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Light-cone Gauge NSR Strings in Noncritical Dimensions II —Ramond Sector—

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Abstract

Light-cone gauge superstring theory in noncritical dimensions corresponds to a worldsheet theory with nonstandard longitudinal part in the conformal gauge. The longitudinal part of the worldsheet theory is a superconformal field theory called $X^{\pm}$ CFT. We show that the $X^{\pm}$ CFT combined with the super-reparametrization ghost system can be described by free variables. It is possible to express the correlation functions in terms of these free variables. Bosonizing the free variables, we construct the spin fields and BRST invariant vertex operators for the Ramond sector in the conformal gauge formulation. By using these vertex operators, we can rewrite the tree amplitudes of the noncritical light-cone gauge string field theory, with external lines in the (R,R) sector as well as those in the (NS,NS) sector, in a BRST invariant way.

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1 Introduction

The light-cone gauge string field theory [1, 2, 3, 4, 5, 6] takes a simple form and it can therefore be a very useful tool to study string theory. Being a gauge fixed theory, it can be formulated in noncritical spacetime dimensions. In the conformal gauge, such noncritical string theories correspond to worldsheet theories with nonstandard longitudinal part. In our previous works [7, 8], we have studied the longitudinal part of the worldsheet theory which is an interacting CFT called $X^{\pm}$ CFT. It has the right value of the Virasoro central charge, so that one can construct a nilpotent BRST charge combined with the transverse part and the reparametrization ghosts.

In the conformal gauge formulation, the amplitudes can be calculated in a BRST invariant manner. In Ref. [7] we have shown that the tree level amplitudes in the light-cone gauge coincide with the BRST invariant ones in the conformal gauge, in the case of the bosonic noncritical strings. For superstrings, the equivalence of the amplitudes in the two gauges has been shown for the cases where all the external lines are in the (NS,NS) sector[1] [9].

We would like to extend this analysis into the case in which external lines in the Ramond sector are involved. In the conformal gauge formulation, the vertex operators corresponding to the external lines in the Ramond sector should involve the spin fields in the $X^{\pm}$ CFT. Since the $X^{\pm}$ CFT is an interacting theory, it is not straightforward to construct spin fields. In this paper, we formulate a free field description of the CFT consisting of the $X^{\pm}$ CFT and the reparametrization ghosts. Namely, we construct free variables which can be expressed in terms of $X^{\pm}$ and the ghosts. We provide a formula to express the correlation functions of this interacting CFT in terms of these free variables. Bosonizing the free variables, we define the spin fields and thereby construct the vertex operators in the Ramond sector.

In the conformal gauge, the amplitudes can be expressed by the vertex operators thus constructed. It turns out that the closed superstring theory in noncritical dimensions generically does not include spacetime fermions. We show that the tree amplitudes with not only external lines in the (NS,NS) sector but also those in the (R,R) sector can be written by using the vertex operators.

This paper is organized as follows. In section 2, we consider the system of the bosonic $X^{\pm}$ CFT combined with the reparametrization ghosts and construct free variables. We show how the correlation functions on the complex plane can be expressed by those of the free variables. In section 3, we supersymmetrize the analyses in section 2 and formulate the free field description of the supersymmetric $X^{\pm}$ CFT. In section 4, we first study how the

[1] In our previous works and in this work, we discuss closed strings.

BRST invariant vertex operators in the Neveu-Schwarz sector can be described in terms of free variables obtained in section 3. Then we construct those in the Ramond sector, using the free variables. In section 5, we show that the tree amplitudes involving external lines in the (R,R) and the (NS,NS) sectors of the noncritical strings can be expressed in a BRST invariant way using the BRST invariant vertex operators. Section 6 is devoted to conclusions and discussions. In appendix A, we explain some details of the action for the strings in the (R,R) and the (NS,NS) sectors of the light-cone gauge string field theory in noncritical dimensions. In appendix B, we present a proof of a relation which we use in section 5.

2 Free variables: bosonic case

As a warm-up, we would like to present the free field description for bosonic $X^{\pm}$ CFT formulated in Ref. [7] and show how the correlation functions are expressed by using the free variables.

2.1 Bosonic $X^{\pm}$ CFT

In the conformal gauge, the longitudinal part of the worldsheet theory for the noncritical light-cone gauge string theory is described by a conformal field theory with the energy-momentum tensor

$$\partial X^+ \partial X^- - \frac{d-26}{12} \left\{ X^+, z \right\} \ , \tag{2.1}$$

where

$$\left\{ X^+, z \right\} \equiv \frac{\partial^3 X^+}{\partial X^+} - \frac{3}{2} \left(\frac{\partial^2 X^+}{\partial X^+} \right)^2 \tag{2.2}$$

is the Schwarzian derivative.

Such a conformal field theory can be studied using the path integral formalism [7]. In order to make the theory well-defined, we always consider the situations where the vertex operators of the form $e^{-ip^+X^-}$ are inserted so that ∂X^+ has an expectation value and it is invertible except for sporadic points on the worldsheet. Indeed, for a functional $F[X^+]$ of X^+, one can calculate the correlation function with the insertion $\prod_{r=1}^{N} e^{-ip_r^+ X^-} \left(Z_r, \bar{Z}_r \right)$ on the complex plane as

$$\left\langle F \left[X^+ \right] \prod_{r=1}^{N} e^{-ip_r^+ X^-} \left(Z_r, \bar{Z}_r \right) \right\rangle = F \left[-\frac{i}{2} \left(\rho + \bar{\rho} \right) \right] \left\langle \prod_{r=1}^{N} e^{-ip_r^+ X^-} \left(Z_r, \bar{Z}_r \right) \right\rangle \ , \tag{2.3}$$

where

$$\rho(z) = \sum_{r=1}^{N} \alpha_r \ln(z - Z_r) \ , \qquad \alpha_r \equiv 2p_r^+ \ . \tag{2.4}$$

Thus one can see that X^+ acquires an expectation value $-\frac{i}{2}(\rho(z) + \bar{\rho}(\bar{z}))$. The expectation value of $\partial X^+(z)$ is proportional to $\partial\rho(z)$. $\partial\rho(z)$ has N poles at $z = Z_r$ and $N-2$ zeros at $z = z_I$ $(I = 1, \cdots N-2)$. $\rho(z)$ coincides with the Mandelstam mapping of a tree light-cone diagram for N strings and z_I are the interaction points.

The variables $X^\pm$ can be shown to satisfy the OPE's

$$\begin{aligned} \partial X^+(z)\partial X^+(z') &\sim \text{regular} \ , \\ \partial X^-(z)\partial X^+(z') &\sim \frac{1}{(z-z')^2} \ , \\ \partial X^-(z)\partial X^-(z') &\sim -\frac{d-26}{12}\partial_z\partial_{z'}\left[\frac{1}{\left(X_L^+(z) - X_L^+(z')\right)^2}\right] \ , \end{aligned} \tag{2.5}$$

where X_L^+ denotes the left-moving part of X^+. Expanding the right hand side of the third equation in terms of $z - z'$ with the assumption $|z - z'| \ll 1$, one gets the form of the OPE given in Ref. [7]. Using these OPE's, one can show that the energy-momentum tensor (2.1) satisfies the Virasoro algebra with central charge $28 - d$. Thus, with the reparametrization ghosts and the transverse part, the worldsheet theory becomes a CFT with the total central charge 0.

2.2 Free fields

Let us consider a 2D CFT which consists of the $X^\pm$ CFT and the system of reparametrization ghosts $b, c, \tilde{b}, \tilde{c}$. One can show that this theory can be described by free variables [7]. Free variables X^+, X'^-, b', c', $\tilde{b}', \tilde{c}'$ are defined as

$$\begin{aligned} b' &\equiv \left(\partial X^+\right)^\alpha b \ , \qquad \tilde{b}' \equiv \left(\bar{\partial} X^+\right)^\alpha \tilde{b} \ , \\ c' &\equiv \left(\partial X^+\right)^{-\alpha} c \ , \qquad \tilde{c}' \equiv \left(\bar{\partial} X^+\right)^{-\alpha} \tilde{c} \ , \\ X'^- &\equiv X^- - \alpha\frac{cb}{\partial X^+} - \frac{3}{2}\alpha\frac{\partial^2 X^+}{\left(\partial X^+\right)^2} - \alpha\frac{\tilde{c}\tilde{b}}{\bar{\partial} X^+} - \frac{3}{2}\alpha\frac{\bar{\partial}^2 X^+}{\left(\bar{\partial} X^+\right)^2} \ , \end{aligned} \tag{2.6}$$

with

$$\alpha(\alpha + 3) = \frac{d-26}{12} \ . \tag{2.7}$$

The OPE's between $X^+, X'^-, b', c', \tilde{b}', \tilde{c}'$ can be derived from the OPE's of $X^\pm, b, c, \tilde{b}, \tilde{c}$ and one can see that they are free variables. It is straightforward to show that the energy-momentum tensor of the system

$$T(z) = \partial X^+ \partial X^- - \frac{d-26}{12}\left\{X^+, z\right\} - 2b\partial c - \partial b c \ , \tag{2.8}$$

can be written as

$$T(z) = \partial X^+ \partial X'^- - b'\partial c' - (1+\alpha)\,\partial\left(b'c'\right) \ , \tag{2.9}$$

in the form of the energy-momentum tensor for the free fields X^+, X'^-, b', c'. The fields b', c' are with conformal weight $(2+\alpha, 0), (-1-\alpha, 0)$ respectively. It is also easy to express $X^\pm, b, c, \tilde{b}, \tilde{c}$ in terms of the free variables.

2.3 Correlation functions

Since one can express all the fields in the theory in terms of the free variables and vice versa, it should be possible to describe the theory using these free variables. Let

$$\left\langle \phi_1 \phi_2 \cdots \phi_N \right\rangle_{X^\pm, b, c} \tag{2.10}$$

denote the correlation function on the complex plane in the CFT we are considering. As we mentioned above, in the $X^\pm$ CFT, we are mainly interested in the the correlation functions with insertions of $e^{-ip^+X^-}$. In our setup, the correlation functions to be considered are of the form

$$\left\langle \left|e^{3\sigma}(\infty)\right|^2 \phi_1(z_1, \bar{z}_1)\, \phi_2(z_2, \bar{z}_2) \cdots \phi_n(z_n, \bar{z}_n) \prod_{r=1}^N e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right) \right\rangle_{X^\pm, b, c} , \tag{2.11}$$

where $\partial\sigma = cb$ and $\phi_i \ (i = 1, \cdots, n)$ are local operators made from $X^+, \partial X^-, \bar{\partial} X^-, b, c, \tilde{b}, \tilde{c}$ and their derivatives. $\left|e^{3\sigma}(\infty)\right|^2$ is inserted to soak up the ghost zero modes.

The correlation function (2.11) should be expressed by using the free variables. Let us define the correlation function for the free theory on the complex plane as

$$\left\langle \phi_1 \phi_2 \cdots \phi_n \right\rangle_{\text{free}} \equiv \frac{\int \left[dX^+ dX'^- db' dc' d\tilde{b}' d\tilde{c}'\right] e^{-S_{\text{free}}\left[X^+, X'^-, b', c', \tilde{b}', \tilde{c}'\right]} \phi_1 \phi_2 \cdots \phi_n}{\int \left[dX^+ dX'^- db' dc' d\tilde{b}' d\tilde{c}'\right] e^{-S_{\text{free}}\left[X^+, X'^-, b', c', \tilde{b}', \tilde{c}'\right]}} \ . \tag{2.12}$$

Naively, one might expect that the correlation function (2.11) should be expressed in terms of the free variables as

$$\left\langle \left|e^{3\sigma}(\infty)\right|^2 \phi_1(z_1, \bar{z}_1)\, \phi_2(z_2, \bar{z}_2) \cdots \phi_n(z_n, \bar{z}_n) \prod_{r=1}^N e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right) \right\rangle_{X^\pm, b, c}$$

$$= \left\langle \left| e^{3\sigma}(\infty) \right|^2 \phi_1(z_1, \bar{z}_1)\, \phi_2(z_2, \bar{z}_2) \cdots \phi_n(z_n, \bar{z}_n) \prod_{r=1}^{N} e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right) \right\rangle_{\text{free}} , \quad (2.13)$$

on the right hand side of which σ, ϕ_i and X^- are considered to be expressed by the free variables using the relations (2.6). Eq.(2.13) would hold if the relations (2.6) were not singular anywhere on the complex plane. However, if the expectation value of ∂X^+ has zeros and poles, the relations (2.6) are not well-defined at these points. Then we need to modify eq.(2.13) and insert operators at these points on the right hand side.

2.4 Operator insertions

The necessity of such insertions can be seen by considering the case where all the ϕ_i do not involve derivatives of X^- in eq.(2.13). Because of eq.(2.3), X^+ in the correlation function can be replaced by its expectation value $-\frac{i}{2}(\rho + \bar{\rho})$ in such a case.

If ∂X^+ is replaced by $-\frac{i}{2}\partial\rho$, the relations between the ghost variables are the ones which were studied in Refs. [10, 11, 12]. They showed that the correlation functions of b, c can be expressed by those of b', c' with extra operator insertions at $z = z_I, Z_r, \infty$. For example, if $b(z)$, $c(z)$ are regular at $z = z_I$,[2] the relation (2.6) implies that $b'(z)$, $c'(z)$ are singular at $z = z_I$, because $\partial\rho(z_I) = 0$. One can see

$$b'(z) \sim (z - z_I)^{\alpha} , \qquad c'(z) \sim (z - z_I)^{-\alpha} , \quad (2.14)$$

for $z \sim z_I$. Such singularities are induced by the insertions $e^{-\alpha\sigma'}(z_I)$, where $\sigma'(z)$ is defined so that $\partial\sigma' = c'b'$. Therefore the correlation functions of b, c with no insertions at $z = z_I$ should correspond to those of b', c' with insertions of $e^{-\alpha\sigma'}(z_I)$. Thus we can see that eq.(2.13) cannot be true as it is. It should at least be modified as

$$\begin{aligned} &\left\langle \left| e^{3\sigma}(\infty) \right|^2 \phi_1\phi_2 \cdots \phi_n \prod_{r=1}^{N} e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right) \right\rangle_{X^\pm, b, c} \\ &\quad \sim \left\langle \left| e^{3\sigma}(\infty) \right|^2 \phi_1\phi_2 \cdots \phi_n \prod_{I} \left| e^{-\alpha\sigma'}(z_I) \right|^2 \prod_{r=1}^{N} e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right) \right\rangle_{\text{free}} , \end{aligned} \quad (2.15)$$

in order to be consistent with the singularities of the ghost variables.

In our case, X^+ is dynamical and eq.(2.15) is still inconsistent. If one inserts the energy-momentum tensor $T(z)$ into the correlation functions in eq.(2.15), the left hand side should

[2] Here we assume in eq.(2.13) the generic configuration in which $z_i \neq Z_r, z_I$ $(i = 1, \cdots, n)$. The special cases where z_i coincides with one of these points are realized as a limit of the generic ones.

be regular at $z = z_I$ [7] but the right hand side is not because of $e^{-\alpha\sigma'}(z_I)$. Instead of $e^{-\alpha\sigma'}(z_I)$, we therefore need to insert an operator which is conformal invariant and induces the same singularities for b', c' as $e^{-\alpha\sigma'}(z_I)$. We find that

$$\mathcal{O}_I \equiv \left| \oint_{z_I} \frac{dz}{2\pi i} \partial\Phi \frac{e^{-\alpha\sigma'}}{(\partial^2 X^+)^{\frac{3}{4}\alpha(\alpha+1)}}(z) \right|^2 , \tag{2.16}$$

where

$$\Phi \equiv \ln \partial X^+ \bar{\partial} X^+ , \tag{2.17}$$

has such properties. Indeed, replacing X^+ by its expectation value $-\frac{i}{2}(\rho + \bar{\rho})$, one can see that $\mathcal{O}_I$ is equivalent to

$$\left| \frac{e^{-\alpha\sigma'}}{(\partial^2 \rho)^{\frac{3}{4}\alpha(\alpha+1)}}(z_I) \right|^2 , \tag{2.18}$$

and b', c' behave as eq.(2.14) in the presence of $\mathcal{O}_I$. Moreover, the OPE with the energy-momentum tensor can be calculated as

$$\begin{aligned} T(z)\,\mathcal{O}_I \;&\sim\; \oint_{z_I} \frac{dw}{2\pi i} \partial_w \left[\frac{1}{z-w} \left(\partial\left(\ln \partial X^+\right) \frac{e^{-\alpha\sigma'}}{(\partial^2 X^+)^{\frac{3}{4}\alpha(\alpha+1)}} \right)(w) \right] \\ &\quad + \left(1 - \frac{3}{4}\alpha(\alpha+1)\right) \oint_{z_I} \frac{dw}{2\pi i} \frac{2}{(z-w)^3} \frac{e^{-\alpha\sigma'}}{(\partial^2 X^+)^{\frac{3}{4}\alpha(\alpha+1)}}(w) \\ &= \left(1 - \frac{3}{4}\alpha(\alpha+1)\right) \oint_{z_I} \frac{dw}{2\pi i} \frac{2}{(z-w)^3} \frac{e^{-\alpha\sigma'}}{(\partial^2 X^+)^{\frac{3}{4}\alpha(\alpha+1)}}(w) \; . \end{aligned} \tag{2.19}$$

On the assumption that one can replace X^+ by its expectation value[3]

$$\begin{aligned} T(z)\,\mathcal{O}_I \;&\sim\; \left(1 - \frac{3}{4}\alpha(\alpha+1)\right) \oint_{z_I} \frac{dw}{2\pi i} \frac{2}{(z-w)^3} \frac{e^{-\alpha\sigma'}}{\left(-\frac{i}{2}\partial^2\rho\right)^{\frac{3}{4}\alpha(\alpha+1)}}(w) \\ &\sim\; \text{regular} \; . \end{aligned} \tag{2.20}$$

Therefore $\mathcal{O}_I$ seems to have the right properties to be inserted in the free field expression. We will prove the fact that the OPE $T(z)\,\mathcal{O}_I$ becomes regular without any assumptions, in the next subsection.

With similar reasonings, one can deduce the singular behaviors of b', c' at the points $z = Z_r$ and ∞ as well, from which one can infer the ghost operators to be inserted at these

[3]Here we have also assumed $\partial^2\rho(z_I) \neq 0$, which is generically true. $\partial^2\rho(z_I) = 0$ implies that z_I coincides with another interaction point $z_{I'}$ ($I' \neq I$). Such cases are considered as a limit of the generic cases, in which we should insert $\mathcal{O}_I$ and $\mathcal{O}_{I'}$ at the same point.

points. For $z \sim Z_r$, we can see that $e^{\alpha\sigma'}(Z_r)$ should be inserted. Combined with the insertion $e^{-ip_r^+X^-}(Z_r,\bar{Z}_r)$ and the operator to be introduced in eq.(2.23) at $z=Z_r$, this ghost operator reproduces the correct OPE with the energy-momentum tensor. For $w\equiv \frac{1}{z}\sim 0$, the ghosts should behave as

$$b'(w)\sim w^{-3}\ , \qquad c'(w)\sim w^{3}\ , \tag{2.21}$$

and one can see that $e^{3\sigma'}(\infty)$, which is of weight -3α, should be inserted. We can define a conformal invariant combination,

$$\mathcal{R}\equiv\left|\oint_{\infty}\frac{dz}{2\pi i}\partial\Phi\left(\partial X^+\right)^{3\alpha}e^{3\sigma'}\right|^2\ , \tag{2.22}$$

to implement such an insertion.

We should also take care of the singular behavior of X^- in the $X^\pm$ CFT. From the results of Ref. [7], one can see that X^- possesses logarithmic singularities at Z_r and $z_{I^{(r)}}$, where $z_{I^{(r)}}$ is the interaction point at which the rth string interacts. Therefore it is necessary to insert

$$\exp\left(\frac{d-26}{24}\frac{i}{p_r^+}X^+\right)\left(Z_r,\bar{Z}_r\right)\ , \qquad \exp\left(-\frac{d-26}{24}\frac{i}{p_r^+}X^+\right)\left(z_{I^{(r)}},\bar{z}_{I^{(r)}}\right)\ . \tag{2.23}$$

The latter should be made into a conformal invariant combination

$$\mathcal{S}_r\equiv\oint_{z_{I^{(r)}}}\frac{dz}{2\pi i}\partial\Phi\oint_{\bar{z}_{I^{(r)}}}\frac{d\bar{z}}{2\pi i}\bar{\partial}\Phi\,\exp\left(-\frac{d-26}{24}\frac{i}{p_r^+}X^+\right). \tag{2.24}$$

The conformal invariance of $\mathcal{R}$ and $\mathcal{S}_r$ can be proved in a similar way to that of $\mathcal{O}_I$ in eq.(2.20) on the assumption that X^+ can be replaced by its expectation value $-\frac{i}{2}(\rho+\bar{\rho})$. In the next subsection, we will prove that this assumption is not necessary as in the $\mathcal{O}_I$ case.

2.5 Correlation functions in terms of the free variables

We have shown what kind of operator insertions are necessary. We would like to show that they are actually enough and the correlation functions of the system can be expressed in terms of the free variables only with the insertions obtained above. To be precise, we will prove

$$\begin{aligned}
&\left\langle\left|e^{3\sigma}(\infty)\right|^2\phi_1\phi_2\cdots\phi_n\prod_{r=1}^{N}e^{-ip_r^+X^-}\left(Z_r,\bar{Z}_r\right)\right\rangle_{X^\pm,b,c}\\
&\quad=\ \mathcal{C}\left\langle\mathcal{R}\phi_1\phi_2\cdots\phi_n\prod_I\mathcal{O}_I\prod_{r=1}^{N}\left[\mathcal{S}_r\left|\alpha_r\right|^{-3\alpha}\left|e^{\alpha\sigma'}(Z_r)\right|^2e^{-ip_r^+X'^-+\frac{d-26}{24}\frac{i}{p_r^+}X^+}\left(Z_r,\bar{Z}_r\right)\right]\right\rangle_{\mathrm{free}}\ ,
\end{aligned} \tag{2.25}$$

where $\mathcal{C}$ is a numerical constant.

Let us first consider the simplest case and check if

$$\left\langle \left|e^{3\sigma}(\infty)\right|^2 \prod_{r=1}^{N} e^{-ip_r^+ X^-}(Z_r,\bar{Z}_r) \right\rangle_{X^\pm,b,c}$$
$$\propto \left\langle \mathcal{R}\prod_I \mathcal{O}_I \prod_{r=1}^{N}\left[\mathcal{S}_r \left|\alpha_r\right|^{-3\alpha}\left|e^{\alpha\sigma'}(Z_r)\right|^2 e^{-ip_r^+ X'^- + \frac{d-26}{24}\frac{i}{p_r^+}X^+}\left(Z_r,\bar{Z}_r\right)\right]\right\rangle_{\text{free}} . \quad (2.26)$$

The left hand side was evaluated to be $\exp\left(-\frac{d-26}{24}\Gamma\right)$ up to a constant multiplicative factor in Ref. [7], where Γ is defined as

$$e^{-\Gamma} = \left|\sum_{r=1}^{N}\alpha_r Z_r\right|^4 \prod_{r=1}^{N}\left(\left|\alpha_r\right|^{-2} e^{-2\,\mathrm{Re}\,\bar{N}_{00}^{rr}}\right)\prod_{I=1}^{N-2}\left|\partial^2\rho(z_I)\right|^{-1} . \quad (2.27)$$

Here $\bar{N}_{00}^{rr}$ is a Neumann coefficient given by

$$\bar{N}_{00}^{rr} = \frac{\rho(z_{I^{(r)}})}{\alpha_r} - \sum_{s\neq r}\frac{\alpha_s}{\alpha_r}\ln(Z_r - Z_s) . \quad (2.28)$$

It is easy to calculate the free field correlation function on the right hand side, and we find that this also becomes $\exp\left(-\frac{d-26}{24}\Gamma\right)$ up to a constant multiplicative factor, using the identity,

$$\frac{\prod_{r>s}\left|Z_r - Z_s\right|^2 \prod_{I>J}\left|z_I - z_J\right|^2}{\prod_{r=1}^{N}\prod_I \left|Z_r - z_I\right|^2} = \left|\sum_{r=1}^{N}\alpha_r Z_r\right|^2 \frac{\prod_I \left|\partial^2\rho\left(z_I\right)\right|}{\prod_{r=1}^{N}\left|\alpha_r\right|} . \quad (2.29)$$

Thus eq.(2.26) holds.

Let us then consider the next simplest case where all the ϕ_i do not include derivatives of X^-. In this case, X^+ in the correlation function can be replaced by its expectation value $-\frac{i}{2}\left(\rho+\bar{\rho}\right)$. Therefore the problem is reduced to the case where ϕ_i are made of ghost fields. Since the operator insertions on the right hand side was fixed so that the ghost variables $b, c, \tilde{b}, \tilde{c}$ have the same singularity structure as the quantity on the left hand side, it is easy to see

$$\left\langle \mathcal{R} b\left(z_1\right)\cdots b\left(z_n\right) c\left(w_1\right)\cdots c\left(w_n\right)\tilde{b}\left(\bar{u}_1\right)\cdots\tilde{b}\left(\bar{u}_m\right)\tilde{c}\left(\bar{v}_1\right)\cdots\tilde{c}\left(\bar{v}_m\right)\prod_I \mathcal{O}_I \right.$$
$$\left. \times \prod_{r=1}^{N}\left[\mathcal{S}_r\left|\alpha_r\right|^{-3\alpha}\left|e^{\alpha\sigma'}(Z_r)\right|^2 e^{-ip_r^+ X'^- + \frac{d-26}{24}\frac{i}{p_r^+}X^+}\left(Z_r,\bar{Z}_r\right)\right]\right\rangle_{\text{free}}$$
$$\propto \left\langle \left|e^{3\sigma}\left(\infty\right)\right|^2 b\left(z_1\right)\cdots b\left(z_n\right) c\left(w_1\right)\cdots c\left(w_n\right)\tilde{b}\left(\bar{u}_1\right)\cdots\tilde{b}\left(\bar{u}_m\right)\tilde{c}\left(\bar{v}_1\right)\cdots\tilde{c}\left(\bar{v}_m\right)\right\rangle_{b,c}$$
$$\times \left\langle \mathcal{R}\prod_I \mathcal{O}_I \prod_{r=1}^{N}\left[\mathcal{S}_r\left|\alpha_r\right|^{-3\alpha}\left|e^{\alpha\sigma'}(Z_r)\right|^2 e^{-ip_r^+ X'^- + \frac{d-26}{24}\frac{i}{p_r^+}X^+}\left(Z_r,\bar{Z}_r\right)\right]\right\rangle_{\text{free}} . \quad (2.30)$$

On the other hand, we have

$$\begin{aligned}
&\left\langle \left|e^{3\sigma}(\infty)\right|^2 b(z_1)\cdots b(z_n)\,c(w_1)\cdots c(w_n)\right.\\
&\qquad\left.\times\tilde{b}(\bar{u}_1)\cdots\tilde{b}(\bar{u}_m)\,\tilde{c}(\bar{v}_1)\cdots\tilde{c}(\bar{v}_m)\prod_{r=1}^{N}e^{-ip_r^+X^-}\left(Z_r,\bar{Z}_r\right)\right\rangle_{X^\pm,b,c}\\
&\propto\left\langle \left|e^{3\sigma}(\infty)\right|^2 b(z_1)\cdots b(z_n)\,c(w_1)\cdots c(w_n)\,\tilde{b}(\bar{u}_1)\cdots\tilde{b}(\bar{u}_m)\,\tilde{c}(\bar{v}_1)\cdots\tilde{c}(\bar{v}_m)\right\rangle_{b,c}\\
&\qquad\times\left\langle \left|e^{3\sigma}(\infty)\right|^2\prod_{r=1}^{N}e^{-ip_r^+X^-}\left(Z_r,\bar{Z}_r\right)\right\rangle_{X^\pm,b,c}. \qquad (2.31)
\end{aligned}$$

Using eq.(2.26) we can show that these two are proportional to each other.

X^- insertions

Now let us turn to the cases where derivatives of X^- are included in ϕ_i. Once eq.(2.25) is proved for ϕ_i made of the derivatives of X^+ and the ghosts, one can get the free field expression of the correlation functions with X^- insertions by differentiating eq.(2.25) with respect to p_r^+ [7]. As an example, let us consider the correlation function with one insertion of ∂X^-

$$\left\langle \left|e^{3\sigma}(\infty)\right|^2\partial X^-(z)\prod_{r=1}^{N}e^{-ip_r^+X^-}\left(Z_r,\bar{Z}_r\right)\right\rangle_{X^\pm,b,c}. \qquad (2.32)$$

It can be expressed in terms of the one with no insertions. One can show

$$\begin{aligned}
&\left\langle \left|e^{3\sigma}(\infty)\right|^2\partial X^-(Z_0)\prod_{r=1}^{N}\left[\left|\alpha_r\right|^{\frac{d-26}{8}}e^{-ip_r^+X^-}\left(Z_r,\bar{Z}_r\right)\right]\right\rangle_{X^\pm,b,c}\\
&\qquad\propto 2i\partial_{Z_0}\partial_{\alpha_0}\left.\left\langle \left|e^{3\sigma}(\infty)\right|^2\prod_{r=0}^{N+1}\left[\left|\alpha_r\right|^{\frac{d-26}{8}}e^{-ip_r^+X^-}\left(Z_r,\bar{Z}_r\right)\right]\right\rangle_{X^\pm,b,c}\right|_{\alpha_0=0}. \qquad (2.33)
\end{aligned}$$

Here the factors $|\alpha_r|^{\frac{d-26}{8}}$ are included so that the limit $\alpha_0\to 0$ becomes smooth. On the right hand side, we have inserted sources $e^{-ip_0^+X^-}(Z_0,\bar{Z}_0)$ and $e^{-ip_{N+1}^+X^-}(Z_{N+1},\bar{Z}_{N+1})$ with $p_{N+1}^+=-p_0^+=-\frac{1}{2}\alpha_0$, to generate the insertions of X^-. With these sources, X^+ has the expectation value $-\frac{i}{2}\left(\hat{\rho}(z)+\bar{\hat{\rho}}(\bar{z})\right)$, where $\hat{\rho}(z)\equiv\sum_{r=0}^{N+1}\alpha_r\ln(z-Z_r)$. $\hat{\rho}(z)$ has N interaction points. In the limit $\alpha_0\to 0$, two of them, which we denote by $\hat{z}_{I^{(0)}}$ and $\hat{z}_{I^{(N+1)}}$, tend to Z_0 and Z_{N+1}, and the other $\hat{z}_I$'s go to the interaction points z_I of $\rho(z)$, which are denoted with the same subscripts [7].

Now let us rewrite the expression on the right hand side using the free variables. By making use of eq.(2.26) with ρ replaced by $\hat{\rho}$, we have

$$
\begin{aligned}
&2i\partial_{Z_0}\partial_{\alpha_0}\left\langle \left|e^{3\sigma}(\infty)\right|^2 \prod_{r=0}^{N+1}\left[\left|\alpha_r\right|^{\frac{d-26}{8}} e^{-ip_r^+X^-}\left(Z_r,\bar{Z}_r\right)\right]\right\rangle_{X^\pm,b,c}\Bigg|_{\alpha_0=0} \\
&= 2i\partial_{Z_0}\partial_{\alpha_0}\Bigg\langle \mathcal{R}\prod_{I=1}^{N-2}\hat{\mathcal{O}}_I\hat{\mathcal{O}}_{I^{(0)}}\hat{\mathcal{O}}_{I^{(N+1)}} \\
&\qquad \times\prod_{r=0}^{N+1}\left[\hat{\mathcal{S}}_r\left|\alpha_r\right|^{\frac{3}{2}\alpha(\alpha+1)}\left|e^{\alpha\sigma'}(Z_r)\right|^2 e^{-ip_r^+X'^-+\frac{d-26}{24}\frac{i}{p_r^+}X^+}\left(Z_r,\bar{Z}_r\right)\right]\Bigg\rangle_{\mathrm{free}}\Bigg|_{\alpha_0=0}
\end{aligned}
\quad ,(2.34)
$$

where $\hat{\mathcal{O}}_I$ and $\hat{\mathcal{S}}_r$ are respectively $\mathcal{O}_I$ and $\mathcal{S}_r$ with z_I being replaced by $\hat{z}_I$. Since $\hat{z}_{I^{(0)}} \to Z_0$ and $\hat{z}_{I^{(N+1)}} \to Z_{N+1}$ in the limit $\alpha_0 \to 0$ which we should eventually take, we get some operator insertions at $z = Z_0$ and $z = Z_{N+1}$ as a result. These insertions should correspond to $X^-(Z_0) - X^-(Z_{N+1})$. The behaviors of $\hat{z}_{I^{(0)}} - Z_0$, $\mathrm{Re}\,\hat{\bar{N}}^{00}_{00}$ and $\partial^2\hat{\rho}\left(\hat{z}_{I^{(0)}}\right)$ in the limit $\alpha_0 \to 0$ are given by eqs.(D.1), (D.2) and (D.4) of Ref. [7] respectively, where $\hat{\bar{N}}^{00}_{00}$ is a Neumann coefficient corresponding to $\hat{\rho}(z)$. In the free field correlation function on the right hand side of eq.(2.34), X'^-'s appear only in the form of the vertex operator $e^{-ip^+X'^-}$. Therefore one can replace X^+ by its expectation value and vice versa. Using these facts, we can show that in the limit $\alpha_0 \to 0$,

$$
\begin{aligned}
&\hat{\mathcal{O}}_{I^{(0)}}\hat{\mathcal{S}}_0\left|\alpha_0\right|^{\frac{3}{2}\alpha(\alpha+1)}\left|e^{\alpha\sigma'}(Z_0)\right|^2 e^{-ip_0^+X'^-+\frac{d-26}{24}\frac{i}{p_0^+}X^+}\left(Z_0,\bar{Z}_0\right) \\
&\sim\ \left|\alpha_0\right|^{\frac{3}{2}\alpha(\alpha+1)}\left|\partial^2\hat{\rho}\left(\hat{z}_{I^{(0)}}\right)\right|^{-\frac{3}{2}\alpha(\alpha+1)}\left|e^{-\alpha\sigma'}\left(\hat{z}_{I^{(0)}}\right)\right|^2 \\
&\qquad \times\left|e^{\alpha\sigma'}(Z_0)\right|^2 e^{-ip_0^+X'^-}\left(Z_0,\bar{Z}_0\right)e^{-\frac{d-26}{12}\mathrm{Re}\,\hat{\bar{N}}^{00}_{00}} \\
&\sim\ 1-ip_0^+X'^-\left(Z_0,\bar{Z}_0\right)+\alpha_0\alpha\left(\frac{\partial\sigma'}{\partial\rho}(Z_0)+\frac{\bar{\partial}\tilde{\sigma}'}{\bar{\partial}\bar{\rho}}\left(\bar{Z}_0\right)\right)+\left(2\alpha^2+3\alpha\right)\mathrm{Re}\,\frac{\partial^2\rho}{(\partial\rho)^2}(Z_0) \\
&\sim\ 1-\frac{i}{2}\alpha_0\left[X'^-+\alpha\left(\frac{\partial\sigma'}{\partial X^+}+\frac{\bar{\partial}\tilde{\sigma}'}{\bar{\partial}X^+}\right)+\left(\alpha^2+\frac{3}{2}\alpha\right)\left(\frac{\partial^2X^+}{(\partial X^+)^2}+\frac{\bar{\partial}^2X^+}{\left(\bar{\partial}X^+\right)^2}\right)\right]\left(Z_0,\bar{Z}_0\right) \\
&=\ 1-\frac{i}{2}\alpha_0X^-\left(Z_0,\bar{Z}_0\right)\ ,
\end{aligned}
\qquad (2.35)
$$

and similarly

$$
\begin{aligned}
&\hat{\mathcal{O}}_{I^{(N+1)}}\hat{\mathcal{S}}_{N+1}\left|\alpha_0\right|^{\frac{3}{2}\alpha(\alpha+1)}\left|e^{\alpha\sigma'}(Z_{N+1})\right|^2 e^{ip_0^+X'^--\frac{d-26}{24}\frac{i}{p_0^+}X^+}\left(Z_{N+1},\bar{Z}_{N+1}\right) \\
&\qquad \sim 1+\frac{i}{2}\alpha_0X^-\left(Z_{N+1},\bar{Z}_{N+1}\right)\ .
\end{aligned}
\qquad (2.36)
$$

Substituting eqs.(2.35) and (2.36) into eq.(2.34), we obtain[4]

$$\begin{aligned} & \left\langle \left|e^{3\sigma}(\infty)\right|^2 \partial X^-(Z_0) \prod_{r=1}^{N} e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right) \right\rangle_{X^\pm b,c} && (2.37) \\ & \propto \left\langle \mathcal{R}\partial X^-(Z_0) \prod_{I=1}^{N-2} \mathcal{O}_I \prod_{r=1}^{N} \left[\mathcal{S}_r \left|\alpha_r\right|^{-3\alpha} \left|e^{\alpha\sigma'}(Z_r)\right|^2 e^{-ip_r^+ X'^- + \frac{d-26}{24}\frac{i}{p_r^+}X^+}\left(Z_r, \bar{Z}_r\right)\right] \right\rangle_{\text{free}} . \end{aligned}$$

It is straightforward to prove eq.(2.25) for more general insertions. The key relation is eq.(2.35), which is valid in the free field correlation functions in which X'^-'s appear only in the form of the vertex operator $e^{-ip^+X'^-}$. For example, let us consider the correlation functions with two insertions of ∂X^-

$$\begin{aligned} & \left\langle \left|e^{3\sigma}(\infty)\right|^2 \partial X^-(z)\, \partial X^-(Z_0) \prod_{r=1}^{N} \left[\left|\alpha_r\right|^{\frac{d-26}{8}} e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right)\right] \right\rangle_{X^\pm,b,c} \\ & \quad \propto 2i\partial_{Z_0}\partial_{\alpha_0} \left\langle \left|e^{3\sigma}(\infty)\right|^2 \partial X^-(z) \prod_{r=0}^{N+1} \left[\left|\alpha_r\right|^{\frac{d-26}{8}} e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right)\right] \right\rangle_{X^\pm,b,c} \Bigg|_{\alpha_0=0} . \qquad (2.38) \end{aligned}$$

Using eq.(2.37), the right hand side is expressed as

$$\begin{aligned} & 2i\partial_{Z_0}\partial_{\alpha_0} \Bigg\langle \mathcal{R} \prod_{I=1}^{N-2} \hat{\mathcal{O}}_I \hat{\mathcal{O}}_{I^{(0)}} \hat{\mathcal{O}}_{I^{(N+1)}} \partial X^-(z) \\ & \quad \times \prod_{r=0}^{N+1} \left[\hat{\mathcal{S}}_r \left|\alpha_r\right|^{\frac{3}{2}\alpha(\alpha+1)} \left|e^{\alpha\sigma'}(Z_r)\right|^2 e^{-ip_r^+ X'^- + \frac{d-26}{24}\frac{i}{p_r^+}X^+}\left(Z_r, \bar{Z}_r\right)\right] \Bigg\rangle_{\text{free}} \Bigg|_{\alpha_0=0} . \qquad (2.39) \end{aligned}$$

Here we would like to use eq.(2.35) to deal with the limit $\alpha_0 \to 0$. In this form, eq.(2.35) does not hold apparently, because of the presence of $\partial X^-(z)$. However, $\partial X^-(z)$ can be rewritten as

$$\begin{aligned} \partial X^-(z) &= \partial X'^-(z) + \cdots \\ &= i\partial_z \partial_{p^+} e^{-ip^+X'^-}(z)\Big|_{p^+=0} + \cdots , \qquad (2.40) \end{aligned}$$

where $\cdots$ denotes the quantities which does not involve X'^-. Substituting this into eq.(2.39), we can make it into the form where X'^- is exponentiated so that eq.(2.35) holds. Thus we

[4]Here (and in eqs.(2.25)(2.41)) z_I which appears in the definitions of $\mathcal{O}_I, \mathcal{S}_r$ are taken to be the interaction point which correspond to the Mandelstam mapping $\rho(z) = \sum_{r=1}^{N} \alpha_r \ln(z - Z_r)$.

can show

$$
\begin{aligned}
&\left\langle \left|e^{3\sigma}(\infty)\right|^2 \partial X^-(z)\,\partial X^-(Z_0) \prod_{r=1}^{N} e^{-ip_r^+ X^-}\left(Z_r, \bar{Z}_r\right) \right\rangle_{X^\pm b,c} \\
&\propto \left\langle \mathcal{R}\partial X^-(z)\,\partial X^-(Z_0) \prod_{I=1}^{N-2} \mathcal{O}_I \right. \\
&\qquad \left. \times \prod_{r=1}^{N} \left[\mathcal{S}_r \left|\alpha_r\right|^{-3\alpha} \left|e^{\alpha\sigma'}(Z_r)\right|^2 e^{-ip_r^+ X'^- + \frac{d-26}{24}\frac{i}{p_r^+}X^+}\left(Z_r, \bar{Z}_r\right) \right] \right\rangle_{\mathrm{free}} . \qquad (2.41)
\end{aligned}
$$

Proceeding in this way, we can show that eq.(2.25) holds for general ϕ_i and the correlation functions can be expressed by using the free variables.

From eq.(2.25) one can see that $\mathcal{O}_I$, $\mathcal{R}$, $\mathcal{S}_r$ are conformal invariant, which has been proved in the previous subsection assuming that X^+ can be replaced by its expectation value. Indeed, when one of ϕ_i is the energy-momentum tensor $T(z)$, the left hand side is not singular in the limit $z \to z_I, \infty$. On the right hand side, this fact implies that $\mathcal{O}_I$, $\mathcal{R}$, $\mathcal{S}_r$ are conformal invariant.

3 Free variables: supersymmetric case

Supersymmetric case can be dealt with in a similar way, using the superspace formulation. In this section, we denote some superfields by using the same symbols as those in the bosonic case. We think that this does not cause any confusion.

3.1 Supersymmetric $X^\pm$ CFT

Supersymmetric $X^\pm$ CFT can be defined by using the superspace formalism. It is described by the superfield variables

$$
\mathcal{X}^\pm(\mathbf{z}, \bar{\mathbf{z}}) \equiv X^\pm(z, \bar{z}) + i\theta\psi^\pm(z) + i\theta\tilde{\psi}^\pm(\bar{z}) + i\theta\bar{\theta}F^\pm(z, \bar{z}) \ , \qquad (3.1)
$$

where $\mathbf{z} = (z, \theta)$ is the superspace coordinate. The energy-momentum tensor is given as [8]

$$
\frac{1}{2}D\mathcal{X}^+\partial\mathcal{X}^- + \frac{1}{2}D\mathcal{X}^-\partial\mathcal{X}^+ - \frac{d-10}{4}S(\mathbf{z}, \mathbf{X}_L^+) \ . \qquad (3.2)
$$

Here $S(\mathbf{z}, \mathbf{X}_L^+)$ is the super Schwarzian derivative,

$$
S(\mathbf{z}, \mathbf{X}_L^+) \equiv \frac{D^4\Theta^+}{D\Theta^+} - 2\frac{D^3\Theta^+ D^2\Theta^+}{(D\Theta^+)^2} = -\frac{1}{4}D\Phi\partial\Phi + \frac{1}{2}\partial D\Phi \ . \qquad (3.3)
$$

The superspace coordinate $\mathbf{X}_L^+$ is defined as $\mathbf{X}_L^+ \equiv \left(\mathcal{X}_L^+, \Theta^+\right)$, where $\mathcal{X}_L^+$ denotes the left-moving part of $\mathcal{X}^+$ and

$$\Theta^+(\mathbf{z}) \equiv \frac{D\mathcal{X}^+}{(\partial\mathcal{X}^+)^{\frac{1}{2}}}(\mathbf{z}) \ , \qquad \Phi\left(\mathbf{z}, \bar{\mathbf{z}}\right) \equiv \ln\left(-4\left(D\Theta^+\right)^2\left(\bar{D}\tilde{\Theta}^+\right)^2\right) \ . \tag{3.4}$$

Similarly to the bosonic case, we consider the correlation functions with the insertion $\prod_{r=1}^N e^{-p_r^+\mathcal{X}^-}\left(\mathbf{Z}_r, \bar{\mathbf{Z}}_r\right)$. With this insertion, $\mathcal{X}^+\left(\mathbf{z}, \bar{\mathbf{z}}\right)$ has an expectation value $-\frac{i}{2}\left(\rho\left(\mathbf{z}\right) + \bar{\rho}\left(\mathbf{z}\right)\right)$ where $\rho\left(\mathbf{z}\right) \equiv \sum_{r=1}^N \alpha_r \ln\left(\mathbf{z} - \mathbf{Z}_r\right)$ is the super Mandelstam mapping [8].

The variables $\mathcal{X}^\pm$ satisfy the OPE's

$$\begin{aligned}
D\mathcal{X}^+\left(\mathbf{z}\right) D\mathcal{X}^+\left(\mathbf{z}'\right) &\sim \text{regular} \ , \\
D\mathcal{X}^-\left(\mathbf{z}\right) D\mathcal{X}^+\left(\mathbf{z}'\right) &\sim \frac{1}{\mathbf{z} - \mathbf{z}'} \ , \\
D\mathcal{X}^-\left(\mathbf{z}\right) D\mathcal{X}^-\left(\mathbf{z}'\right) &\sim -\frac{d-10}{4} DD' \left[\frac{3\Theta^+\left(\mathbf{z}\right)\Theta^+\left(\mathbf{z}'\right)}{\left(\mathcal{X}_L^+\left(\mathbf{z}\right) - \mathcal{X}_L^+\left(\mathbf{z}'\right) - \Theta^+\left(\mathbf{z}\right)\Theta^+\left(\mathbf{z}'\right)\right)^3}\right. \\
&\qquad \left. + \frac{1}{2\left(\mathcal{X}_L^+\left(\mathbf{z}\right) - \mathcal{X}_L^+\left(\mathbf{z}'\right) - \Theta^+\left(\mathbf{z}\right)\Theta^+\left(\mathbf{z}'\right)\right)^2}\right] \ .
\end{aligned} \tag{3.5}$$

The right hand side of the third equation should be treated as in the bosonic case and we get the form of the OPE in Ref. [8]. Using these OPE's, one can show that the energy-momentum tensor in eq.(3.2) satisfies the super Virasoro algebra with $\hat{c} = 12 - d$. It follows that together with the super-reparametrization ghosts and the transverse part, the total central charge becomes 0.

3.2 Free fields

As in the bosonic case, we consider the system which consists of the supersymmetric $X^\pm$ CFT and the super-reparametrization ghosts. The ghost variables are described by the superfields $B\left(\mathbf{z}\right)$, $C\left(\mathbf{z}\right)$, $\tilde{B}\left(\bar{\mathbf{z}}\right)$, $\tilde{C}\left(\mathbf{z}\right)$, which are given as

$$\begin{aligned}
B\left(\mathbf{z}\right) &\equiv \beta\left(z\right) + \theta b\left(z\right) \ , \qquad & \tilde{B}\left(\bar{\mathbf{z}}\right) &\equiv \tilde{\beta}\left(\bar{z}\right) + \bar{\theta}\tilde{b}\left(\bar{z}\right) \ , \\
C\left(\mathbf{z}\right) &\equiv c\left(z\right) + \theta\gamma\left(z\right) \ , \qquad & \tilde{C}\left(\bar{\mathbf{z}}\right) &\equiv \tilde{c}\left(\bar{z}\right) + \bar{\theta}\tilde{\gamma}\left(\bar{z}\right) \ .
\end{aligned} \tag{3.6}$$

The free superfields $\mathcal{X}^+$, $\mathcal{X}'^-$ and the ghosts B', C', $\tilde{B}'$, $\tilde{C}'$ with weights $\left(\frac{3}{2} + \alpha, 0\right)$, $\left(-1 - \alpha, 0\right)$, $\left(0, \frac{3}{2} + \alpha\right)$, $\left(0, -1 - \alpha\right)$ can be defined as

$$B'\left(\mathbf{z}\right) \equiv \left(D\Theta^+\right)^{2\alpha} B\left(\mathbf{z}\right) \ , \qquad \tilde{B}'\left(\bar{\mathbf{z}}\right) \equiv \left(\bar{D}\tilde{\Theta}^+\right)^{2\alpha} \tilde{B}\left(\bar{\mathbf{z}}\right) \ ,$$

$$\begin{aligned}
&C'(\mathbf{z}) \equiv \left(D\Theta^{+}\right)^{-2\alpha} C(\mathbf{z}) \ , \qquad \tilde{C}'(\bar{\mathbf{z}}) \equiv \left(\bar{D}\tilde{\Theta}^{+}\right)^{-2\alpha} \tilde{C}(\bar{\mathbf{z}}) \ , \\
&\mathcal{X}'^{-}(\mathbf{z},\bar{\mathbf{z}}) \equiv \mathcal{X}^{-}(\mathbf{z},\bar{\mathbf{z}}) \\
&\qquad + \alpha \left[\partial D \left(\Sigma' + \frac{1}{2}\Phi \right) \frac{\Theta^{+}}{\left(D\Theta^{+}\right)^{3}} - \partial \left(\Sigma' + \frac{1}{2}\Phi \right) \left(\frac{1}{\left(D\Theta^{+}\right)^{2}} + \frac{\partial\Theta^{+}\Theta^{+}}{\left(D\Theta^{+}\right)^{4}} \right) \right. \\
&\qquad\qquad - D \left(\Sigma' + \frac{1}{2}\Phi \right) \left(\frac{\partial\Theta^{+}}{\left(D\Theta^{+}\right)^{3}} + \frac{\partial D\Theta^{+}\Theta^{+}}{\left(D\Theta^{+}\right)^{4}} \right) \\
&\qquad\qquad \left. + \, \text{c.c.} \right] \ .
\end{aligned} \tag{3.7}$$

Here

$$\alpha = \frac{d-10}{8} \ , \tag{3.8}$$

and

$$\Sigma'(\mathbf{z}) \equiv \sigma'(z) - \phi'(z) - \theta\beta' c'(z) \ , \tag{3.9}$$

where σ' and ϕ' are defined so that $\partial\sigma' = c'b'$ and

$$\beta'(z) = e^{-\phi'}\partial\xi'(z) \ , \qquad \gamma'(z) = \eta' e^{\phi'}(z) \ . \tag{3.10}$$

We note that

$$CB(\mathbf{z}) = C'B'(\mathbf{z}) = -D\Sigma'(\mathbf{z}) \ . \tag{3.11}$$

The OPE's between $\mathcal{X}^{+}$, $\mathcal{X}'^{-}$, B', C', $\tilde{B}'$, $\tilde{C}'$ can be derived from the OPE's of $\mathcal{X}^{\pm}$, B, C, $\tilde{B}$, $\tilde{C}$ and one can see that they are free variables.

The total energy-momentum tensor,

$$\begin{aligned}
T(\mathbf{z}) \ &= \ \frac{1}{2}D\mathcal{X}^{+}\partial\mathcal{X}^{-} + \frac{1}{2}D\mathcal{X}^{-}\partial\mathcal{X}^{+} - \frac{d-10}{4}S\left(\mathbf{z}, \ \boldsymbol{\mathcal{X}}_{L}^{+}\right) \\
&\quad + \frac{1}{2}DCDB - \frac{3}{2}\partial CB - C\partial B \ ,
\end{aligned} \tag{3.12}$$

can be rewritten in terms of the free fields as

$$\begin{aligned}
T(\mathbf{z}) \ &= \ \frac{1}{2}D\mathcal{X}^{+}\partial\mathcal{X}'^{-} + \frac{1}{2}D\mathcal{X}'^{-}\partial\mathcal{X}^{+} \\
&\quad + \frac{1}{2}DC'DB' - \frac{1}{2}\partial C'B' - (1+\alpha)\,\partial\left(C'B'\right) \ ,
\end{aligned} \tag{3.13}$$

which is the energy-momentum tensor for the free fields $\mathcal{X}^{+}$, $\mathcal{X}'^{-}$, B', C'. It is also possible to express $\mathcal{X}^{+}$, $\mathcal{X}^{-}$, B, C, $\tilde{B}$, $\tilde{C}$ in terms of the free variables.

3.3 Operator insertions

Let us define the correlation functions on the complex plane $\langle \phi_1 \cdots \phi_n \rangle_{\mathcal{X}^\pm,B,C}$ and $\langle \phi_1 \cdots \phi_n \rangle_{\rm free}$ as in the bosonic case. The correlation functions that we are interested in are of the form

$$\left\langle \left| e^{3\sigma-2\phi}(\infty) \right|^2 \phi_1\phi_2\cdots\phi_n \prod_{r=1}^{N} e^{-ip_r^+\mathcal{X}^-} \left(\mathbf{Z}_r, \bar{\mathbf{Z}}_r\right) \right\rangle_{\mathcal{X}^\pm,B,C} . \tag{3.14}$$

The ghosts are bosonized in the usual way and $\left| e^{3\sigma-2\phi}\left(\infty\right) \right|^2$ is inserted to soak up the ghost zero modes.

As in the bosonic case, the correlation functions of the superconformal field theory for $\mathcal{X}^\pm$, B, C, $\tilde{B}$, $\tilde{C}$ can be expressed as the correlation functions of the free field theory with operator insertions at $\mathbf{z} = \tilde{\mathbf{z}}_I$, $\mathbf{Z}_r$ and ∞. Here $\tilde{\mathbf{z}}_I$ $(I = 1, \ldots, N-2)$ denote the points determined by $\partial\rho(\tilde{\mathbf{z}}_I) = \partial D\rho(\tilde{\mathbf{z}}_I) = 0$ [13, 14, 15].

Let us consider the operator which should be inserted at $\mathbf{z} = \tilde{\mathbf{z}}_I$ to realize the singular behaviors of the ghost fields at this point. It is a little bit complicated, compared with the bosonic case, but straightforward to show that in order for the variables $B\left(\mathbf{z}\right)$ and $C\left(\mathbf{z}\right)$ to be regular at $\mathbf{z} = \tilde{\mathbf{z}}_I$, we need to insert

$$\left[1 - \alpha\frac{D\rho}{\partial^2\rho}\partial D\Sigma' - \alpha\frac{\partial^2 D\rho D\rho}{\left(\partial^2\rho\right)^2}\partial\Sigma'\right] e^{-\alpha\Sigma'}\left(\tilde{\mathbf{z}}_I\right) . \tag{3.15}$$

We should make up a superconformal invariant operator insertions whose ghost part is eq.(3.15). The form of such insertions can be read off from the partition function. The partition function of the $X^\pm$ CFT with the insertion $\prod_{r=1}^{N} e^{-ip_r^+\mathcal{X}^-}(\mathbf{Z}_r, \bar{\mathbf{Z}}_r)$ becomes $e^{-\alpha\Gamma_{\rm super}}$ [8, 13, 14], where

$$e^{-\Gamma_{\rm super}} = |A|^2 \prod_{I=1}^{N-2} \left| \left(\partial^2\rho - \frac{5}{3}\frac{\partial^3 D\rho D\rho}{\partial^2\rho} + 3\frac{\partial^3\rho\partial^2 D\rho D\rho}{\left(\partial^2\rho\right)^2}\right)\left(\tilde{\mathbf{z}}_I\right) \right|^{-\frac{1}{2}} \prod_{r=1}^{N} \left(\left|\alpha_r\right|^{-1} e^{-\mathrm{Re}\,\bar{N}_{00}^{rr}} \right) , \tag{3.16}$$

with

$$\begin{aligned} A &\equiv \sum_r \alpha_r Z_r - \frac{\sum_r \alpha_r\Theta_r \sum_r \alpha_r\Theta_r Z_r}{\sum_r \alpha_r Z_r} , \\ \bar{N}_{00}^{rr} &\equiv \frac{\rho(\tilde{\mathbf{z}}_{I^{(r)}})}{\alpha_r} - \sum_{s\neq r}\frac{\alpha_s}{\alpha_r}\ln\left(\mathbf{Z}_r - \mathbf{Z}_s\right) , \end{aligned} \tag{3.17}$$

and $\tilde{\mathbf{z}}_{I^{(r)}}$ denotes one of $\tilde{\mathbf{z}}_I$'s such that at $\rho = \rho(\tilde{\mathbf{z}}_{I^{(r)}})$ the rth string interacts. From this, one can see that the insertion (3.15) should come with

$$\left(\partial^2\rho - \frac{5}{3}\frac{\partial^3 D\rho D\rho}{\partial^2\rho} + 3\frac{\partial^3\rho\partial^2 D\rho D\rho}{\left(\partial^2\rho\right)^2}\right)^{-\frac{\alpha}{4}}\left(\tilde{\mathbf{z}}_I\right) . \tag{3.18}$$

Indeed, as we will see later, with the factor so arranged, one can obtain eq.(3.26). Therefore we define the following operator

$$\begin{aligned}
\mathcal{O}_I \equiv & \left| \oint_{\tilde{\mathbf{z}}_I} \frac{d\mathbf{z}}{2\pi i} D\Phi \left[1 + \frac{\alpha}{12} \frac{\partial^3 D\mathcal{X}^+ D\mathcal{X}^+}{\left(\partial^2 \mathcal{X}^+\right)^2} + \alpha \left(\frac{\alpha}{32} - \frac{1}{8} \right) \frac{\partial^3 \mathcal{X}^+ \partial^2 D\mathcal{X}^+ D\mathcal{X}^+}{\left(\partial^2 \mathcal{X}^+\right)^3} \right. \right. \\
& - \frac{\alpha^2}{8} \frac{\partial^3 \mathcal{X}^+ D\mathcal{X}^+}{\left(\partial^2 \mathcal{X}^+\right)^2} D\Sigma' + \frac{\alpha^2}{8} \frac{\partial^2 D\mathcal{X}^+ D\mathcal{X}^+}{\left(\partial^2 \mathcal{X}^+\right)^2} \partial\Sigma' \\
& \left. \left. - \frac{\alpha^2}{2} \frac{D\mathcal{X}^+}{\partial^2 \mathcal{X}^+} \partial\Sigma' D\Sigma' \right] \frac{e^{-\alpha\Sigma'}}{\left(\partial^2 \mathcal{X}^+\right)^{\frac{\alpha}{4}}} \right|^2 . \qquad (3.19)
\end{aligned}$$

Replacing $\mathcal{X}^+$ by its expectation value, one can see that $\mathcal{O}_I$ is equivalent to the insertion of

$$\left| \frac{\left[1 - \alpha \frac{D\rho}{\partial^2 \rho} \partial D\Sigma' - \alpha \frac{\partial^2 D\rho D\rho}{\left(\partial^2\rho\right)^2} \partial\Sigma' \right] e^{-\alpha\Sigma'}}{\left(\partial^2 \rho - \frac{5}{3} \frac{\partial^3 D\rho D\rho}{\partial^2 \rho} + 3 \frac{\partial^3 \rho \partial^2 D\rho D\rho}{\left(\partial^2\rho\right)^2} \right)^{\frac{\alpha}{4}}} \left(\tilde{\mathbf{z}}_I\right) \right|^2 . \qquad (3.20)$$

The ghost operator that should be inserted at $z = \infty$ can be seen to be $e^{3\sigma' - 2\phi'}$, whose superspace form is

$$\left(1 + \theta\gamma b\right) e^{3\sigma' - 2\phi'} (z) . \qquad (3.21)$$

The conformal invariant combination which implements such an insertion is given as

$$\mathcal{R} \equiv \left| \oint_\infty \frac{d\mathbf{z}}{2\pi i} D\Phi \left(D\Theta^+\right)^{2\alpha} \left(1 + \theta\gamma b\right) e^{3\sigma' - 2\phi'} (z) \right|^2 . \qquad (3.22)$$

At $\mathbf{z} = \mathbf{Z}_r$, one can see that $e^{\alpha\Sigma'}\left(\mathbf{Z}_r\right)$ should be inserted.

The logarithmic singularities for $\mathcal{X}^-$ can be taken care of by inserting

$$\exp\left(\frac{d-10}{16} \frac{i}{p_r^+} \mathcal{X}^+ \right) \left(\mathbf{Z}_r, \bar{\mathbf{Z}}_r\right) , \qquad \exp\left(-\frac{d-10}{16} \frac{i}{p_r^+} \mathcal{X}^+ \right) \left(\tilde{\mathbf{z}}_{I^{(r)}}, \bar{\tilde{\mathbf{z}}}_{I^{(r)}}\right) . \qquad (3.23)$$

The latter should be made into the conformal invariant combination

$$\mathcal{S}_r \equiv \oint_{\tilde{\mathbf{z}}_{I^{(r)}}} \frac{d\mathbf{z}}{2\pi i} D\Phi \oint_{\bar{\tilde{\mathbf{z}}}_{I^{(r)}}} \frac{d\bar{\mathbf{z}}}{2\pi i} \bar{D}\tilde{\Phi} \exp\left(-\frac{d-10}{16} \frac{i}{p_r^+} \mathcal{X}^+ \right) \left(\mathbf{z}, \bar{\mathbf{z}}\right) . \qquad (3.24)$$

3.4 Correlation functions in terms of the free variables

The correlation functions can be expressed by using the free variables with the insertions of the operators defined above. One can show the supersymmetric version of eq.(2.25):

$$\left\langle \left| e^{3\sigma - 2\phi} (\infty) \right|^2 \phi_1 \left(\mathbf{z}_1, \bar{\mathbf{z}}_1\right) \phi_2 \left(\mathbf{z}_2, \bar{\mathbf{z}}_2\right) \cdots \phi_n \left(\mathbf{z}_n, \bar{\mathbf{z}}_n\right) \prod_{r=1}^N e^{-ip_r^+ \mathcal{X}^-} \left(\mathbf{Z}_r, \bar{\mathbf{Z}}_r\right) \right\rangle_{\mathcal{X}^\pm, B, C}$$

$$
\begin{aligned}
&= \mathcal{C} \left\langle \mathcal{R}\phi_1\left(\mathbf{z}_1\bar{\mathbf{z}}_1\right)\phi_2\left(\mathbf{z}_2,\bar{\mathbf{z}}_2\right)\cdots\phi_n\left(\mathbf{z}_n,\bar{\mathbf{z}}_n\right)\prod_{I=1}^{N-2}\mathcal{O}_I \right.\\
&\qquad \left. \times \prod_{r=1}^{N}\left[\mathcal{S}_r\left|\alpha_r\right|^{-\alpha}\left|e^{\alpha\Sigma'}(\mathbf{Z}_r)\right|^2 e^{-ip_r^+\mathcal{X}'^- + \frac{d-10}{16}\frac{i}{p_r^+}\mathcal{X}^+}\left(\mathbf{Z}_r,\bar{\mathbf{Z}}_r\right)\right]\right\rangle_{\mathrm{free}} .
\end{aligned}
\tag{3.25}
$$

Here $\phi_i \ (i=1,\cdots,n)$ are made of $\mathcal{X}^+$, $D\mathcal{X}^-$, $\bar{D}\mathcal{X}^-$, B, C, $\tilde{B}$, $\tilde{C}$ and their covariant derivatives and $\mathcal{C}$ is a numerical constant.

This formula can be shown in the same way as eq.(2.25) in the bosonic case. One can first prove the simplest case

$$
\begin{aligned}
&\left\langle \left|e^{3\sigma-2\phi}\left(\infty\right)\right|^2 \prod_{r=1}^{N} e^{-ip_r^+\mathcal{X}^-}\left(\mathbf{Z}_r,\bar{\mathbf{Z}}_r\right)\right\rangle_{\mathcal{X}^\pm,B,C}\\
&\quad \propto \left\langle \mathcal{R}\prod_{I=1}^{N-2}\mathcal{O}_I\prod_{r=1}^{N}\left[\mathcal{S}_r\left|\alpha_r\right|^{-\alpha}\left|e^{\alpha\Sigma'}(\mathbf{Z}_r)\right|^2 e^{-ip_r^+\mathcal{X}'^- + \frac{d-10}{16}\frac{i}{p_r^+}\mathcal{X}^+}\left(\mathbf{Z}_r,\bar{\mathbf{Z}}_r\right)\right]\right\rangle_{\mathrm{free}} .
\end{aligned}
\tag{3.26}
$$

The left hand side was evaluated in Ref. [8] to be $e^{-\alpha\Gamma_{\mathrm{super}}}$ and the right hand side is easily computed to be proportional to it. It is much more complicated but straightforward to prove the supersymmetric versions of eqs.(2.35) and (2.40) and show that eq.(3.25) holds. From eq.(3.25), one can see that $\mathcal{R}, \mathcal{O}_I, \mathcal{S}_r$ given in the previous subsection are conformal invariant.

Using eq.(3.25), one can rewrite arbitrary correlation functions in the $X^\pm$ CFT using the free theory and vice versa. In particular, one can modify the vertex operator at $\mathbf{z} = \mathbf{Z}_r$, by taking the limit $\mathbf{z}_i \to \mathbf{Z}_r$ appropriately. Thus we can get various vertex operators in the $X^\pm$ CFT and their free field versions. Hence, to any vertex operator in the free field description there exists a corresponding vertex operator in the $X^\pm$ CFT, and vice versa.

4 Vertex operators

In the conformal gauge, the light-cone gauge noncritical superstrings can be described by worldsheet theory which consists of the supersymmetric $X^\pm$ CFT, super-reparametrization ghosts and the transverse variables. We have shown that the former two systems combined can be described by free variables in the previous section. Vertex operators are made from these variables and should be BRST invariant. We would like to construct BRST invariant vertex operators in the Ramond sector using the free variables. Before doing so, we examine how the BRST invariant vertex operators in the Neveu-Schwarz sector are expressed in terms of the free variables. We construct the BRST invariant vertex operators in the Ramond sector imitating those in the Neveu-Schwarz sector.

4.1 Vertex operators in the Neveu-Schwarz sector

Let us consider the left-moving part of a state in the Neveu-Schwarz sector of the light-cone gauge superstrings:

$$\alpha^{i_1}_{-n_1}\cdots\psi^{j_1}_{-s_1}\cdots|\vec{p}\rangle_L \ . \tag{4.1}$$

Here n_i are positive integers and s_i are positive half odd integers. $|\vec{p}\rangle_L$ is the state which corresponds to the operator $e^{i\vec{p}\cdot\vec{X}_L}$, where $\vec{p}=(p^i)$ $(i=1,\ldots,d-2)$ denotes the transverse $(d-2)$-momentum and $\vec{X}_L$ denotes the left-moving part of the transverse variables $\vec{X}=(X^i)$.

The left-moving BRST invariant vertex operator in the conformal gauge corresponding to this state is given as [9]

$$\begin{aligned} V^{(-1)}_L(z) \ \equiv\ & A^{i_1}_{-n_1}\cdots B^{j_1}_{-s_1}\cdots \\ & \times e^{\sigma-\phi}\exp\left[-ip^+X^-_L - i\left(p^- - \frac{\mathcal{N}}{p^+}+\frac{d-10}{16}\frac{1}{p^+}\right)X^+_L + i\vec{p}\cdot\vec{X}_L\right](z)\,, \end{aligned} \tag{4.2}$$

where A^i_{-n} and B^i_{-s} are the DDF operators defined as

$$\begin{aligned} A^i_{-n} \ &\equiv\ \oint_z \frac{dz'}{2\pi i}\left(i\partial X^i+\frac{n}{p^+}\psi^i\psi^+\right)e^{-i\frac{n}{p^+}X^+_L}(z')\ , \\ B^i_{-s} \ &\equiv\ \oint_z \frac{dz'}{2\pi i}\left(\psi^i-\partial X^i\frac{\psi^+}{\partial X^+}-\frac{1}{2}\psi^i\frac{\psi^+\partial\psi^+}{(\partial X^+)^2}\right)\left(\frac{i\partial X^+}{p^+}\right)^{\frac{1}{2}}e^{-i\frac{s}{p^+}X^+_L}(z')\ , \end{aligned} \tag{4.3}$$

$\mathcal{N}\equiv\sum_i n_i+\sum_j s_j$, and p^- is taken to satisfy the on-shell condition

$$p^- = \frac{1}{p^+}\left(\frac{1}{2}\vec{p}^2+\mathcal{N}-\frac{d-2}{16}\right)\ . \tag{4.4}$$

The superscript (-1) on the left hand side of eq.(4.2) indicates the picture number. $V^{(-1)}_L$ appears to have the momentum in the $-$ direction shifted as $p^-+\frac{d-10}{16}\frac{1}{p^+}$ instead of p^-. Because of the shift, the vertex operator $V^{(-1)}_L$ becomes of dimension 0 and BRST invariant. In the scattering amplitudes, this shift comes with the insertion of $\exp\left(i\frac{d-10}{16}\frac{1}{p^+}X^+\right)$ at the interaction points [7], and thus the momentum conserved is p^-.

The free field expression $V'^{(-1)}_L$ for $V^{(-1)}_L$ can be read off from eq.(3.25) and

$$\begin{aligned} V'^{(-1)}_L(z) \ =\ & A^{i_1}_{-n_1}\cdots B^{j_1}_{-s_1}\cdots \\ & \times e^{(1+\alpha)(\sigma'-\phi')}\exp\left[-ip^+X'^-_L - i\left(p^- - \frac{\mathcal{N}}{p^+}\right)X^+_L + i\vec{p}\cdot\vec{X}_L\right](z)\,, \end{aligned} \tag{4.5}$$

up to a factor which depends on α_r. Since the variable $\mathcal{X}^+$ is common between the $X^\pm$ CFT and the free theory, the DDF operators A^i_{-n} and B^i_{-s} can be defined in the same way for the both theories. One good feature of $V_L^{\prime(-1)}$ is that unlike $V_L^{(-1)}$ the momentum p^- is not shifted. With the ghost factor $e^{(1+\alpha)(\sigma'-\phi')}$, one can see that the dimension of $V_L^{\prime(-1)}$ is 0 when the on-shell condition (4.4) is satisfied. The insertions of $\mathcal{S}_r$ in the free field description take care of the momentum conservation.

The right-moving vertex operator $V_R^{\prime(-1)}$ can be constructed in the same way.

4.2 Vertex operators in the Ramond sector

Using the free variables, it is easy to construct the spin fields. We can bosonize ψ^+ and ψ'^- as

$$\psi^+(z) = e^{iH'}(z) \ , \qquad \psi'^-(z) = -e^{-iH'}(z) \ , \tag{4.6}$$

with $H'(z)\, H'(w) \sim -\ln(z-w)$. This yields

$$\psi'^-\psi^+(z) = i\partial H'(z) \ . \tag{4.7}$$

Using H', we can construct the spin field for the longitudinal variables as

$$e^{\pm\frac{i}{2}H'} \ . \tag{4.8}$$

The ghost variables b', c', β', γ' are bosonized as in eq.(3.10). The ghost part of the Ramond vertex operators is given by the conformal primary field

$$e^{(1+\alpha)(\sigma'-\phi')\pm\frac{1}{2}\phi'} \tag{4.9}$$

with weight $-\frac{1}{2}\alpha - \frac{5}{8}$.

Eq.(3.25) holds even when some of ϕ_i involve such spin fields. For example, let us consider a pair of spin fields $e^{\frac{i}{2}H'}(z)\, e^{-\frac{i}{2}H'}(w)$. It can be expressed as

$$e^{\frac{i}{2}H'}(z)\, e^{-\frac{i}{2}H'}(w) = (z-w)^{-\frac{1}{4}} \sum_{n=0}^{\infty} \frac{1}{n!} : \left(\frac{1}{2} \int_w^z dz'\, i\partial H'(z') \right)^n : \ , \tag{4.10}$$

in terms of $i\partial H'$, to which eq.(3.25) is applicable. Since spin fields always appear in such pairs in the correlation function, one can see that eq.(3.25) holds even in the presence of the spin fields. Taking the limit mentioned at the end of the last section, we can get the vertex operators containing the spin fields.

BRST invariant vertex operator

With these spin fields, we can construct BRST invariant vertex operators in the Ramond sector. Let us consider the left-moving part of a state in the Ramond sector of the light-cone gauge superstring of the form

$$\alpha^{i_1}_{-n_1}\cdots\psi^{j_1}_{-m_1}\cdots|\vec{p},\vec{s}\rangle_L\ , \tag{4.11}$$

where n_i and m_i are positive integers, and $|\vec{p},\vec{s}\rangle_L$ is the state corresponding to the operator $e^{i\vec{p}\cdot\vec{X}_L+i\vec{s}\cdot\vec{H}}$. Here $\vec{H}=(H^A)$ $\left(A=1,\ldots,\frac{d-2}{2}\right)$ are defined by using the transverse fermions as

$$e^{\pm iH^A}=\frac{1}{\sqrt{2}}\left(\psi^{2A-1}\pm i\psi^{2A}\right)\ , \tag{4.12}$$

and $\vec{s}=(s^A)$ with $s^A=\frac{1}{2}$ or $-\frac{1}{2}$.

In order to express the vertex operator for the Ramond sector state (4.11), we need to use the free fields. Imitating $V_L^{\prime(-1)}$ in eq.(4.5), we construct

$$\begin{aligned}
V_L^{\prime\left(-\frac{3}{2}\right)}(z) &\equiv A^{i_1}_{-n_1}\cdots B^{j_1}_{-m_1}\cdots\exp\left[\frac{i}{2}H'+i\vec{s}\cdot\vec{H}\right](z)\\
&\quad\times e^{(1+\alpha)(\sigma'-\phi')-\frac{1}{2}\phi'}\exp\left[-ip^+X_L^{\prime -}-i\left(p^--\frac{\mathcal{N}}{p^+}\right)X_L^++i\vec{p}\cdot\vec{X}_L\right](z)\,,\\
V_L^{\prime\prime\left(-\frac{1}{2}\right)}(z) &\equiv A^{i_1}_{-n_1}\cdots B^{j_1}_{-m_1}\cdots\exp\left[-\frac{i}{2}H'+i\vec{s}\cdot\vec{H}\right](z)\\
&\quad\times e^{(1+\alpha)(\sigma'-\phi')+\frac{1}{2}\phi'}\exp\left[-ip^+X_L^{\prime -}-i\left(p^--\frac{\mathcal{N}}{p^+}\right)X_L^++i\vec{p}\cdot\vec{X}_L\right](z)\,,
\end{aligned} \tag{4.13}$$

where $\mathcal{N}\equiv\sum_i n_i+\sum_j m_j$, and A^i_{-n} and B^i_{-m} are the DDF operators defined in eq.(4.3) with s replaced by an integer m. The reason why we adopt the notation $V_L^{\prime\prime\left(-\frac{1}{2}\right)}(z)$ for the second one will become clear later. Since the conformal dimension of the transverse spin field $e^{i\vec{s}\cdot\vec{H}}$ is $\frac{d-2}{16}$, this time the on-shell condition is

$$p^-=\frac{1}{p^+}\left(\frac{1}{2}\vec{p}^2+\mathcal{N}\right)\ . \tag{4.14}$$

One can construct the right-moving counterparts $V_R^{\prime\left(-\frac{3}{2}\right)}(\bar{z})$ and $V_R^{\prime\prime\left(-\frac{1}{2}\right)}(\bar{z})$ in the same way.

As we argued above, there should exist a vertex operator $V_L^{\left(-\frac{3}{2}\right)}$ in the $X^\pm$ CFT which corresponds to $V_L^{\prime\left(-\frac{3}{2}\right)}$ in the free field description. The explicit form of $V_L^{\left(-\frac{3}{2}\right)}$ will be complicated because of the presence of the spin fields, but even without the explicit form, we can read off its properties from its free field form $V_L'^{\left(-\frac{3}{2}\right)}$. In particular, one can show that $V_L^{\left(-\frac{3}{2}\right)}$ is BRST invariant. Using the explicit form of $D\Theta^+$ in terms of the component fields,

$$D\Theta^+ = \left(\partial X^+\right)^{\frac{1}{2}} \left[1 + \frac{\partial\psi^+\psi^+}{2\left(\partial X^+\right)^2} + \theta\left(\frac{i\partial\psi^+}{\partial X^+} - \frac{i}{2}\frac{\partial^2 X^+}{\left(\partial X^+\right)^2}\psi^+\right)\right] , \tag{4.15}$$

one can obtain

$$D\Theta^+ (z) V_L'^{\left(-\frac{3}{2}\right)} (0) \sim \left(\frac{-ip^+}{z}\right)^{\frac{1}{2}} V_L'^{\left(-\frac{3}{2}\right)} (0) . \tag{4.16}$$

Combining this relation with eqs.(3.6), (3.7), and the OPE's between the primed ghost fields and $V_r'^{\left(-\frac{3}{2}\right)}$, we have

$$\begin{aligned} b(z) V_L^{\left(-\frac{3}{2}\right)}(0) &\sim z^{-1} b_{-1} V_L^{\left(-\frac{3}{2}\right)}(0) , \\ c(z) V_L^{\left(-\frac{3}{2}\right)}(0) &\sim z c_0 V_L^{\left(-\frac{3}{2}\right)}(0) , \\ \beta(z) V_L^{\left(-\frac{3}{2}\right)}(0) &\sim z^{-\frac{3}{2}} \beta_0 V_L^{\left(-\frac{3}{2}\right)}(0) , \\ \gamma(z) V_L^{\left(-\frac{3}{2}\right)}(0) &\sim z^{\frac{3}{2}} \gamma_{-1} V_L^{\left(-\frac{3}{2}\right)}(0) . \end{aligned} \tag{4.17}$$

These relations imply that $V_L^{\left(-\frac{3}{2}\right)}$ should be of the form

$$e^{\sigma - \frac{3}{2}\phi} \mathcal{O}_X , \tag{4.18}$$

where $\mathcal{O}_X$ is a conformal field made of the unprimed longitudinal and the transverse variables, which satisfies

$$\begin{aligned} T_B^X(z)\mathcal{O}_X(0) &\sim \frac{\frac{1}{2}p^2 + \mathcal{N} + \frac{5}{8}}{z^2}\mathcal{O}_X(0) + \frac{1}{z}\partial\mathcal{O}_X(0) , \\ T_F^X(z)\mathcal{O}_X(0) &\sim z^{-\frac{3}{2}} G_0^X \mathcal{O}_X(0) , \end{aligned} \tag{4.19}$$

where T_B^X, T_F^X, G_0^X are the energy-momentum tensor, the supercurrent and the supercharge in the matter respectively. Using eqs.(4.18) and (4.19), we can prove that $V^{\left(-\frac{3}{2}\right)}$ commutes with the BRST operator

$$Q_{\mathrm{B}} = \oint \frac{dz}{2\pi i} \left[c T_B^X - \gamma T_F^X - \frac{1}{2}c\gamma\partial\beta - \frac{3}{2}c\partial\gamma\beta - bc\partial c - \frac{1}{4}b\gamma^2\right] + \text{c.c.} , \tag{4.20}$$

if the on-shell condition (4.14) is satisfied.

Unfortunately, $V_L''^{\left(-\frac{1}{2}\right)}$ does not correspond to a BRST invariant vertex operator. We can obtain the left-moving BRST invariant vertex operator $V_L^{\left(-\frac{1}{2}\right)}$ in the $-\frac{1}{2}$ picture by applying to $V_L^{\left(-\frac{3}{2}\right)}$ the picture changing operator $X(z)$ defined as

$$X(z) \equiv \{Q_{\mathrm{B}}\, , \xi(z)\} = c\partial\xi - e^{\phi} T_F^X + \frac{1}{4}\partial b \eta e^{2\phi} + \frac{1}{4} b\left(2\partial\eta e^{2\phi} + \eta\partial e^{2\phi}\right) , \tag{4.21}$$

namely

$$V_L^{\left(-\frac{1}{2}\right)}(0) \equiv \lim_{z\to 0} X(z) V_L^{\left(-\frac{3}{2}\right)}(0) = \lim_{z\to 0} \left(-e^{\phi} T_F^X\right)(z) V_L^{\left(-\frac{3}{2}\right)}(0) \ . \tag{4.22}$$

We define $V_L'^{\left(-\frac{1}{2}\right)}$ to be the free field version of $V_L^{\left(-\frac{1}{2}\right)}$.

One can obtain the BRST invariant right-moving parts $V_R^{\left(-\frac{3}{2}\right)}(\bar{z})$, $V_R^{\left(-\frac{1}{2}\right)}(\bar{z})$ and their free field versions in the same way.

5 Amplitudes

In this section, we would like to show that the tree level amplitudes in the noncritical light-cone gauge string field theory can be expressed by using the BRST invariant vertex operators constructed in the previous section. Our procedure is as follows. We start from the light-cone gauge amplitudes. We rewrite them by adding the longitudinal and ghost degrees of freedom. Then we reach the BRST invariant conformal gauge expression.

5.1 Light-cone gauge superstring field theory in d dimensions

The light-cone gauge superstring field theory can be defined in noncritical dimensions by just considering the action with three-string interactions like that for the (NS,NS) strings in Ref. [9]. However, putting naively $d \neq 10$ makes the physical content of the theory quite different from that in the critical case. From the on-shell conditions (4.4) and (4.14) for the Neveu-Schwarz and the Ramond sectors, the level matching condition for the (NS,R) sector becomes

$$\mathcal{N} = \tilde{\mathcal{N}} + \frac{d-2}{16} \ , \tag{5.1}$$

where $\mathcal{N}$ and $\tilde{\mathcal{N}}$ denote the level numbers in the left- and the right-moving parts. Since $\mathcal{N} - \tilde{\mathcal{N}}$ is a half-integer, there exist no states satisfying this condition for generic d. The situation is the same for the (R,NS) sector and one can see that the theory does not include any spacetime fermions for generic d. This fact is problematic if one wants to use the noncritical string theory to dimensionally regularize the critical theory. We need to modify the worldsheet theory for such applications. We will deal with this problem elsewhere. Here we take the theory as it is and consider the theory for generic $d \neq 10$ with only (NS,NS) and (R,R) sectors[5] and calculate the tree amplitudes. In appendix A, we present the string field theory action in this situation.

[5] We might have to consider Type 0 theory for $d \neq 10$ for the modular invariance but as far as we are discussing tree amplitudes, there is not so big difference between Type II theory and Type 0 theory.

The tree level N-string amplitudes $\mathcal{A}_N$ are perturbatively computed in the same way as those in Ref. [9]. Starting from the action (A.2) of string field theory, we obtain

$$\mathcal{A}_N = (4ig)^{N-2} \int \left(\prod_{\mathcal{I}=1}^{N-3} \frac{d^2 \mathcal{T}_{\mathcal{I}}}{4\pi} \right) F_N \left(\mathcal{T}_{\mathcal{I}}, \bar{\mathcal{T}}_{\mathcal{I}} \right) , \tag{5.2}$$

where $\mathcal{T}_{\mathcal{I}}$ denotes the complex Schwinger parameter of the $\mathcal{I}$th internal propagator ($\mathcal{I} = 1, \ldots, N-3$), which consists of the $N-3$ complex moduli parameters of the amplitude $\mathcal{A}_N$. As was discussed in Ref. [9], on the right hand side the integration region is taken to cover the whole moduli space and the integrand F_N is described by the correlation function of the superconformal field theory for the light-cone gauge superstrings on the z-plane:

$$\begin{aligned} F_N \left(\mathcal{T}_{\mathcal{I}}, \bar{\mathcal{T}}_{\mathcal{I}} \right) &= (2\pi)^2 \delta \left(\sum_{r=1}^{N} p_r^+ \right) \delta \left(\sum_{r=1}^{N} p_r^- \right) \mathrm{sgn} \left(\prod_{r=1}^{N} \alpha_r \right) e^{-\frac{d-2}{16}\Gamma} \\ & \quad \times \left\langle \prod_{I=1}^{N-2} \left| \left(\partial^2 \rho \right)^{-\frac{3}{4}} T_F^{\mathrm{LC}} (z_I) \right|^2 \prod_{r=1}^{N} V_r^{\mathrm{LC}} \right\rangle_{X^i} . \end{aligned} \tag{5.3}$$

Here Γ is given in eq.(2.27), and V_r^{LC} denotes the vertex operators for the rth external string in the light-cone gauge. An external state in the (NS,NS) sector is obtained by multiplying the state (4.1) by a similar one in the right-moving sector:

$$\alpha_{-n_1}^{i_1(r)} \cdots \tilde{\alpha}_{-\tilde{n}_1}^{\tilde{i}_1(r)} \cdots \psi_{-s_1}^{j_1(r)} \cdots \tilde{\psi}_{-\tilde{s}_1}^{\tilde{j}_1(r)} \cdots |\vec{p}_r\rangle_r \ . \tag{5.4}$$

To this state corresponds a vertex operator

$$\begin{aligned} V_r^{\mathrm{LC}} &= \alpha_r \oint_0 \frac{dw_r}{2\pi i} \partial_{w_r} X^{i_1} (w_r) \, w_r^{-n_1} \cdots \oint_0 \frac{d\bar{w}_r}{2\pi i} \partial_{\bar{w}_r} X^{\tilde{i}_1} (\bar{w}_r) \, \bar{w}_r^{-\tilde{n}_1} \cdots \\ & \quad \times \oint_0 \frac{dw_r}{2\pi i} \psi^{j_1} (w_r) \, w_r^{-s_1-\frac{1}{2}} \cdots \oint_0 \frac{d\bar{w}_r}{2\pi i} \tilde{\psi}^{\tilde{j}_1} (\bar{w}_r) \, \bar{w}_r^{-\tilde{s}_1-\frac{1}{2}} \cdots \\ & \quad \times e^{i\vec{p}_r \cdot \vec{X}} (w_r = 0, \bar{w}_r = 0) \, e^{-p_r^- \tau_0^{(r)}} , \end{aligned} \tag{5.5}$$

where w_r is the local coordinate, introduced in the region $z \sim Z_r$ as

$$w_r \equiv \exp \left[\frac{1}{\alpha_r} \left(\rho - \tau_0^{(r)} - i\beta_r \right) \right] , \qquad \tau_0^{(r)} + i\beta_r \equiv \rho(z_{I^{(r)}}) \ . \tag{5.6}$$

Similarly, an (R,R) external state is obtained by multiplying the state (4.11) by a similar one in the right-moving sector:

$$\alpha_{-n_1}^{i_1(r)} \cdots \tilde{\alpha}_{-\tilde{n}_1}^{\tilde{i}_1(r)} \cdots \psi_{-m_1}^{j_1(r)} \cdots \tilde{\psi}_{-\tilde{m}_1}^{\tilde{j}_1(r)} \cdots \left| \vec{p}_r, \vec{s}_r, \vec{\tilde{s}}_r \right\rangle_r \ . \tag{5.7}$$

For this state, we should take

$$
\begin{aligned}
V_r^{\mathrm{LC}} &= \alpha_r \oint_0 \frac{dw_r}{2\pi i}\partial_{w_r}X^{i_1}\left(w_r\right)w_r^{-n_1}\cdots\oint_0 \frac{d\bar{w}_r}{2\pi i}\partial_{\bar{w}_r}X^{\tilde{i}_1}\left(\bar{w}_r\right)\bar{w}_r^{-\tilde{n}_1}\cdots \\
&\quad\times\oint_0 \frac{dw_r}{2\pi i}\psi^{j_1}\left(w_r\right)w_r^{-m_1-\frac{1}{2}}\cdots\oint_0 \frac{d\bar{w}_r}{2\pi i}\tilde{\psi}^{\tilde{j}_1}\left(\bar{w}_r\right)\bar{w}_r^{-\tilde{m}_1-\frac{1}{2}}\cdots \\
&\quad\times e^{i\vec{p}_r\cdot\vec{X}+i\vec{s}_r\cdot\vec{H}+i\vec{\tilde{s}}_r\cdot\vec{\tilde{H}}}\left(w_r=0,\bar{w}_r=0\right)e^{-p_r^-\tau_0^{(r)}}\ .
\end{aligned}
\tag{5.8}
$$

5.2 Longitudinal variables and ghosts

We rewrite the light-cone gauge expression (5.2) by adding the longitudinal variables and the super-reparametrization ghosts to the worldsheet theory. Suppose that V_r^{LC} $(r=1,\cdots,2f)$ are in the (R,R) sector and the other V_r^{LC}'s are in the (NS,NS) sector. It is straightforward to show that the quantity which appears on the right hand side of eq.(5.3) can be expressed as a correlation function in the system of the free variables defined in section 3:

$$
\begin{aligned}
&(2\pi)^2\delta\left(\sum_{r=1}^N p_r^+\right)\delta\left(\sum_{r=1}^N p_r^-\right)e^{-\frac{d-2}{16}\Gamma}\prod_{I=1}^{N-2}\left|\partial^2\rho\left(z_I\right)\right|^{-\frac{3}{2}}\prod_{r=1}^N V_r^{\mathrm{LC}} \\
&\sim\left\langle\left|(\partial\rho)^{1+\alpha}e^{\sigma'}(\infty)\right|^2\prod_{I=1}^{N-2}\left|\frac{e^{-(1+\alpha)(\sigma'-\phi')}}{\left(\partial^2\rho\right)^{1+\frac{\alpha}{4}}}(z_I)\right|^2\prod_{r=1}^f\left(|\alpha_r|^{-\alpha}V_r^{\prime\left(-\frac{3}{2},-\frac{3}{2}\right)}\left(Z_r,\bar{Z}_r\right)\right)\right. \\
&\quad\left.\times\prod_{r=f+1}^{2f}\left(|\alpha_r|^{-\alpha}V_r^{\prime\prime\left(-\frac{1}{2},-\frac{1}{2}\right)}\left(Z_r,\bar{Z}_r\right)\right)\prod_{r=2f+1}^N\left(|\alpha_r|^{-\alpha}V_r^{\prime(-1,-1)}\left(Z_r,\bar{Z}_r\right)\right)\right\rangle_{\mathrm{free}}.
\end{aligned}
\tag{5.9}
$$

Here

$$
\begin{aligned}
V_r^{\prime(-1,-1)}(Z_r,\bar{Z}_r) &\equiv V_{L,r}^{\prime(-1)}(Z_r)V_{R,r}^{\prime(-1)}(\bar{Z}_r)\ , \\
V_r^{\prime\left(-\frac{3}{2},-\frac{3}{2}\right)}(Z_r,\bar{Z}_r) &\equiv |\alpha_r|^{-(\alpha+1)}V_{L,r}^{\prime\left(-\frac{3}{2}\right)}(Z_r)V_{R,r}^{\prime\left(-\frac{3}{2}\right)}(\bar{Z}_r)\ , \\
V_r^{\prime\prime\left(-\frac{1}{2},-\frac{1}{2}\right)}(Z_r,\bar{Z}_r) &\equiv |\alpha_r|^{\alpha+1}V_{L,r}^{\prime\prime\left(-\frac{1}{2}\right)}(Z_r)V_{R,r}^{\prime\prime\left(-\frac{1}{2}\right)}(\bar{Z}_r)\ ,
\end{aligned}
\tag{5.10}
$$

where $V_{L,r}^{\prime(-1)}$, $V_{L,r}^{\prime\left(-\frac{3}{2}\right)}$ and $V_{L,r}^{\prime\prime\left(-\frac{1}{2}\right)}$ are the vertex operators $V_L^{\prime(-1)}$, $V_L^{\prime\left(-\frac{3}{2}\right)}$ and $V_L^{\prime\prime\left(-\frac{1}{2}\right)}$ defined in eqs.(4.5) and (4.13) for the r-th external string, and similarly for the right moving sector ones $V_{R,r}^{\prime(-1)}$, $V_{R,r}^{\prime\left(-\frac{3}{2}\right)}$ and $V_{R,r}^{\prime\prime\left(-\frac{1}{2}\right)}$. In deriving eq.(5.9), we have used the relation

$$
\frac{\prod_{f+1\le s<r\le 2f}|Z_r-Z_s|^2\cdot\prod_{r=1}^f\prod_I|Z_r-z_I|\cdot\prod_{r=f+1}^{2f}\prod_{s=2f+1}^N|Z_r-Z_s|}{\prod_{1\le s<r\le f}|Z_r-Z_s|^2\cdot\prod_{r=f+1}^{2f}\prod_I|Z_r-z_I|\cdot\prod_{r=1}^f\prod_{s=2f+1}^N|Z_r-Z_s|}=\frac{\prod_{r=1}^f|\alpha_r|}{\prod_{r=f+1}^{2f}|\alpha_r|}\ .
\tag{5.11}
$$

On the right hand side of eq.(5.9), X'^{-} appears only in the form of the vertex operator $e^{-ip^{+}X'^{-}}$ and $\psi'^{-}, \tilde{\psi}'^{-}$ do not appear. Therefore we can replace $X^{+}, \psi^{+}, \tilde{\psi}^{+}$ by their expectation values $-\frac{i}{2}(\rho+\bar{\rho}), 0, 0$ in the correlation function, and vice versa. The insertions at $z = z_I$ and ∞ can be rearranged as

$$\begin{aligned}
\left|\frac{e^{-(1+\alpha)(\sigma'-\phi')}}{(\partial^2\rho)^{1+\frac{\alpha}{4}}}(z_I)\right|^2 &= \left|\oint_{z_I}\frac{dz}{2\pi i}\frac{e^{-\sigma'}}{(\partial\rho)^{1+\alpha}}(z)\lim_{w\to z_I}\left((\partial\rho)^{\alpha}e^{\phi'}\right)(w)\frac{e^{-\alpha(\sigma'-\phi')}}{(\partial^2\rho)^{\frac{\alpha}{4}}}(z_I)\right|^2 \\
&\sim \left|\oint_{z_I}\frac{dz}{2\pi i}\frac{e^{-\sigma'}}{(\partial\rho)^{1+\alpha}}(z)\lim_{w\to z_I}\left((\partial\rho)^{\alpha}e^{\phi'}\right)(w)\right|\mathcal{O}_I\ , \\
\left|(\partial\rho)^{1+\alpha}e^{\sigma'}(\infty)\right|^2 &= \left|\left(\sum_r\alpha_r Z_r\right)\lim_{z\to\infty}e^{-2(\sigma'-\phi')}(z)\left((\partial\rho)^{\alpha}e^{3\sigma'-2\phi'}\right)(\infty)\right|^2 \\
&\sim \left|\left(\sum_r\alpha_r Z_r\right)\lim_{z\to\infty}e^{-2(\sigma'-\phi')}(z)\right|^2\mathcal{R}\ . \qquad (5.12)
\end{aligned}$$

We also modify the vertex operator $V_r''^{\left(-\frac{1}{2},-\frac{1}{2}\right)}$ appearing on the right hand side of eq.(5.9) into $V_r'^{\left(-\frac{1}{2},-\frac{1}{2}\right)}$, which is composed of the free field versions $V_{L,r}'^{\left(-\frac{1}{2}\right)}$ and $V_{R,r}'^{\left(-\frac{1}{2}\right)}$ of the vertex operators $V_{L,r}^{\left(-\frac{1}{2}\right)}$ and $V_{R,r}^{\left(-\frac{1}{2}\right)}$, instead of $V_{L,r}''^{\left(-\frac{1}{2}\right)}$ and $V_{R,r}''^{\left(-\frac{1}{2}\right)}$. $V_r'^{\left(-\frac{1}{2},-\frac{1}{2}\right)}$ is defined to be obtained by applying the picture changing operators to $V_r'^{\left(-\frac{3}{2},-\frac{3}{2}\right)}(Z_r,\bar{Z}_r)$ and

$$\begin{aligned}
V_r'^{\left(-\frac{1}{2},-\frac{1}{2}\right)} &\equiv X\tilde{X}V'^{\left(-\frac{3}{2},-\frac{3}{2}\right)} \\
&= \left|-\frac{i}{2}(\partial\rho)^{\alpha}e^{\phi'}\partial X^{+}\psi'^{-}\right|^2 V_r'^{\left(-\frac{3}{2},-\frac{3}{2}\right)}+\cdots \\
&= \left|-\frac{1}{2}(\alpha_r)^{\alpha}p_r^{+}\right|^2|\alpha_r|^{-(\alpha+1)}V_{L,r}''^{\left(-\frac{1}{2}\right)}V_{R,r}''^{\left(-\frac{1}{2}\right)}+\cdots \\
&\propto V_r''^{\left(-\frac{1}{2},-\frac{1}{2}\right)}+\cdots\ . \qquad (5.13)
\end{aligned}$$

Here $\cdots$ denotes the terms which either include derivatives of $\partial X^{+}+\frac{i}{2}\partial\rho$, $\bar{\partial}X^{+}+\frac{i}{2}\bar{\partial}\bar{\rho}$ or are with the fermion numbers $\oint\frac{dz}{2\pi i}i\partial H'(z)$, $\oint\frac{d\bar{z}}{2\pi i}i\bar{\partial}\tilde{H}'(\bar{z})$ bigger than those of $V_r''^{\left(-\frac{1}{2},-\frac{1}{2}\right)}$. Therefore we can replace $V_r''^{\left(-\frac{1}{2},-\frac{1}{2}\right)}$ in eq.(5.9) by $V_r'^{\left(-\frac{1}{2},-\frac{1}{2}\right)}$ without changing the value of the correlation function up to a constant multiplicative factor.

Substituting these into eq.(5.9), we get

$$\begin{aligned}
&(2\pi)^2\delta\left(\sum_{r=1}^{N}p_r^{+}\right)\delta\left(\sum_{r=1}^{N}p_r^{-}\right)e^{-\frac{d-2}{16}\Gamma}\prod_{I=1}^{N-2}\left|\partial^2\rho(z_I)\right|^{-\frac{3}{2}}\prod_{r=1}^{N}V_r^{\mathrm{LC}} \\
&\sim \left\langle\left|\left(\sum_r\alpha_r Z_r\right)\lim_{z\to\infty}e^{-2(\sigma'-\phi')}(z)\right|^2\mathcal{R}\right.
\end{aligned}$$

$$\times \prod_{I=1}^{N-2} \left| \oint_{z_I} \frac{dz}{2\pi i} \frac{e^{-\sigma'}}{(\partial\rho)^{1+\alpha}} (z) \lim_{w\to z_I} \left((\partial\rho)^{\alpha} e^{\phi'} \right) (w) \right| \mathcal{O}_I$$

$$\times \prod_{r=1}^{N} \left(|\alpha_r|^{-\alpha} V_r'^{\left(p_{L,r},p_{R,r}\right)} \left(Z_r, \bar{Z}_r\right) \right) \Bigg\rangle_{\mathrm{free}} , \tag{5.14}$$

where $p_{L,r}, p_{R,r} = -\frac{1}{2}, -1, -\frac{3}{2}$ indicate the picture of the vertex operator. The choice of picture is obvious from eq.(5.9). To this equation we can easily apply the formula (3.25) and express the right hand side using the $X^{\pm}$ CFT and the unprimed ghost fields. Substituting it into eq.(5.3), we obtain

$$F_N \sim \left\langle |\partial\rho c(\infty)|^2 \prod_I \left| \oint_{z_I} \frac{dz}{2\pi i} \frac{b}{\partial\rho} (z) e^{\phi} T_F^{\mathrm{LC}} (z_I) \right|^2 \prod_{r=1}^{N} \mathcal{S}_r^{-1} \prod_{r=1}^{N} V_r^{\left(p_{L,r},p_{R,r}\right)} \left(Z_r, \bar{Z}_r\right) \right\rangle . \tag{5.15}$$

Here $\langle \cdots \rangle$ denotes the correlation function of the CFT for the longitudinal and transverse variables and the super-reparametrization ghosts. $V_r^{\left(p_{L,r},p_{R,r}\right)}$ is the unprimed field version of $V_r'^{\left(p_{L,r},p_{R,r}\right)}$ and it is BRST invariant. $\mathcal{S}_r^{-1}$ is defined as

$$\mathcal{S}_r^{-1} \equiv \oint_{z_{I(r)}} \frac{d\mathbf{z}}{2\pi i} D\Phi(\mathbf{z}) \oint_{\bar{z}_{I(r)}} \frac{d\bar{\mathbf{z}}}{2\pi i} \bar{D}\Phi(\bar{\mathbf{z}}) e^{\frac{d-10}{16} \frac{i}{p_r^+} \mathcal{X}^+} (\mathbf{z}, \bar{\mathbf{z}}) , \tag{5.16}$$

which can be shown to be the inverse of $\mathcal{S}_r$ in eq.(2.24) by replacing $\mathcal{X}^+$ by its expectation value. $\mathcal{S}_r^{-1}$ coincides with the BRST invariant form of $e^{\frac{d-10}{16}\frac{i}{p_r^+}X^+} (z_{I(r)}, \bar{z}_{I(r)})$ introduced in Ref. [9].

5.3 BRST invariant form of the amplitudes

In eq.(5.15), the right hand side is expressed by the variables in the conformal gauge, but it is not manifestly BRST invariant. In order to get a BRST invariant form of the amplitudes, we would like to show that $e^{\phi} T_F(z_I)$ in eq.(5.15) can be turned into the picture changing operator $X(z_I)$ and

$$F_N \sim \left\langle |\partial\rho c(\infty)|^2 \prod_I \left| \oint_{z_I} \frac{dz}{2\pi i} \frac{b}{\partial\rho} (z) X(z_I) \right|^2 \prod_{r=1}^{N} \mathcal{S}_r^{-1} \prod_{r=1}^{N} V_r^{\left(p_{L,r},p_{R,r}\right)} \right\rangle . \tag{5.17}$$

picture changing operator

Let us introduce a nilpotent fermionic charge Q defined as

$$Q \equiv \oint \frac{dz}{2\pi i} \left[-\frac{1}{4} \frac{b}{\partial\rho} \left(iX_L^+ - \frac{1}{2}\rho \right) + \frac{1}{2} \frac{e^{-\phi}\partial\xi}{\partial\rho} \psi^+ \right] (z) . \tag{5.18}$$

Here we define $X_L^+(z)$ so that

$$\left(iX_L^+ - \frac{1}{2}\rho\right)(z) = \int_\infty^z dz' \left(i\partial X^+ - \frac{1}{2}\partial\rho\right)(z') \ . \tag{5.19}$$

One can show

$$\begin{aligned}
\oint_{z_I} \frac{dz}{2\pi i}\frac{b}{\partial\rho}(z)\, X(z_I) \quad = \quad & -\oint_{z_I} \frac{dz}{2\pi i}\frac{b}{\partial\rho}(z)\, e^{\phi}T_F^{\mathrm{LC}}(z_I) \\
& + \left[Q\,,\,\oint_{z_I,w}\frac{dz}{2\pi i}\frac{b}{\partial\rho}(z)\oint_{z_I}\frac{dw}{2\pi i}\frac{A(w)}{w-z_I}e^{\phi}(z_I)\right] \\
& + \frac{1}{4}\oint_{z_I,w}\frac{dz}{2\pi i}\frac{b}{\partial\rho}(z)\oint_{z_I}\frac{dw}{2\pi i}\frac{\partial\rho\psi^-(w)}{w-z_I}e^{\phi}(z_I)\,,
\end{aligned} \tag{5.20}$$

where

$$\begin{aligned}
A(w) \quad \equiv \quad & -i\partial X^+\partial\rho\eta e^{\phi}(w) - 2\partial\left(\partial\rho c\right)\psi^-(w) \\
& -\frac{d-10}{4}i\left[\left(\frac{5\left(\partial^2X^+\right)^2}{4\left(\partial X^+\right)^3} - \frac{\partial^3X^+}{2\left(\partial X^+\right)^2}\right)\left(-2\partial\rho\eta e^{\phi}\right) - \frac{2\partial^2X^+}{\left(\partial X^{+2}\right)}\partial\left(-2\partial\rho\eta e^{\phi}\right)\right. \\
& \qquad\left. + \frac{\partial^2\left(-2\partial\rho\eta e^{\phi}\right)}{\partial X^+} - \frac{\left(-2\partial\rho\eta e^{\phi}\right)\partial\psi^+\partial^2\psi^+}{2\left(\partial X^+\right)^3}\right](w) \ .
\end{aligned} \tag{5.21}$$

Substituting eq.(5.20) into the right hand side of eq.(5.17) and comparing it with that of eq.(5.15), one can see that in order to prove eq.(5.17), one should show that the second and the third terms on the right hand side of eq.(5.20) do not contribute to the correlation function.

One can prove the third term does not contribute to the correlation function by describing the insertion (5.20) in terms of the free variables. The proof is given in appendix B.[6] The second term is Q-exact. We can therefore prove that this is also irrelevant, by showing that Q (anti)commutes with all the operators in the correlation function (5.17). It is straightforward to show that Q (anti)commutes with the vertex operators. Moreover, Q (anti)commutes with other insertions:

$$\begin{aligned}
&\left\{Q\,,\,\partial\rho c(\infty)\right\} = -\frac{1}{4}\left(iX_L^+ - \frac{1}{2}\rho\right)(\infty) = 0\ , \qquad \left\{Q\,,\,\oint_{z_I}\frac{dz}{2\pi i}\frac{b}{\partial\rho}\right\} = 0\ , \\
&\left[Q\,,\,e^{\phi}T_F^{\mathrm{LC}}(z_I)\right] = \left[Q\,,\,\oint_{z_I}\frac{dw}{2\pi i}\frac{1}{w-z_I}\partial\rho\psi^-(w)\,e^{\phi}(z_I)\right] = 0\ .
\end{aligned} \tag{5.22}$$

Thus we obtain the expression (5.17) for F_N.

[6] Actually this term has the same structure as the third term on the right hand side of eq.(3.25) in Ref. [9]. In Ref. [9], we have given another proof that the contributions of such terms vanish.

BRST invariant form

By deforming the contour of $\oint_{z_I} \frac{dz}{2\pi i} \frac{b}{\partial\rho}(z)$ in eq.(5.17) as was done in Ref. [9], we can obtain a manifestly BRST invariant form of the amplitude $\mathcal{A}_N$:

$$\mathcal{A}_N \sim \int \prod_{\mathcal{I}=1}^{N-3} d^2\mathcal{T}_{\mathcal{I}} \left\langle \prod_{\mathcal{I}=1}^{N-3} \left[\oint_{C_{\mathcal{I}}} \frac{dz}{2\pi i} \frac{b}{\partial\rho}(z) \oint_{C_{\mathcal{I}}} \frac{d\bar{z}}{2\pi i} \frac{\tilde{b}}{\bar{\partial}\bar{\rho}}(\bar{z}) \right] \prod_I |X(z_I)|^2 \right.$$
$$\left. \times \prod_{r=1}^{N} \mathcal{S}_r^{-1} \prod_{r=1}^{N} V_r^{(p_{L,r}, p_{R,r})} \right\rangle . \qquad (5.23)$$

Here $C_{\mathcal{I}}$ denotes a contour which goes around the $\mathcal{I}$th internal propagator. Compared with the form of the tree amplitudes in the critical case, the difference is in the insertions of $\mathcal{S}_r^{-1}$. These insertions are peculiar to the noncritical strings [7].

6 Conclusions and discussions

In this paper, we have formulated a free field description of the $X^{\pm}$ CFT combined with the reparametrization ghosts, and provided a formula to express the correlation functions in terms of the free variables. Since the $X^{\pm}$ CFT is an interacting theory, it is not straightforward to construct spin fields and thus the vertex operators in the Ramond sector. We have given the spin fields via the free variables, and thereby we have constructed the BRST invariant vertex operators in the Ramond sector. We have shown how one can calculate tree amplitudes with the external lines in the (R,R) sector as well as those in the (NS,NS) sector in the noncritical string theory using these vertex operators in the conformal gauge.

We study such noncritical string field theories, in order to dimensionally regularize the string field theory to deal with the divergences of the theory [16, 9]. One occasion in which such regularization is useful is when we deal with the contact term problem [17, 18, 19, 20, 21]. In the light-cone gauge superstring field theory, even tree amplitudes are divergent because of the existence of the supercurrent insertions at the interaction points. Using the results obtained in this paper, we can show that the dimensional regularization can be employed to deal with the contact term problem for the tree amplitudes when the external lines are in the (R,R) and the (NS,NS) sectors.

In order to generalize our regularization scheme to the amplitudes involving external lines in the (R,NS) and the (NS,R) sectors, there are several issues to be resolved. As we have pointed out, if we take $d \neq 10$ naively, we get a theory with no spacetime fermions. In order to deal with this problem, we need to modify the worldsheet theory. Moreover the dimensional regularization in usual field theory for point particles has some problems in treating fermions. We encounter similar problems when we try to apply the regularization to superstring field theory. We will discuss these points elsewhere.

Another thing to be examined is the Green-Schwarz formalism. As was commented in Ref. [22], the results in Ref. [7] seems to be useful in constructing vertex operators in the semi-light-cone gauge formulation of the Green-Schwarz formalism, recently re-examined in Refs. [22, 23, 24, 25, 26]. Moreover, the similarity transformation given in Ref. [22] looks similar to the field redefinition (2.6) [27]. It will be interesting to examine how the results in this paper are related to these developments.

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A String field theory action in d dimensions

In this appendix, we explain some details of the action of the light-cone gauge superstring field theory in noncritical dimensions.

We represent the string field $|\Phi_\lambda(t)\rangle$ by a wave function for the bosonic zero modes $(t, \alpha, \vec{p})$ and a Fock state for the other modes. We denote the integration measure of the momentum zero modes of the rth string by dr, which is defined as

$$dr = \frac{\alpha_r d\alpha_r}{4\pi} \frac{d^{d-2}p_r}{(2\pi)^{d-2}} \ . \tag{A.1}$$

The string fields are taken to be GSO even and satisfy the level-matching condition. The subscript λ of the string field labels the sector to which the string field belongs. As was stated in section 5, we concentrate on the strings in the (NS,NS) and the (R,R) sectors. Therefore the subscript λ takes only (NS,NS) and (R,R) and the string fields Φ_λ are Grassmann even.

The action for the string fields in these sectors of the light-cone gauge superstring field theory in d dimensions ($d \neq 10$) takes the form

$$
\begin{aligned}
S \;=\; & \int dt \left[\frac{1}{2} \sum_{\lambda} \int d1d2 \left\langle R_{\lambda}\left(1,2\right) \middle| \Phi_{\lambda}(t) \right\rangle_1 \left(i\frac{\partial}{\partial t} - \frac{L_0^{\mathrm{LC}(2)} + \tilde{L}_0^{\mathrm{LC}(2)} - \frac{d-2}{8}}{\alpha_2} \right) \left| \Phi_{\lambda}(t) \right\rangle_2 \right. \\
& + \frac{2g}{3} \int d1d2d3 \left\langle V_3\left(1_{\mathrm{NSNS}}, 2_{\mathrm{NSNS}}, 3_{\mathrm{NSNS}}\right) \middle| \Phi_{\mathrm{NSNS}}(t) \right\rangle_1 \left| \Phi_{\mathrm{NSNS}}(t) \right\rangle_2 \left| \Phi_{\mathrm{NSNS}}(t) \right\rangle_3 \\
& \left. + 2g \int d1d2d3 \left\langle V_3\left(1_{\mathrm{NSNS}}, 2_{\mathrm{RR}}, 3_{\mathrm{RR}}\right) \middle| \Phi_{\mathrm{NSNS}}(t) \right\rangle_1 \left| \Phi_{\mathrm{RR}}(t) \right\rangle_2 \left| \Phi_{\mathrm{RR}}(t) \right\rangle_3 \right] \; . \qquad \text{(A.2)}
\end{aligned}
$$

Here $\langle R_\lambda(1,2)|$ denotes the reflector for the string fields in sector λ. $\langle V_3(1_{\lambda_1}, 2_{\lambda_2}, 3_{\lambda_3})|$ denotes the interaction vertex for the three strings in sector λ_r ($r = 1, 2, 3$). This is invariant under the permutation of the string fields and takes the form

$$
\begin{aligned}
\langle V_3(1_{\lambda_1}, 2_{\lambda_2}, 3_{\lambda_3})| \;=\; & 4\pi\delta\left(\sum_{r=1}^{3}\alpha_r\right)(2\pi)^{d-2}\delta^{d-2}\left(\sum_{r=1}^{3}p_r\right) \\
& \times \langle V_3^{\mathrm{LPP}}(1_{\lambda_1}, 2_{\lambda_2}, 3_{\lambda_3})| P_{123} e^{-\Gamma^{[3]}(1,2,3)} \; . \qquad \text{(A.3)}
\end{aligned}
$$

Here $\langle V_3^{\mathrm{LPP}}(1,2,3)|$ denotes the LPP vertex [28], which satisfies eq.(A.5) of Ref. [9]. $\Gamma^{[3]}(1,2,3)$ and P_{123} are defined in eqs.(A.4) and (A.6) of Ref. [9] respectively.

B Correlation functions of ψ^-

In this appendix, we show that the third term on the right hand side of eq.(5.20) does not contribute the correlation function. In terms of free fields, the third term on the right hand side of eq.(5.20) turns out to be

$$
\frac{1}{\partial^2\rho(z_I)} \frac{1}{4} \oint_{z_I} \frac{dw}{2\pi i} \frac{\partial\rho\psi^-(w)}{w - z_I} e^{-\sigma'+\phi'}(z_I)\,\mathcal{O}_I \; , \qquad \text{(B.1)}
$$

where $\psi^-(w)$ is written using the free fields as

$$
\psi^-(w) = \psi'^-(w) + \delta\psi^-(w) \; . \qquad \text{(B.2)}
$$

The explicit form of $\delta\psi^-(w)$ can be deduced from eq.(3.7) but we do not need it here.

We would like to show that the integral $\oint_{z_I} \frac{dw}{2\pi i} \frac{\partial\rho\psi^-(w)}{w-z_I}$ does not contribute to the correlation function. It can have nonzero contributions when $\psi^-(w)$ has singularities at $w = z_I$. Using the expression (B.2), one can see that such singularities come either from contracting

ψ'^- with ψ^+ included in $\mathcal{O}_I$ or from $\delta\psi^-(w)$. $\delta\psi^-(w)$ involves factors $(\partial X^+)^{-n}(w)$, which has the expectation value $\left(-\frac{i}{2}\partial\rho\right)^{-n}(w)$ and singular at $w = z_I$. In order to give a nonvanishing contribution, ψ'^- contained in $\oint_{z_I}\frac{dw}{2\pi i}\frac{\partial\rho\psi^-(w)}{w-z_I}$ should be contracted with ψ^+ contained in $\mathcal{O}_I$ and not in $\mathcal{O}_J$ with $J \neq I$. Since the Grassmann odd quantities in $\mathcal{O}_I$ and $\delta\psi^-$ are made from ψ^+ and βc, these terms necessarily involves derivatives of ψ^+ and βc. If we take contractions of all ψ'^- with appropriate ψ^+'s, the resulting contributions of the second term on the right hand side of eq.(B.1) to the correlation functions can be seen to vanish because of the conservation of the fermion number $\oint\frac{dz}{2\pi i}i\partial H'$ and the bc ghost number.

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