}{24\pi^2 r^2}\,\frac{4-7\xi-4v^2\sin^2\theta+3\gamma^2\xi^2}{\gamma^2\xi^6}, \tag{6.5}$$

which is indeed the same as the result for the radiated energy density in [6] (cf. eq. (3.72) there). The energy density (6.5) is proportional to $1/\xi^6$, hence in the ultrarelativistic limit $v \simeq 1$ or

$\gamma \gg 1$, it is strongly peaked at the minima of ξ, defined by $\sin\theta\sin(\phi-\omega t_r)=1$. This condition describes a spiral in the plane $\theta=\pi/2$, located at (recall that $t_r \simeq t-r$)

$$\phi(t,r) \simeq \frac{\pi}{2}+\omega(t-r)\,. \tag{6.6}$$

Using $1-\upsilon\sin\theta \simeq (1/2)(1/\gamma^2+\alpha^2)$ where $\alpha \equiv \pi/2-\theta$, it is obvious that the energy is localized within a small angle $\alpha \sim 1/\gamma$ around $\alpha=0$, or $\theta=\pi/2$. This is the expected collimation due to the Lorentz boost, as already discussed in relation with the general formulæ (5.3) and (5.7). Furthermore, using (6.5) one can check that the arms of the spiral have a tiny radial width[3] $\Delta r \sim R_0/\gamma^3$ [6], exactly like in the corresponding classical problem [14]. As explained in the Introduction, this feature is very surprising in a quantum theory at strong coupling, where one would rather expect broadening due to the virtual quantum fluctuations. In the context of our calculation, this follows from the fact that, as explained on Sec. 4, the whole backreaction arises from the string endpoint at $z=0$. (See also [18] for a different perspective of this problem.)

We now compute the radiated power in terms of the quark's own time. From eq. (5.11), it is clear that $P^{(2)}_{\rm rad}=0$ in this case, so the only contribution comes from eq. (5.10) and reads

$$P_{\rm rad}=\frac{\sqrt{\lambda}}{2\pi}\,\gamma^4 a^2\,. \tag{6.7}$$

This coincides with the respective result in [6] and also with the result of the world–sheet analysis in [15].

(ii) Non–uniform circular motion: It is of course straightforward to apply our general expressions to an arbitrary circular motion, but to be more precise we shall focus on the specific motion

$$\boldsymbol{r}_q(t_r)=(R_0,\pi/2,\phi_q(t_r)) \quad \text{with} \quad \phi_q(t_r)=\sqrt{t_r^2+b^2}/R_0\,, \tag{6.8}$$

for which we shall directly compute the radiated power. The (angular) velocity, whose magnitude approaches the speed of light at large times, and the acceleration are given by (below $\hat{e}_r$ and $\hat{e}_\phi$ are the respective unit vectors)

$$\boldsymbol{\upsilon}_q(t_r)=\frac{t_r}{\sqrt{t_r^2+b^2}}\,\hat{e}_\phi \qquad \text{and} \qquad \boldsymbol{a}_q=-\frac{1}{R_0}\frac{t_r^2}{t_r^2+b^2}\,\hat{e}_r+\frac{b^2}{(t_r^2+b^2)^{3/2}}\,\hat{e}_\phi, \tag{6.9}$$

so in particular $\gamma_q=\sqrt{t_r^2+b^2}/b$ and $\boldsymbol{\upsilon}_q\cdot\boldsymbol{a}_q$ is not vanishing anymore (in contrast to the case of uniform rotation). Hence both terms contributing to the radiated power, (5.10) and (5.11), are now non–zero, and this is interesting as it allows us to observe an hierarchy among these terms, that we believe to be generic. Namely, $P^{(1)}_{\rm rad}$ dominates over $P^{(2)}_{\rm rad}$ in the ultrarelativistic limit $\gamma_q \gg 1$, and hence also for sufficiently large times (in the problems where the velocity grows with time, due to acceleration). Specifically, for the motion in eq. (6.8), one finds

$$P^{(1)}_{\rm rad}=\frac{\sqrt{\lambda}}{2\pi}\frac{t_r^4+b^2R_0^2}{b^4R_0^2} \qquad \text{and} \qquad P^{(2)}_{\rm rad}=-\frac{\sqrt{\lambda}}{18\pi}\frac{1}{b^2}. \tag{6.10}$$

[3] In order to deduce this property from eq. (6.5), it is not enough to use the simplified version of the retardation time $t_r \simeq t-r$; rather, one needs a more precise analysis of the retardation condition, which also takes into account the collimation of the radiation along the direction of emission; see [6, 14].

As anticipated, the first term $P^{(1)}_{\rm rad}$ is the dominant one for large enough times such as $t_r^2 \gg bR_0$. This example also gives us some insight into the physical origin of this hierarchy: eq. (5.10) for $P^{(1)}_{\rm rad}$ involves the component of the acceleration which is *transverse* to the velocity[4] (the radial component of $\boldsymbol{a}_q$ in eq. (6.9)), whereas $P^{(2)}_{\rm rad}$ in eq. (5.11) rather involves the respective *longitudinal* component (tangential in the case of eq. (6.9)). We thus recover a feature familiar in the context of classical radiation [14]: rotation is much more effective than tangential acceleration in producing radiation, since the velocity $\boldsymbol{v}_q$ changes rapidly in direction while the particle rotates, even though its change in magnitude is relatively small (or even zero for uniform rotation).

(iii) Uniform linear acceleration: A classical particle subjected to a constant force $\boldsymbol{F} = F\hat{e}_1$ follows a trajectory $x_q(t_r) = \sqrt{t_r^2 + b^2}$ where $x \equiv x^1$, $b = m/F$ and we selected convenient initial conditions at $t = 0$. Clearly, this motion is unbounded: the trajectory will eventually cross the sphere 'at infinity' that we use to define the radiated power. In view of that, we do not expect our previous results for the power to also cover this case. Notwithstanding, let us first see what these results yield if naively applied to this case. Using

$$v_q = \frac{t_r}{\sqrt{t_r^2 + b^2}}, \qquad \gamma_q = \frac{\sqrt{t_r^2 + b^2}}{b}, \qquad a_q = \frac{b^2}{(t_r^2 + b^2)^{3/2}}, \tag{6.11}$$

and hence $\gamma_q^4 v_q a_q = t_r/b^2$, one easily finds that the two contributions in eq. (5.10) and respectively eq. (5.11) are now of the same parametric order and thus contribute on the same footing to the final result for the power, in contradiction with our general expectations and also with the previous examples. Namely, one (naively) has

$$P^{(1)}_{\rm rad} = \frac{\sqrt{\lambda}}{2\pi b^2} \qquad \text{and} \qquad P^{(2)}_{\rm rad} = -\frac{\sqrt{\lambda}}{18\pi b^2} \quad \Longrightarrow \quad P_{\rm rad} = \frac{4\sqrt{\lambda}}{9\pi b^2}\,. \tag{6.12}$$

Moreover, this result also contradicts the independent calculation in Refs. [10, 19], where the radiated power has been extracted from a world–sheet analysis, as the energy flow across the induced horizon at $z = b$. That previous calculation furnished a result equal to $P^{(1)}_{\rm rad}$ in the above equation, which if course would be also the prediction of Mikhailov's analysis for the problem at hand [15] (since in that analysis the total power reduces to our $P^{(1)}_{\rm rad}$). Clearly, these mismatches shed further doubts on the validity of the above calculation, that is anyway transgressing the validity limits of our general calculation.

Let us therefore redo our analysis of this problem, but in such a way to stay within the limits of our general discussion. To that aim, we assume that the quark is under uniform acceleration only for a finite period of time t_0, and we measure the radiated energy at a distance $r \gg t_0$. By doing this we effectively reduce the motion to a bounded one. Let us be more specific and consider the one-dimensional motion

$$x_q(t_r) = \Theta(-t_r)\, b + \Theta(t_r)\Theta(t_0 - t_r)\sqrt{t_r^2 + b^2} + \Theta(t_r - t_0)\, \frac{t_0 t_r + b^2}{\sqrt{t_0^2 + b^2}}, \tag{6.13}$$

where t_0 can be taken to be much larger than b so that the quark becomes eventually ultra-relativistic. (The last term in eq. (6.13) describes a constant velocity motion with the velocity

[4] More precisely, the term $\boldsymbol{a}_q^2$ in eq. (5.10) receives contributions from both the radial and the azimuthal components of the acceleration in eq. (6.9), but the dominant contribution at large times, represented by the term proportional to t_r^4 in the numerator of $P^{(1)}_{\rm rad}$ in eq. (6.10), is generated by the radial piece of $\boldsymbol{a}_q$.

acquired at $t_r = t_0$.) In this last example we shall evaluate the total energy radiated, and therefore we need to integrated the total power over t_r. Then it is straightforward to see that eq. (5.11) will not contribute to the final results, since it involves a total derivative w.r.t. t_r and the acceleration vanishes outside the interval $[0, t_0]$. Thus the power in eq. (5.10) will determine the total energy radiated which is

$$E_{\rm rad} = \frac{\sqrt{\lambda}}{2\pi}\,\frac{t_0}{b^2}, \tag{6.14}$$

in agreement with [10, 19].

This last calculation also illustrates a rather curious feature of $P^{(2)}_{\rm rad}$ in eq. (5.11), which makes us feel uncomfortable about its physical interpretation as radiation: this term yields no contribution to the radiated energy for any motion where the acceleration is non–zero over only a finite interval of time, and also for any periodic motion for which one computes the total radiation over one period, or an integer multiple of it. As already mentioned after eq. (5.11), a term with a similar structure appears in the world–sheet calculation of the total energy of the moving quark (which is the same as the total energy carried by the string) in Ref. [16, 17]. Specifically, by taking the limit of a very heavy quark ($z_m \to 0$) in eqs. (2.32)–(2.33) of Ref. [17], one obtains the following expression for the quark energy

$$E_q(t) \,=\, m_q\gamma_q(t) \,-\, \frac{\sqrt{\lambda}}{2\pi}\,\gamma_q^4 \boldsymbol{v}_q\!\cdot\!\boldsymbol{a}_q \,+\, \frac{\sqrt{\lambda}}{2\pi}\int_{-\infty}^{t} {\rm d}t_q\,\gamma_q^6\,[\boldsymbol{a}_q^2 - (\boldsymbol{v}_q\times\boldsymbol{a}_q)^2]\,, \tag{6.15}$$

where the first two terms in the r.h.s. are interpreted [16, 17] as the quark proper (or kinetic) energy, while the third one, which is the time–integral of $P^{(1)}_{\rm rad}$ in eq. (5.10), as the radiation. As anticipated, the second term in the above equation has the same structure as the contribution to the 'radiated energy' that would be obtained from $P^{(2)}_{\rm rad}$, eq. (5.11), but with a different numerical coefficient. Recall that the reason why we have identified this term as radiation in Sect. 5 was because it arises from a piece in the energy density which falls off like $1/R^2$ at large R. This suggests the interesting possibility that a piece of the quark proper energy have a tail at large R which cannot be distinguished from radiation. We leave this question, as well the calculation of the total energy via the backreaction, for further studies.

7 The classical result

In the previous discussion, we have already anticipated some similarities between the predictions of the supergravity approximation for the strong coupling limit and the corresponding results in the classical approximation, which are also the leading order results at weak coupling. For this comparison to be more precise and in preparation of the physical discussion in the next section, in this section we shall explicitly solve the corresponding classical problem — the radiation by a heavy quark undergoing some arbitrary motion in the $\mathcal{N} = 4$ SYM theory at weak coupling. To our knowledge, the general result that we shall derive here has not been presented elsewhere, except for the case of the uniform circular motion that was discussed in [6]. But even in that case, the final results and the associated physical discussion have been plagued by some mistakes in the numerical factors that we shall here correct.

The general structure of the classical theory describing a massive test quark[5] propagating through the vacuum of the $\mathcal{N}=4$ SYM theory has been clarified in Refs. [6, 20]. As explained there, the heavy quark radiates both vector (gauge) fields and scalar fields, and in the limits of interest here (arbitrarily weak coupling and very large quark mass) this radiation is described by decoupled, linear equations, which generalize Maxwell equations to the theory at hand. These equations are then solved in the standard way, to give

$$A^\mu = \frac{e_{\rm eff}}{4\pi(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)R}\,(1,\boldsymbol{v}_q) \quad \text{and} \quad \chi = \frac{e_{\rm eff}}{4\pi\gamma_q(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)R}\,, \tag{7.1}$$

with A^μ the vector field, χ the scalar field, $\boldsymbol{R}=\boldsymbol{r}-\boldsymbol{r}_q$ and $\boldsymbol{n}$ the unit vector along $\boldsymbol{R}$ as earlier, and the proper counting of the color degrees of freedom for the radiated field is encoded in $e_{\rm eff}^2 \equiv \lambda/2$. As before, the above expressions are to be evaluated at the retarded time t_r which is the solution to eq. (4.13). In general the energy density is obtained from

$$\mathcal{E}_{\rm vector} = \frac{1}{2}\left(\boldsymbol{E}^2+\boldsymbol{B}^2\right) \quad \text{and} \quad \mathcal{E}_{\rm scalar} = \frac{1}{2}\left[(\partial_t\chi)^2+(\nabla\chi)^2\right], \tag{7.2}$$

with $\boldsymbol{B}$ the magnetic field. Since we are interested in the radiated energy, we keep only the contributions which fall like $1/R$ when computing the electric field and the derivative with respect to time of the scalar field. This yields

$$\boldsymbol{E}_{\rm rad} = \frac{e_{\rm eff}}{4\pi R}\left[-\frac{\boldsymbol{a}_q}{(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^2}+\frac{(\boldsymbol{n}\cdot\boldsymbol{a}_q)(\boldsymbol{n}-\boldsymbol{v}_q)}{(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^3}\right], \tag{7.3}$$

$$(\partial_t\chi)_{\rm rad} = \frac{e_{\rm eff}}{4\pi R}\left[-\frac{\gamma_q\boldsymbol{v}_q\cdot\boldsymbol{a}_q}{(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^2}+\frac{\boldsymbol{n}\cdot\boldsymbol{a}_q}{\gamma_q(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^3}\right]. \tag{7.4}$$

Since moreover $|\boldsymbol{B}_{\rm rad}|=|\boldsymbol{E}_{\rm rad}|$ and $|(\partial_t\chi)_{\rm rad}|=|(\nabla\chi)_{\rm rad}|$ for the radiation, we deduce that

$$\mathcal{E}_{\rm vector} = \frac{\lambda}{32\pi^2R^2}\left[\frac{\boldsymbol{a}_q^2}{(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^4}+2\,\frac{(\boldsymbol{v}_q\cdot\boldsymbol{a}_q)(\boldsymbol{n}\cdot\boldsymbol{a}_q)}{(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^5}-\frac{(\boldsymbol{n}\cdot\boldsymbol{a}_q)^2}{\gamma_q^2(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^6}\right], \tag{7.5}$$

$$\mathcal{E}_{\rm scalar} = \frac{\lambda}{32\pi^2R^2}\left[\frac{\gamma_q^2(\boldsymbol{v}_q\cdot\boldsymbol{a}_q)^2}{(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^4}-2\,\frac{(\boldsymbol{v}_q\cdot\boldsymbol{a}_q)(\boldsymbol{n}\cdot\boldsymbol{a}_q)}{(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^5}+\frac{(\boldsymbol{n}\cdot\boldsymbol{a}_q)^2}{\gamma_q^2(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^6}\right], \tag{7.6}$$

where we have substituted $e_{\rm eff}^2=\lambda/2$. Adding the two contributions the terms depending on $\boldsymbol{n}\cdot\boldsymbol{a}_q$ cancel[6] and we obtain a simple result:

$$\mathcal{E}_{\rm rad}^{\rm class} = \frac{\lambda}{32\pi^2}\,\frac{\gamma_q^2[\boldsymbol{a}_q^2-(\boldsymbol{v}_q\times\boldsymbol{a}_q)^2]}{(\boldsymbol{r}-\boldsymbol{r}_q)^2(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^4}\,. \tag{7.7}$$

It is very interesting to notice that with the replacement $\lambda\to 4\sqrt{\lambda}$, this is exactly the same as the $\mathcal{E}_{\rm rad}^{(1)}$ piece of the strong coupling result in eq. (5.3). Since by assumption $r\gg r_q$ and thus the retarded time can be approximated as $t_r\simeq t-r$, we see that the only angular dependence is

[5]More precisely an infinitely massive spin–1/2 particle from the $\mathcal{N}=2$ hypermultiplet, that is in the fundamental representation of the $\mathrm{SU}(N_c)$ gauge group.

[6]This cancelation for the particular case of uniform circular motion was not realized by the authors of [6], because their corresponding expressions (2.20a) and (2.20b) for $\mathcal{E}_{\rm vector}$ and $\mathcal{E}_{\rm scalar}$ miss a numerical factor of 1/2 and 2 respectively.

the boost factor $(1-\boldsymbol{n}\cdot\boldsymbol{v}_q)^4$ in the denominator. This means that in the non–relativistic limit eq. (7.7) becomes isotropic and reduces to

$$\mathcal{E}_{\rm rad}^{\rm class} \simeq \frac{\lambda}{32\pi^2}\,\frac{\boldsymbol{a}_q^2}{r^2}. \tag{7.8}$$

Obviously this is a property which is not shared by QED, where only vector fields are radiated, and the radiated energy as given in eq. (7.5) contains anisotropic pieces.

One can compute separately the vector and scalar contributions to the power (see again Appendix A for the corresponding integrals), which read

$$P_{\rm vector} = \frac{\lambda}{12\pi}\,\gamma_q^6\,[\boldsymbol{a}_q^2 - (\boldsymbol{v}_q\times\boldsymbol{a}_q)^2] \quad \text{and} \quad P_{\rm scalar} = \frac{\lambda}{24\pi}\,\gamma_q^6\,[\boldsymbol{a}_q^2 - (\boldsymbol{v}_q\times\boldsymbol{a}_q)^2]\,, \tag{7.9}$$

leading to a total power

$$P_{\rm rad}^{\rm class} = \frac{\lambda}{8\pi}\,\gamma_q^6\,[\boldsymbol{a}_q^2 - (\boldsymbol{v}_q\times\boldsymbol{a}_q)^2]\,, \tag{7.10}$$

which, up to the replacement $\lambda \to 4\sqrt{\lambda}$, is the same as the piece $P_{\rm rad}^{(1)}$ of the corresponding supergravity result, cf. eq. (5.10).

8 Discussion and open issues

One of the main results of this paper is the verification of a conjecture made in [7] that for an arbitrary relativistic motion of a heavy quark, and in the supergravity approximation to the dual string theory, it is only the endpoint of the dual string at $z=0$ which contributes to the radiated energy. That was first observed in [6] for the uniform circular motion and then extended in [7] to a general non–relativistic motion and also, *mutatis mutandis*, to other types of radiation, like the decay of a time–like wave–packet. (The dual description of the time–like wave–packet is a supergravity field falling into AdS_5. Then the corresponding statement is that the radiation is generated only from the starting point of the trajectory at $z=0$.) This property implies that the radiation propagates at the speed of light and therefore the space–time distribution of this radiated energy is very similar to that of the corresponding classical radiation, without any sign of quantum broadening [6, 7]. As argued in [7] this leads to a radial distribution which is to difficult to reconcile with quantum mechanics, which sheds doubts on the validity of the supergravity approximation as the correct, dual, description of the strong–coupling limit. Moreover, it was shown there, for a specific example and via an admittedly heuristic calculation, that there are particular string corrections (which in the light–cone gauge appear as fluctuations in the longitudinal coordinates of the string points) which are not suppressed when $\lambda\to\infty$, and hence should be treated as a part of the leading order theory at strong coupling. One effect of those fluctuations (at least within the limits of the calculation in [7]) is to provide a radial broadening for the energy distribution, in agreement with expectations from both quantum mechanics and the UV/IR correspondence.

The proper calculation of string fluctuations in a curved space–time is an outstanding open problem, that we shall not attempt to address here. Rather, we would like to emphasize some curious features of the previous results obtained in the supergravity approximation, which look rather implausible to us on physical grounds and may represent additional shortcomings of this approximation (besides the lack of quantum broadening). The peculiarities to be discussed here are all associated with the second contribution to the energy density, $\mathcal{E}^{(2)}_{\rm rad}$ in eq. (5.7).

(i) *The lack of isotropy in the non–relativistic limit*

As mentioned in the Introduction, at strong coupling one expects the radiation to be isotropically distributed at large distances away from the source [4, 5], except for the trivial anisotropy introduced by the Lorentz boost. Indeed, the radiation should typically proceed via the emission of off–shell quanta[7], which then should evacuate their virtuality via successive branchings. This gives rise to a partonic cascade through which the original energy and momentum get divided among many quanta. Due to their large number and to the absence of any preferred pattern in the process of branching (at strong coupling), these quanta should have an isotropic distribution. This is generally not the case at weak coupling (say, in classical electrodynamics), although it happens to be the case in the weak coupling limit of the $\mathcal{N}=4$ SYM theory, as shown in Sect. 7 (see also [5]), because of the additional symmetries of this theory.

In view of the above, we find it extremely surprising that the supergravity result for the radiated energy density is *not* isotropic in the non–relativistic limit, especially in the context of the $\mathcal{N}=4$ SYM theory, where the isotropy is realized already at weak coupling. Indeed, when $v_q \ll 1$, Eqs. (5.3) and (5.7) reduce to

$$\mathcal{E}^{(1)}_{\rm rad} \simeq \frac{\sqrt{\lambda}}{8\pi^2}\,\frac{\boldsymbol{a}_q^2}{r^2}\,, \tag{8.1}$$

which is isotropic and similar in structure to eq. (7.8), and respectively

$$\mathcal{E}^{(2)}_{\rm rad} \simeq -\frac{\sqrt{\lambda}}{24\pi^2}\,\frac{\boldsymbol{n}\cdot\dot{\boldsymbol{a}}_q - \big[\boldsymbol{a}_q^2 - 4(\boldsymbol{n}\cdot\boldsymbol{a}_q)^2 + \boldsymbol{v}_q\cdot\dot{\boldsymbol{a}}_q + (\boldsymbol{n}\cdot\boldsymbol{v}_q)(\boldsymbol{n}\cdot\dot{\boldsymbol{a}}_q)\big]}{r^2}. \tag{8.2}$$

which is manifestly not isotropic. Moreover, this last, anisotropic, term can even dominate over the first one in some cases, as shown by the example of the uniform rotation: there, $a=\omega^2 R_0=\omega v$ and $\dot a=\omega^3 R_0=\omega^2 v$, so clearly $\dot a \gg a^2$ when $v\ll 1$. Since moreover the sign of $\boldsymbol{n}\cdot\dot{\boldsymbol{a}}_q$ is oscillating when changing the direction of observation, one sees that the radiated energy density is *negative* in some regions, which brings us to our second puzzle.

(ii) *The negativity of the radiated energy density*

In Ref. [6] already the authors noticed that the energy density (6.5) radiated in the case of uniform rotation can become negative in some regions of space–time. From our present discussion we know that this behaviour must be associated with the second piece (6.3) of the radiation, which in some regions can become negative and also larger in magnitude than the first piece (6.2). In Ref. [6], where only the relativistic case was considered, the regions of negative energy were relatively small (and localized near the edges of the arm of the spiral) and besides the negative values reached by the energy in those regions were much smaller than its positive values towards the middle of the spiral arm. In the non–relativistic case, however, we have just seen that (for the case of rotation at least), the second piece (8.2) dominates over the first one

[7]This follows from the uncertainty principle: quanta with a large virtuality Q have a short formation time $\Delta t \sim \omega/Q^2$, where ω is the energy carried by the quanta. However, after being emitted, such quanta need to further radiate to become on–shell, which explains why their emission is suppressed at weak coupling.

(8.1) anywhere except at the particular points where $\boldsymbol{n}\cdot\dot{\boldsymbol{a}}_q$ vanishes, and that the sign of this second piece oscillates. Specifically, the non–relativistic limit of (6.5) reads

$$\mathcal{E}_{\rm rad} = \frac{\sqrt{\lambda}\omega^2 \upsilon}{24\pi^2 r^2}\,\sin\theta\sin(\phi-\omega t_r)\,, \tag{8.3}$$

which arises, as expected, from the $\boldsymbol{n}\cdot\dot{\boldsymbol{a}}_q$ term of $\mathcal{E}^{(2)}_{\rm rad}$ in eq. (8.2). Clearly, this (dominant) contribution to the radiated energy density oscillates around zero with the positive and negative maxima being of equal magnitude.

In principle, regions of negative energy density can occur in a quantum field theory, in the process of subtraction of the ultraviolet (UV) divergences. For the problem under consideration, such UV issues could affect the proper energy of the heavy quark, as carried by its near field, but on the other hand we find them rather unnatural in relation with the far fields and the radiation. As manifest say in eq. (8.3), the length scale associated with such space–time variations in the radiated energy is not some UV cutoff, but rather is determined by the external force that is giving the quark the specified motion.

Note also that eq. (8.3), and more generally the $\boldsymbol{n}\cdot\dot{\boldsymbol{a}}_q$ term of eq. (8.2), do not contribute to the radiated *power*, since they integrate to zero. The power appears to be dominated by the first term (5.10), and thus be positive, for all the examples that we have investigated.

To summarize, the anisotropy and the negativity of the energy density associated with the contribution $\mathcal{E}^{(2)}_{\rm rad}$ in eq. (5.7) look very unnatural to us and make us feel skeptical about this particular term. In our opinion, these unappealing features are merely an artifact of the supergravity approximation which will be corrected after including string fluctuations. It is also possible that this term, or at least a part of it, represent the tail of the quark proper energy at large distances, as suggested by the comparison between the associated power, eq. (5.11), and the results in [16, 17] (cf. the discussion after eq. (6.15)).

Also, the fact that the problems alluded to above are solely generated by the second piece, eq. (5.7), of the energy density does not mean that we fully trust the other piece in eq. (5.3). In spite of its rather appealing structure and of its similarity with the corresponding classical result, this term too has been produced from the string endpoint at $z = 0$ and thus it shows no quantum (radial) broadening. We therefore believe that also this term will be modified by string fluctuations, in the sense of acquiring a spread, but in such a way that its spatial integral giving the power will remain unchanged. Indeed, we believe that the correct result for the power (at least for a bounded motion and sufficiently large times) is given by $P^{(1)}_{\rm rad}$ in eq. (5.10), because this expression has been suggested by independent considerations (based on a world–sheet analysis) in Refs. [15–17] and because it coincides with the energy flow at the world–sheet horizon in all the examples that have been worked out in the literature.

At this point, we should recall that Ref. [5] has studied the angular distribution of the energy density produced by the decay of a time–like wave–packet within AdS/CFT, and found that this is isotropic in the supergravity approximation and it is only weakly affected by string fluctuations (at least, in a heuristic treatment of the latter inspired by flat–space string quantization). However, in that particular problem there was no kinematical scale which could induce an anisotropy (the wave–packet was spherically symmetric and at rest), unlike in the heavy quark problem under present consideration, where there are such scales. For us, our present results signal that, in general, the supergravity predictions cannot be trusted neither for the angular distribution of the radiation, nor for the radial one. Therefore, any progress towards better understanding the effects of the string fluctuations would be of paramount importance.

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A Useful integrals

Here we list some standard integrals which are useful when calculating the total power. With $\boldsymbol{v}$ the velocity, γ the Lorentz boost factor, $\boldsymbol{a}$ an arbitrary vector and $\boldsymbol{n}$ the unit vector

$$\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{A.1}$$

we have

$$\int \mathrm{d}\Omega\, \frac{1}{(1-\boldsymbol{n}\cdot\boldsymbol{v})^3} = 4\pi\gamma^4, \tag{A.2}$$

$$\int \mathrm{d}\Omega\, \frac{\boldsymbol{n}\cdot\boldsymbol{a}}{(1-\boldsymbol{n}\cdot\boldsymbol{v})^4} = \frac{16\pi}{3}\,\gamma^6\, \boldsymbol{v}\cdot\boldsymbol{a}, \tag{A.3}$$

$$\int \mathrm{d}\Omega\, \frac{(\boldsymbol{n}\cdot\boldsymbol{a})^2}{(1-\boldsymbol{n}\cdot\boldsymbol{v})^5} = \frac{4\pi}{3}\,\gamma^6 a^2 + 8\pi\gamma^8(\boldsymbol{v}\cdot\boldsymbol{a})^2, \tag{A.4}$$

and also

$$\int \mathrm{d}\Omega\, \frac{1+\boldsymbol{n}\cdot\boldsymbol{v}}{(1-\boldsymbol{n}\cdot\boldsymbol{v})^2} = 8\pi\gamma^2 - \frac{4\pi}{v}\,\tanh^{-1} v, \tag{A.5}$$

$$\int \mathrm{d}\Omega\, \frac{(\boldsymbol{n}\times\boldsymbol{v})^2}{(1-\boldsymbol{n}\cdot\boldsymbol{v})^3} = 4\pi\gamma^2 - \frac{4\pi}{v}\,\tanh^{-1} v. \tag{A.6}$$

An easy way to perform all the above or similar integrations is to assume, without any loss of generality, that instantaneously the particle is moving along the third axis, that is $\boldsymbol{v} = (0, 0, v)$. Then it is straightforward to perform the integral $\int \mathrm{d}\Omega\, [(\boldsymbol{n}\cdot\boldsymbol{a})^p/(\alpha - v\cos\theta)]$ with integer $p \geq 0$ and arbitrary $\alpha > v$. The desired integrals follow by an appropriate number of differentiations with respect to α evaluated at $\alpha = 1$.

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