Sequential Interval Estimation of a Location Parameter with the Fixed Width in the Non-regular Case

Dedicated to Professor Masafumi Akahira on his 60th birthday

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Abstract: For a location-scale parameter family of distributions with a finite support, a sequential confidence interval with a fixed width is obtained for the location parameter, and its asymptotic consistency and efficiency are shown. Some comparisons with the Chow-Robbins procedure are also done.

Keywords: Coverage probability; Extreme value; Non-regular case; Sequential interval estimation.

Subject Classifications: 62L12; 62F25.

1. INTRODUCTION

Suppose that we are to estimate a location parameter θ of a sequence of random observations $X_1, X_2, \ldots, X_n, \ldots$ with unknown scale ξ. We would like to obtain sequentially a confidence interval of fixed width $2d$ with confidence coefficient $1-\alpha$. Obviously we can not obtain a fixed sample size procedure if ξ is unknown. There are many works on the fixed-width interval estimation of normal mean (see, e.g. Ghosh et al. (1997)).

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Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent and identically distributed (i.i.d.) random variables according to the uniform distribution on the interval $(\theta - (\xi/2), \theta + (\xi/2))$, where $\theta(\in \mathbb{R}^1)$ and ξ (> 0) are unknown. Let $X_{(1)} := \min_{1\le i\le n} X_i$, $X_{(n)} := \max_{1\le i\le n} X_i$. Then the midrange and the range are $M_n := \left(X_{(1)} + X_{(n)}\right)/2$, $R_n := X_{(n)} - X_{(1)}$, respectively. Akahira and Koike (2005) considered a stopping rule:

$$\tau_1 := \inf\left\{ n \geq n_0 \;\middle|\; \frac{R_n}{n-1} \leq -\frac{2d}{\log \alpha} \right\},$$

where $n_0(\geq 2)$ is an initial size of sample. They showed the asymptotic consistency and efficiency of the estimation procedure $(\tau_1, [M_{\tau_1} - d, M_{\tau_1} + d])$.

In this paper, we consider the case of a location-scale parameter family of distributions with a finite support on the interval $(\theta - \xi a, \theta + \xi a)$, where θ and ξ are unknown, and obtain a sequential confidence interval of θ with fixed width $2d$ and confidence coefficient $1-\alpha$, and show its asymptotic consistency and efficiency. Some comparisons with the Chow-Robbins procedure are also done.

2. ASYMPTOTIC DISTRIBUTIONS OF THE EXTREME VALUES

In this section we consider the asymptotic distributions of the extreme values for distributions with a finite support, in a similar way to Akahira (1991) and Akahira and Takeuchi (1995).

Let $Z_1, Z_2, \ldots,$ be a sequence of independent and identically distributed (i.i.d.) random variables according to the density function $f_0(x-\theta)$ $(\theta \in \mathbb{R}^1)$ with respect to the Lebesgue measure. We assume the following conditions:
(A1) $f_0(x)$ has a finite support $(-a, a)$[1] $(a > 0)$, i.e., $f_0(x) > 0$ for $-a < x < a$, and $f_0(x) = 0$ otherwise.
(A2) $f_0(x)$ is continuously differentiable in the open interval $(-a, a)$ and

$$\lim_{x\to -a+0} f_0(x) = c, \lim_{x\to a-0} f_0(x) = c',$$

[1] If the support of f_0 is $(-a, b)$ $(a \neq b)$, then the normalized midrange does not converge to θ in probability as $n \to \infty$.

where c and c' are some positive constants.
(A3) $f_0(x)$ satisfies

$$f_0(x) \approx g(x+a)^\gamma \quad (x \to -a+0),$$
$$f_0(x) \approx g'|x-a|^\gamma \quad (x \to a-0),$$

where γ, g and g' are some positive constants[2].

Putting $Z_{(1)} := \min_{1\le i\le n} Z_i$, $Z_{(n)} := \max_{1\le i\le n} Z_i$, $U := n(Z_{(1)} + a - \theta)$ and $V := n(Z_{(n)} - a - \theta)$, we have the following lemma (cf. Akahira (1991), Akahira and Takeuchi (1995)).

Lemma 1. *Under the conditions (A1)and (A2), the joint(j.) p.d.f. $f_{U,V}^{(n)}(u,v)$ of (U,V) satisfies*

$$f_{U,V}^{(n)}(u,v) \to \begin{cases} cc' \exp\{c'v - cu\} & (v < 0 < u), \\ 0 & (\textit{otherwise}). \end{cases} \tag{2.1}$$

as $n \to \infty$.

Proof. The j.p.d.f. $f_{U,V}^{(n)}(u,v)$ of (U,V) is

$$\begin{aligned} &f_{U,V}^{(n)}(u,v) \\ &= \begin{cases} \dfrac{n-1}{n}\left\{F\left(a+\dfrac{v}{n}\right) - F\left(-a+\dfrac{u}{n}\right)\right\}^{n-2} f_0\left(-a+\dfrac{u}{n}\right) f_0\left(a+\dfrac{v}{n}\right) \\ \hfill (v<0<u), \\ 0 \hfill (\text{otherwise}), \end{cases} \end{aligned}$$

where $F(x) = \int_{-\infty}^x f_0(u)du$. Hence, by its expansion, we have the desired result. □

Next, we consider the location-scale parameter family of distributions with a finite support $(\theta - \xi a, \theta + \xi a)$. Suppose that $X_1, X_2, \dots, X_n, \dots$ is a sequence of i.i.d. random variables with the p.d.f. $(1/\xi) f_0((x-\theta)/\xi)$, where $\theta \in \mathbb{R}$ and $\xi > 0$. Put $Y_i := (X_i - \theta)/\xi$ for each $i = 1, 2, \dots$, and $Y_{(1)} := \min_{1\le i\le n} Y_i$, $Y_{(n)} := \max_{1\le i\le n} Y_i$. Letting $S := n(Y_{(1)} + Y_{(n)})/2$ and $T = n(Y_{(1)} - Y_{(n)} + 2a)/2$, we have the asymptotic (as.) j.p.d.f. of (S,T)

$$f_{S,T}(s,t) = \begin{cases} 2cc' \exp\{-(c-c')s - (c+c')t\} & (t > |s|), \\ 0 & (\text{otherwise}). \end{cases}$$

[2]If the converging order γ is different, then the normalized midrange does not converge to θ in probability as $n \to \infty$.

Then the as. marginal(m.) p.d.f.'s of S and T are given by

$$f_S(s) = \begin{cases} Ke^{-2cs} & (s \geq 0), \\ Ke^{2c's} & (s < 0), \end{cases} \tag{2.2}$$

$$f_T(t) = \begin{cases} \frac{2cc'}{c'-c}\left(e^{-2ct} - e^{-2c't}\right) & (t > 0 \text{ and } c \neq c'), \\ 4c^2te^{-2ct} & (t > 0 \text{ and } c = c'), \\ 0 & (\text{otherwise}), \end{cases}$$

respectively, where $K = 2cc'/(c+c')$.

In the case when $\lim_{x\to -a+0} f_0(x) = \lim_{x\to a-0} f_0(x) = 0$, we need another lemma. Putting $U' := n^{1/(\gamma+1)}(Z_{(1)} + a - \theta)$ and $V' := n^{1/(\gamma+1)}(Z_{(n)} - a - \theta)$, we have the following lemma in a similar way to Lemma 1.

Lemma 2. *Under the conditions (A1) and (A3), the j.p.d.f. $f^{(n)}_{U',V'}(u,v)$ of (U',V') satisfies*

$$f^{(n)}_{U',V'}(u,v) \to \begin{cases} gg'(-uv)^{\gamma} \exp\{-\frac{g'}{\gamma+1}(-v)^{\gamma+1} - \frac{g}{\gamma+1}u^{\gamma+1}\} & (v < 0 < u), \\ 0 & (\textit{otherwise}). \end{cases}$$

as $n \to \infty$.

The proof is omitted since it is similar to the one of Lemma 1.

From Lemma 2, U' and $(-V')$ are asymptotically, independently distributed according to Weibull distributions.

3. CONSTRUCTING CONFIDENCE INTERVAL

In this section we construct a sequential confidence interval for θ. In the first place, we consider the case under the conditions (A1) and (A2). For $0 < \alpha < 1$, let l_0 be the solution[3] of l for the equation

$$\frac{c+c'}{cc'}\alpha = \frac{e^{-2cl}}{c} + \frac{e^{-2c'l}}{c'}.$$

[3]It can be shown easily that such l_0 exists uniquely.

If ξ is known, we have from (2.2) that

$$\begin{aligned} P\{|M_n - \theta| \le d\} =& P\{n|M_n - \theta|/\xi \le dn/\xi\} \\ \approx& \int_{-dn/\xi}^{dn/\xi} f_S(s)ds \\ =& 1 - \frac{cc'}{c+c'}\left(\frac{e^{-2cnd/\xi}}{c} + \frac{e^{-2c'nd/\xi}}{c'}\right), \end{aligned}$$

where "$\approx$" means that the distribution of $n|M_n - \theta|/\xi$ is approximated by the asymptotic distribution. Letting $n^* = l_0\xi/d$, we have for $n \ge n^*$

$$1 - \frac{cc'}{c+c'}\left(\frac{e^{-2cnd/\xi}}{c} + \frac{e^{-2c'nd/\xi}}{c'}\right) \ge 1 - \alpha.$$

n^* is referred as the asymptotically *optimal* size of samples if ξ is known. Note that $n(M_n - \theta)/\xi = S$ and $R_n/\xi = -(T/n) + 2a$. Now we take as the stopping rule

$$\tau_2 := \inf\left\{ n \ge n_0 \;\middle|\; \frac{R_n}{n-1} \le \frac{2ad}{l_0} \right\}, \tag{3.1}$$

where $n_0(\ge 2)$ is an initial size of sample. Then we obtain the asymptotic properties of the estimation procedure $(\tau_2, [M_{\tau_2} - d, M_{\tau_2} + d])$ as follows.

Theorem 1. *For the sequential estimation procedure* $(\tau_2, [M_{\tau_2} - d, M_{\tau_2} + d])$, *the following hold.*
(i) $\lim_{d\to 0+} P\{|M_{\tau_2} - \theta| \le d) = 1 - \alpha$ *(asymptotic consistency).*
(ii) $\tau_2/n^* \overset{\text{a.s.}}{\to} 1 \quad (d \to 0+)$.
(iii) $E(\tau_2)/n^* \to 1 \ (d \to 0+)$ *(asymptotic efficiency).*

Proof. (i) From Lemma 1 of Chow and Robbins (1965), the stopping rule τ_2 given by (3.1) satisfies

$$\lim_{d\to 0+} \frac{d\tau_2}{\xi l_0} = 1 \quad \text{a.s.} \tag{3.2}$$

Since $S = n(M_n - \theta)/\xi$ converges in distribution to a distribution with the density given by (2.2) as $n \to \infty$, it follows from Theorem 1 of Anscombe (1952) that $\tau_2(M_{\tau_2} - \theta)$ converges in distribution to the same distribution as $d \to 0+$. Hence, since $d\tau_2/\xi \overset{\text{a.s.}}{\to} l_0$ as $d \to 0+$ from (3.2), it follows that

$$\begin{aligned} \lim_{d\to 0+} P\{|M_{\tau_2} - \theta| \le d\} &= \lim_{d\to 0+} P\{\tau_2|M_{\tau_2} - \theta|/\xi \le d\tau_2/\xi\} \\ &= \int_{-l_0}^{l_0} f_S(s)ds = 1 - \alpha. \end{aligned} \tag{3.3}$$

(ii) From (3.2) and the definition of l_0, we have $\tau_2/n^* = \tau_2 d/(l_0\xi) \stackrel{\text{a.s.}}{\to} 1$ as $d \to 0+$.
(iii) From Lemma 2 of Chow and Robbins (1965), we have the desired result. $\square$

Remark. In particular, if $c = c'$, then $l_0 = -\log\alpha/(2c)$ and τ_2 given in (3.1) is expressed as

$$\tau_2 = \inf\left\{ n \geq n_0 \;\middle|\; \frac{R_n}{n-1} \leq -\frac{4acd}{\log\alpha} \right\},$$

which is equal to τ_1 when the underlying distribution is uniform distribution on the interval $(\theta - (\xi/2), \theta + (\xi/2))$.

In the second place, we compare this with the Chow-Robbins procedure. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with the mean θ and the variance σ^2. Let $\bar{X}_n := \sum_{i=1}^n X_i/n$, $s_n^2 = \sum_{i=1}^n (X_i - \bar{X}_n)^2/(n-1)$. Chow and Robbins (1965) considered a stopping rule defined by

$$\tau_{CR} := \inf\left\{ n \geq n_0 \mid n \geq u_{\alpha/2}^2 d^{-2} s_n^2 \right\},$$

where $u_{\alpha/2}$ is the upper $\alpha/2$ point of $N(0,1)$ and $n_0(\geq 2)$ is an initial size of samples. They showed the asymptotic consistency and efficiency of the estimation procedure $(\tau_{CR}, [\bar{X}_{\tau_{CR}} - d, \bar{X}_{\tau_{CR}} + d])$.

Since, from Theorem 2.2 of Akahira and Koike (2005), Theorem 1 and Theorem of Chow and Robbins (1965),

$$\tau_1 \approx \frac{\log\alpha}{\log(1 - (2d/\xi))} \approx \frac{-\xi\log\alpha}{2d}, \quad \tau_2 \approx l_0\xi/d, \quad \tau_{CR} \approx u_{\alpha/2}^2\sigma^2/d^2,$$

as $d \to 0+$, we have $\tau_1/\tau_{CR}, \tau_2/\tau_{CR} \to 0$ $(d \to 0+)$. Therefore τ_1, τ_2 is asymptotically better than τ_{CR} in the sense of the average size of sample.

Furthermore, we consider the case under the conditions (A1) and (A3). By putting $S' := n^{1/(\gamma+1)}(Y_{(1)} + Y_{(n)})/2$ and $T' := n^{1/(\gamma+1)}(Y_{(1)} - Y_{(n)} + 2a)/2$, the as.j.p.d.f. of (S', T') and the as.m.p.d.f.'s of S' and T' are obtained from Lemma 2. In a similar way to (3.3), we take l_0 satisfying $\int_{-l_0}^{l_0} f_{S'}(s)ds = 1 - \alpha$ for the as.m.p.d.f. $f_{S'}(s)$ of S'.

If ξ is known, we have

$$\begin{aligned} P\{|M_n-\theta|\le d\} =& P\{n^{1/(\gamma+1)}|M_n-\theta|/\xi \le dn^{1/(\gamma+1)}/\xi\} \\ \approx& \int_{-dn^{1/(\gamma+1)}/\xi}^{dn^{1/(\gamma+1)}/\xi} f_{S'}(s)ds, \end{aligned}$$

where "$\approx$" means that the distribution of $n^{1/(\gamma+1)}|M_n-\theta|/\xi$ is approximated by the asymptotic distribution. The optimal size of sample required for attaining the preassigned coverage probability $1-\alpha$ is the smallest positive integer $\ge (l_0\xi/d)^{\gamma+1} =: n^{**}$ (say). Define a stopping rule as

$$\tau_3 := \inf\left\{ n \ge n_0 \;\middle|\; \frac{R_n}{n^{1/(\gamma+1)}} \le \frac{2ad}{l_0} \right\},$$

where $n_0(\ge 2)$ is an initial size of samples. Then the next theorem follows.

Theorem 2. *For the sequential estimation procedure* $(\tau_3, [M_{\tau_3}-d, M_{\tau_3}+d])$, *the following hold.*
(i) $\lim_{d\to 0+} P\{|M_{\tau_3}-\theta|\le d) = 1-\alpha$ *(asymptotic consistency).*
(ii) $\tau_3/n^{**} \overset{\text{a.s.}}{\to} 1 \quad (d\to 0+)$.
(iii) $E(\tau_3)/n^{**} \to 1$ $(d\to 0+)$ *(asymptotic efficiency).*

Proof. The proof for (i) is similar to the one of Theorem 1 (i). (ii) follows from $(\tau_3/n^{**})^{1/(\gamma+1)} \overset{\text{a.s.}}{\to} 1$ as $d\to 0+$.
(iii) From (ii), by Fatou's lemma,

$$\liminf_{d\to 0+} \frac{E(\tau_3)}{n^{**}} \ge E\left(\liminf_{d\to 0+} \frac{\tau_3}{n^{**}}\right) = 1. \tag{3.4}$$

On the other hand, since $0\le R_n \le 2a\xi$ with probability 1 for any $n\in\mathbb{N}$, we have $0 \le (R_n l_0/(2ad))^{\gamma+1} \le (2a\xi l_0/(2ad))^{\gamma+1} = (l_0\xi/d)^{\gamma+1}$ with probability 1 for any $n\in\mathbb{N}$. So, $0\le (R_n l_0/(2ad))^{\gamma+1} \le n$ with probability 1 for n satisfying $n \ge (l_0\xi/d)^{\gamma+1}+1$. Therefore, since $\tau_3 =$ $\inf\left\{n\ge n_0 \mid (R_n l_0/(2ad))^{\gamma+1} \le n\right\}$, we have $\tau_3 \le \left(\frac{l_0\xi}{d}\right)^{\gamma+1}+1$. Then, using the definition of n^{**}, we have

$$\frac{E(\tau_3)}{n^{**}} \le \left\{\left(\frac{l_0\xi}{d}\right)^{\gamma+1} + 1\right\}\left(\frac{l_0\xi}{d}\right)^{-(\gamma+1)} = 1 + \left(\frac{d}{l_0\xi}\right)^{\gamma+1},$$

hence

$$\limsup_{d \to 0+} \frac{E(\tau_3)}{n^{**}} \leq 1. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain (iii). □

From Theorem 2 and Theorem of Chow and Robbins (1965), $\tau_3 \approx (l_0 \xi / d)^{\gamma+1}$ and $\tau_{CR} \approx u_{\alpha/2}^2 \sigma^2 / d^2$ as $d \to 0+$. Therefore,

$$\tau_3/\tau_{CR} \begin{cases} = o(1) & (0 < \gamma < 1), \\ = O(1) & (\gamma = 1), \\ \to \infty & (\gamma > 1) \end{cases}$$

as $d \to 0+$. Therefore, τ_3 is asymptotically better than τ_{CR} in the sense of the average size of sample if $0 < \gamma < 1$.

In this paper, we considered the cases when the values at the endpoints of the support of the p.d.f. are positive simultaneously, or tend to 0 at the same speed. In the meantime, if the either value at the endpoints of the support of the p.d.f. is positive, or tend to 0 at a different speed, then the coefficients of $n^\gamma(X_{(1)} - a - \theta)$ and $n^\delta(X_{(n)} - b - \theta)$ converging to nontrivial random variables are different and estimation by using the midrange M_n is inappropriate.

4. NUMERICAL EXAMPLE

In this section we examine the coverage probability of the procedure $[M_{\tau_2} - d, M_{\tau_2} + d]$ by simulation based on 100000 repetitions. Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of i.i.d. random variables with the p.d.f. $(1/\xi) f_0((x - \theta)/\xi)$, where $\theta \in \mathbb{R}$, $\xi > 0$ and $f_0(\cdot)$ is a trapezoid-shape p.d.f. given by

$$f_0(x) = \begin{cases} (\frac{1}{2} - c)x + \frac{1}{2} & (x \in (-1, 1)), \\ 0 & (\text{otherwise}) \end{cases}$$

with $0 < c < 1$. Note that, f_0 is the p.d.f. of the uniform distribution over $(-1, 1)$ and an asymmetric p.d.f. over $(-1, 1)$ for $c = 0.5$ and a sufficiently small $c > 0$, respectively. Since M_{τ_2} is location equivariant, we may assume $\theta = 0$ without loss of generality.

When $\alpha = 0.10$, $d = 0.01(0.01)0.05$, $\xi = 1(1)5$ and $n_0 = 5$, Tables 1 and 2 show the values of coverage probabilities of the sequential estimation procedure $(\tau_2, [M_{\tau_2} - d, M_{\tau_2} + d])$ for $c = 0.1$ and $c = 0.5$, respectively. The result suggests that the estimation procedure is consistent for this case.

Table 1. Coverage probabilities of $[M_{\tau_2} - d, M_{\tau_2} + d]$ for $c = 0.1$

$\xi \setminus d$	0.01	0.02	0.03	0.04	0.05
1	0.90637	0.91545	0.92348	0.93092	0.93758
2	0.89830	0.90544	0.90960	0.91424	0.92017
3	0.90123	0.90313	0.90713	0.90832	0.91030
4	0.89926	0.90117	0.90333	0.90615	0.90804
5	0.89817	0.89952	0.90318	0.90421	0.90561

Table 2. Coverage probabilities of $[M_{\tau_2} - d, M_{\tau_2} + d]$ for $c = 0.5$

$\xi \setminus d$	0.01	0.02	0.03	0.04	0.05
1	0.90210	0.90727	0.91183	0.91328	0.91988
2	0.89929	0.90131	0.90330	0.90628	0.91176
3	0.89849	0.89947	0.90221	0.90235	0.90525
4	0.89729	0.89729	0.89982	0.90169	0.90322
5	0.89785	0.8998	0.89906	0.89862	0.90054

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