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PARAMETERIZED FAIRNESS AXIOMS ON CYCLE-FREE GRAPH GAMES

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Abstract. In this paper we study cooperative transferable utility games with communication structure represented by an undirected graph, i.e., a group of players can cooperate only if they are connected on the graph. This type of games is called graph games and the best-known solution for them is the Myerson value, characterized by component efficiency and fairness. Recently on cycle-free graph games, the average tree solution has been proposed and it is characterized by component efficiency and component fairness. We propose ϵ-parameterized fairness that incorporates the preceding fairness axioms on cycle-free graph games and show the existence and the uniqueness of a solution satisfying component efficiency and our fairness for any nonnegative parameter ϵ. We then discuss a relationship between the existing and our proposed solutions by a numerical example.

1. Introduction

In many situations, a group of players obtain profits or save costs by their cooperation. A subgroup of the players is called a *coalition* and the total profit of a coalition is called *worth* of the coalition if all the members in the coalition agree to cooperate. The problem of how much payoff should be allocated to each player then arises if we know the worth of all possible coalitions. A classical set-valued solution is the core, see Aumann and Hart [1], being the set of payoffs at which the worth of the whole set of players is distributed among the players and no coalition receives less than its worth. The best-known single-valued solution is the Shapley value, defined by Shapley [8], which is the average of all his/her marginal contributions to every coalition that he/she is a member of.

In this paper we consider this problem with restricted cooperation structure. Restricted cooperation means, for example, the friendship: there is no friendship between A and B, however they can form it in the presence of C. In this case we see that C mediates between A and B. We do not know the worth of the coalition consisting of A and B, while that consisting of A, B and C is conceivable. This restricted cooperation structure is often represented by undirected graphs and cooperative transferable utility games with such structure are called *graph games*. Myerson [6] introduced the Myerson value for the games and characterized it by component efficiency and fairness axioms. Myerson [7] showed that fairness can be replaced by stronger requirement of balanced contribution. The position value is another solution for the games proposed by Borm et al. [3]. This value is characterized by component efficiency and balanced link contribution, which is in the same spirit as balanced contribution, see Slikker [9]. Both the Myerson value and the position value are based on the Shapley value. In Slikker and van den Nouweland [10] the properties of these values are described in detail. Recently Herings et al. [5] proposed

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the average tree solution on the class of cycle-free graph games and characterized it by component efficiency and component fairness. They also showed that the solution is in the core if the game exhibits super-additivity, while the Myerson value or the position value may not be. The condition of super-additivity was relaxed to a weaker one by Talman and Yamamoto [11]. In Herings et al. [4] the average tree solution was generalized on the whole class of graph games, however no characterization was given there.

Fairness seen in Myerson [6] means that, when a link is deleted from the underlying graph, the two players on the ends of the link will get the same loss in payoff. Meanwhile in Herings et al. [5], component fairness requires that the divided two components will get the same average loss in payoff. The aim of our research is to propose a new axiom that incorporates the preceding ones concerning fairness on cycle-free graph games. We introduce ϵ-parameterized fairness where ϵ is a nonnegative parameter, and show the existence and the uniqueness of a solution satisfying component efficiency and our axiom for any ϵ. We then discuss a relationship among the existing solutions, our proposed solutions and the core by a numerical example.

The rest of the paper is organized as follows. Section 2 introduces notations of games and undirected graphs, and then lay out graph games. In Section 3 we give the definitions and the axioms characterizing the Myerson value and the average tree solution. The relationship between the latter solution and the core is also presented. We propose a new fairness axiom in Section 4 that incorporates the preceding two on cycle-free graph games and discuss the existence and the uniqueness of a solution satisfying our axiom. In section 5, an example of graph games by three players is given. We compute and observe the existing solutions, our proposed solutions and the core. Section 6 concludes the paper.

2. Preliminaries

A *cooperative game with transferable utility* or simply a TU-game, is defined by a pair (N, v), where N is a finite set of players, i.e., $N = \{1, 2, \dots, n\}$, and $v : 2^N \to \mathbb{R}$ is a characteristic function such that $v(\varnothing) = 0$. The worth of a coalition $S \in 2^N$ is denoted by $v(S)$. We denote the game (N, v) by v for short and the collection of all characteristic functions by $\mathcal{V}$. A payoff vector $\boldsymbol{x} \in \mathbb{R}^n$ is an n-dimensional vector, and we let $x(S) = \sum_{i\in S} x_i$ for each $S \in 2^N$ where x_i is player i's payoff and ith component of $\boldsymbol{x}$. A payoff vector $\boldsymbol{x}$ of n players is said to be *efficient* if $x(N) = v(N)$. A solution for games is a mapping $F : \mathcal{V} \to \mathbb{R}^n$, i.e., it returns a payoff vector when a game is input. If for any $v \in \mathcal{V}$ it holds that $\sum_{i\in N} F_i(v) = v(N)$, then the solution F is said to be efficient. The core is the classical set of efficient solutions for this type of games, see Aumann and Hart [1]. For a game $v \in \mathcal{V}$, the core is defined by

$$C(v) := \{\, \boldsymbol{x} \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \in 2^N \,\}. \tag{2.1}$$

The core is the set of payoffs which are not rejected by any coalition. The best-known single-valued solution is the Shapley value defined by Shapley [8], and it is represented by

$$\psi(v) := \frac{1}{n!} \sum_{\pi\in\Pi} \boldsymbol{m}^\pi(v), \tag{2.2}$$

where $\boldsymbol{m}^\pi(v) \in \mathbb{R}^n$ is called the marginal vector corresponding to a permutation π and Π is the set of all permutations on N. The ith component $m_i^\pi(v)$ of $\boldsymbol{m}^\pi(v)$ is defined as $m_i^\pi(v) := v(\pi^i \cup \{i\}) - v(\pi^i)$ where $\pi^i = \{\, j \in N \mid \pi(j) < \pi(i) \,\}$, i.e., the set of players preceding i in permutation π.

Next we give several notations for undirected graphs. An undirected graph is denoted by a pair (N, L) where N is a set of nodes and L is a set of edges, i.e., $L \subseteq \{\,\{i,j\} \mid i,j \in N, i \neq j\,\}$. The collection of sets of edges is denoted by $\mathcal{L}$. For $K \in 2^N$, the graph $(K, L(K))$, where $L(K) = \{\,\{i,j\} \mid i,j \in K, \{i,j\} \in L\,\}$, is called the *subgraph* of (N, L) on K. A sequence of edges $(\{i_0, i_1\}, \{i_1, i_2\}, \dots, \{i_{l-1}, i_l\})$ is a *path* in (N, L) if $\{i_h, i_{h+1}\} \in L$ for $h = 0, \dots, l-1$. Two nodes $i, j \in N$ are *connected* in (N, L) if either $i = j$ or there exists a path $(\{i_0, i_1\}, \{i_1, i_2\}, \dots, \{i_{l-1}, i_l\})$ such that $i_0 = i$ and $i_l = j$. A graph (N, L) or simply N is connected if any two nodes $i, j \in N$ are connected in (N, L). A subset $K \subseteq N$ is said to be a *connected subset* of N when the subgraph $(K, L(K))$ is connected. The collection of all connected subsets of K in $(K, L(K))$ is denoted by $\mathcal{C}(K, L(K))$, i.e., $\mathcal{C}(K, L(K)) := \{\, H \mid H \subseteq K \text{ and } H \text{ is a connected subset of } K\,\}$. A subset K of N is a *component* of (N, L) if K is maximally connected, i.e., K is connected but the subset $K \cup \{j\}$ is not for any $j \in N \setminus K$. The collection of all components of $(K, L(K))$ is denoted by $\mathcal{C}_m(K, L(K))$, i.e., $\mathcal{C}_m(K, L(K)) := \{\, H \mid H \subseteq K \text{ and } H \text{ is a component of } K\,\}$. A sequence of edges $(\{i_0, i_1\}, \{i_1, i_2\}, \dots, \{i_l, i_{l+1}\})$ is a *cycle* in (N, L) if

(i) $l \geq 1$,
(ii) $i_{l+1} = i_0$, and
(iii) $\{i_h, i_{h+1}\} \in L$ for $h = 0, \dots, l$.

A graph (N, L) is cycle-free if it does not contain any cycle. Here we define the collection of sets of edges forming a cycle-free graph as $\mathcal{M}$.

A graph game is given by a triple (N, v, L) where N is a set of players, v is a characteristic function and L is a set of edges of the graph (N, L). Omitting N, we denote the graph game by (v, L), the collection of all graph games by $\mathcal{V} \times \mathcal{L}$ and that of cycle-free graph games by $\mathcal{V} \times \mathcal{M}$. On graph games only connected subsets of the players are able to cooperate, hence the set of admissible coalitions is $\mathcal{C}(N, L)$. In this paper we assume without loss of generality that N is connected in (N, L), i.e., $N \in \mathcal{C}(N, L)$. Otherwise we have only to discuss each component of the graph analogously. A function $f : \mathcal{V} \times \mathcal{L} \to \mathbb{R}^n$ is called a *solution* for graph games and $f_i(v, L)$ is called a player i's *allocation* by solution f. The core of the graph game is given by

$$C(v, L) := \{\, \boldsymbol{x} \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \in \mathcal{C}(N, L)\,\}. \tag{2.3}$$

When the graph (N, L) is complete, i.e., $\mathcal{C}(N, L) = 2^N$, then $C(v, L) = C(v)$ holds and therefore we can see (2.1) as a special case of (2.3). Moreover the core $C(v, L)$ of the graph game is equal to the core $C(v^L)$ where the so-called restricted game v^L is defined by Myerson [6] as

$$v^L(S) := \sum_{T \in \mathcal{C}_m(S, L(S))} v(T) \quad \text{for each } S \in 2^N. \tag{2.4}$$

3. Existing Solutions

In this section we introduce a pair of existing solutions and the axioms which characterize them for graph games: the Myerson value and the average tree solution. The Myerson value is the best-known single-valued solution for graph games, defined by Myerson [6].

Definition 3.1. On the class of all graph games, the Myerson value is the Shapley value of the restricted game defined by (2.4), i.e.,

$$\mu(v, L) := \psi(v^L).$$

On the Myerson value the following two axioms are introduced, where $f : \mathcal{V} \times \mathcal{L} \to \mathbb{R}^n$ is a solution for graph games.

Axiom 3.2 (component efficiency)**.** For any graph game (v, L) it holds that

$$\sum_{i \in K} f_i(v, L) = v(K) \text{ for each } K \in \mathcal{C}_m(N, L).$$

Axiom 3.3 (fairness)**.** For any $(v, L) \in \mathcal{V} \times \mathcal{L}$ and $\{i, j\} \in L$, it holds that

$$f_i(v, L) - f_i(v, L \setminus \{i, j\}) = f_j(v, L) - f_j(v, L \setminus \{i, j\}).$$

Component efficiency means that the sum of the players' allocations in a component is equal to the worth of the component. This axiom is satisfied by all the solutions that we introduce in this paper. Fairness means that the both players connected by an edge obtain the same change of allocations if the edge is deleted. Myerson [6] gave the following theorem.

Theorem 3.4 (Myerson [6])**.** *On the class of all graph games, the Myerson value μ is the unique solution that satisfies Axiom 3.2 and Axiom 3.3.*

The average tree solution is a solution for cycle-free graph games, which is introduced by Herings et al. [5]. To describe the solution we give here some definitions on directed graphs, following Berge [2]. A directed graph is given by a pair (N, D) with $D \subseteq N \times N$, i.e., D is a set of ordered pairs of nodes called *arcs*. If $(i, j) \in D$, then we say that node j is a *successor* of i and we denoted by $S'_D(i)$ the set of i's successors. If $(j, i) \in D$, then we say that node j is a *predecessor* of i and we denoted by $P'_D(i)$ the set of i's predecessors. A sequence of arcs $((i_0, i_1), (i_1.i_2), \dots, (i_{l-1}, i_l))$ is a *directed path* from i_0 to i_l in (N, D) if $(i_h, i_{h+1}) \in D$ for $h = 0, \dots l - 1$. We say that j is a *subordinate* of i when there is a directed path from i to j. The set of subordinates of i is defined by $S_D(i)$ and we let $\overline{S}_D(i) := S_D(i) \cup \{i\}$. A node r is called a *root* if all the nodes of $N \setminus \{r\}$ can be reached by directed paths starting from r and a *leaf* is a node from which no other nodes can be reached. Finally we define an *arborescence* D^r with respect to node $r \in N$ as an edge set of graph (N, D^r) such that r is a root and the other nodes have only one predecessor, i.e., $|P'_{D^r}(i)| = 1$ for all $i \in N \setminus \{r\}$, or in other words the graph is a rooted tree at r.

Given a cycle-free graph game (v, L), a *tree solution* with respect to $r \in N$, denoted by $\boldsymbol{x}^r$, is defined as follows: make an arborescence D^r with node r as the root by giving a direction to all the edges in L. Note that D^r is uniquely determined for any $r \in N$ since the original undirected graph (N, L) is cycle-free and we have assumed that $N \in \mathcal{C}(N, L)$. For each node $i \in N$ the ith component of tree solution $\boldsymbol{x}^r$ is given by

$$x_i^r := v(\overline{S}_{D^r}(i)) - \sum_{j \in S_{D^r}(i)} x_j^r. \tag{3.1}$$

We can determine x_i^r inductively by (3.1) since $x_i^r = v(\{i\})$ holds when i is a leaf. There are n different tree solutions obtained and the average tree solution is defined as follows.

Definition 3.5. On the class of cycle-free graph games, the average tree solution is the average of all tree solutions, i.e.,

$$AT(v, L) := \frac{1}{n} \sum_{r \in N} \boldsymbol{x}^r.$$

On the axiomatic characterization of the solution, component fairness instead of fairness is introduced in Herings et al. [5]. Consider a solution $f : \mathcal{V} \times \mathcal{M} \to \mathbb{R}^n$ for cycle-free graph games, and for a cycle-free undirected graph (N, L) and an edge $\{i, j\} \in L$ let K^i and K^j be the components of $(N, L \setminus \{i, j\})$ containing i and j, respectively.

Axiom 3.6 (component fairness)**.** For any $(v, L) \in \mathcal{V} \times \mathcal{M}$, $K \in \mathcal{C}_m(N, L)$ and $\{i, j\} \in L(K)$, it holds that

$$\frac{1}{|K^i|} \sum_{h \in K^i} (f_h(v, L) - f_h(v, L \setminus \{i, j\})) = \frac{1}{|K^j|} \sum_{h \in K^j} (f_h(v, L) - f_h(v, L \setminus \{i, j\})).$$

Component fairness means that deleting the edge between two nodes yields the same average loss in payoff between the divided two components. Herings et al. [5] gives the following result.

Theorem 3.7 (Herings et al. [5])**.** *On the class of cycle-free graph games, the average tree solution AT is the unique solution that satisfies Axiom 3.2 and Axiom 3.6.*

They also discuss the relationship between the average tree solution and the core of graph games. A graph game is said to be *super-additive* if

$$v(S \cup T) \geq v(S) + v(T)$$

holds for all $S, T \in \mathcal{C}(N, L)$ such that $S \cap T = \varnothing$ and $S \cup T \in \mathcal{C}(N, L)$. They show that tree solution $\boldsymbol{x}^r$ is one of the extreme points of the core for any r on super-additive cycle-free graph games. Hence the average tree solution is in the core of the graph game, i.e., $AT(v, L) \in C(v, L)$, while the Myerson value is not always.

4. Parameterized Fairness Axiom and Existence of Solution

This section presents a new axiom on fairness that incorporates the preceding ones on cycle-free graph games and shows the existence and the uniqueness of a solution satisfying component efficiency and our fairness. Concerning the equations appearing in Axiom 3.3 and Axiom 3.6, we focus on the difference of the coefficient of each $f_h(v, L)$. Fairness requires the distribution of loss in payoff between player i and j caused by the deletion of edge $\{i, j\}$. In other words, the loss coefficient is one for i and j and zero for the rest of the players. It is natural to think that all the players should accept a certain amount of loss. Component fairness distributes the loss among the whole players in this sense. The loss coefficients are all identical among the players in the same component, $1/|K^i|$ or $1/|K^j|$ according to the component that he/she belongs to. It seems, however, unfair that a player away from the deleted edge is given the same loss coefficient as a player close to the edge. Therefore we propose here a new fairness axiom under which the coefficient varies according to the distance from the deleted edge by introducing one nonnegative parameter ϵ. We name it ϵ-parameterized fairness.

Axiom 4.1 (ϵ-parameterized fairness)**.** For any $(v, L) \in \mathcal{V} \times \mathcal{M}$, $K \in \mathcal{C}_m(N, L)$ and $\{i, j\} \in L(K)$, it holds that

$$\frac{1}{\sum_{h \in K^i} \epsilon^{\tau^i(h)}} \sum_{h \in K^i} \epsilon^{\tau^i(h)} (f_h(v, L) - f_h(v, L \setminus \{i, j\})) = \frac{1}{\sum_{h \in K^j} \epsilon^{\tau^j(h)}} \sum_{h \in K^j} \epsilon^{\tau^j(h)} (f_h(v, L) - f_h(v, L \setminus \{i, j\})),$$

where $\tau^i(h)$ is defined as follows: for component K^i make an arborescence D^i with node i as the root and set

$$\tau^i(h) := \begin{cases} 0 & \text{if } P'_{D^i}(h) = \varnothing, \\ 1 + \tau^i(k) \text{ with } \{k\} = P'_{D^i}(h) & \text{otherwise.} \end{cases}$$

The function $\tau^i(h)$ is the depth of node h on the arborescence with i as the root, and the loss coefficient of a node is ϵ times that of his/her predecessor. Hence for $\epsilon < 1$ the coefficient of a player is relatively high if he/she is close to the deleted edge, while that of another is low if he/she is far from it. Each coefficient is normalized so that the sum of the coefficients in the same component is to be one. If we set ϵ either to 0 or to 1 we obtain the preceding axioms.

Corollary 4.2. *In Axiom 4.1, if we set $\epsilon = 0$ then we obtain Axiom 3.3* [1] *for $L \in \mathcal{M}$ and if we set $\epsilon = 1$ then we obtain Axiom 3.6.*

The following theorem shows the existence and the uniqueness of the solution satisfying component efficiency and ϵ-parameterized fairness.

Theorem 4.3. *On the class of cycle-free graph games, for any $\epsilon \geq 0$ there exists a unique solution that satisfies Axiom 3.2 and Axiom 4.1.*

Given a cycle-free connected graph (N, L), we create, from the one equation appearing in Axiom 3.2 and the $(n-1)$ equations appearing in Axiom 4.1, the following equality system

$$A\, f(v, L) = \boldsymbol{b}$$

where the matrix $A \in \mathbb{R}^{n \times n}$ consists of the coefficients of $f_h(v, L)$s', and $\boldsymbol{b} \in \mathbb{R}^n$ consists of $v(K)$ and the terms of $f_h(v, L \setminus \{i, j\})$s'. Note that each column of the matrix corresponds to the index of the nodes and each row corresponds to the deleted edge in Axiom 4.1 except for that on component efficiency. For inductive proof we assume that $f(v, L \setminus \{i, j\})$ exists and accordingly $\boldsymbol{b}$ is a constant.[2] We show that the system has a unique solution by proving that

$$A\boldsymbol{c} = \boldsymbol{0}$$

holds for $\boldsymbol{c} \in \mathbb{R}^n$ if and only if $\boldsymbol{c} = \boldsymbol{0}$. We see here an element c_i of $\boldsymbol{c}$ correspond to node i. On the cycle-free graph we choose one arbitrary root node and make an arborescence D as we do to define a tree solution in Section 3. We present two lemmas concerning the elements of $\boldsymbol{c}$ between connected nodes.

Lemma 4.4. *Let j_1 and j_2 be siblings whose parent is i, i.e., $\{i\} = P'_D(j_1) = P'_D(j_2)$. If $c_{j_1} = c_p$ for all $p \in \overline{S}_D(j_1)$ and $c_{j_2} = c_q$ for all $q \in \overline{S}_D(j_1)$, then $c_{j_1} = c_{j_2}$.*

Proof. Let $H = N \setminus (\overline{S}_D(j_1) \cup \overline{S}_D(j_2))$. The corresponding equation for row $\{j_1, i\}$ in $Ac = 0$ is, from $K^{j_1} = \overline{S}_D(j_1)$, $K^i = H \cup \overline{S}_D(j_2)$ and our assumption,

$$1 \times c_{j_1} - \frac{\displaystyle\sum_{h \in H} \epsilon^{\tau^i(h)} c_h + \sum_{q \in \overline{S}_D(j_2)} \epsilon^{\tau^i(q)} c_{j_2}}{\displaystyle\sum_{h \in H} \epsilon^{\tau^i(h)} + \sum_{q \in \overline{S}_D(j_2)} \epsilon^{\tau^i(q)}} = 0,$$

[1] We regard 0^0 as 1.

[2] For $f_h(v, \tilde{L})$ with $|\tilde{L}| = 1$ it clearly holds that $f_h(v, \varnothing) = v(h)$.

which is equivalently

$$(\sum_{h\in H}\epsilon^{\tau^i(h)}+\sum_{q\in\overline{S}_D(j_2)}\epsilon^{\tau^i(q)})c_{j_1}-\sum_{h\in H}\epsilon^{\tau^i(h)}c_h-\sum_{q\in\overline{S}_D(j_2)}\epsilon^{\tau^i(q)}c_{j_2}=0. \tag{4.1}$$

For row $\{j_2,i\}$ we analogously have

$$(\sum_{h\in H}\epsilon^{\tau^i(h)}+\sum_{p\in\overline{S}_D(j_1)}\epsilon^{\tau^i(p)})c_{j_2}-\sum_{h\in H}\epsilon^{\tau^i(h)}c_h-\sum_{p\in\overline{S}_D(j_1)}\epsilon^{\tau^i(p)}c_{j_1}=0. \tag{4.2}$$

By subtracting (4.2) from (4.1) we obtain

$$\begin{aligned}&(\sum_{h\in H}\epsilon^{\tau^i(h)}+\sum_{q\in\overline{S}_D(j_2)}\epsilon^{\tau^i(q)}+\sum_{p\in\overline{S}_D(j_1)}\epsilon^{\tau^i(p)})c_{j_1}\\ -\;&(\sum_{h\in H}\epsilon^{\tau^i(h)}+\sum_{p\in\overline{S}_D(j_1)}\epsilon^{\tau^i(p)}+\sum_{q\in\overline{S}_D(j_2)}\epsilon^{\tau^i(q)})c_{j_2}=0\end{aligned}$$

and see that $c_{j_1}=c_{j_2}$. □

Lemma 4.5. *Let i be a node who has child j, i.e., $j\in S'_D(i)$. If $c_j=c_k$ for all $k\in S_D(i)$, then $c_i=c_j$.*

Proof. When i is the root, the values of the nodes other than i are all c_j by the assumption. The equation for row $\{j,i\}$ is then

$$1\times c_j-\frac{1}{\epsilon^0+\sum_{h\in K^i\setminus\{i\}}\epsilon^{\tau^i(h)}}(\epsilon^0c_i+\sum_{h\in K^i\setminus\{i\}}\epsilon^{\tau^i(h)}c_j)=0$$

and it immediately holds that $c_i=c_j$.

When i is not the root node, let $G=N\setminus\overline{S}_D(i)$ and the equation for row $\{j,i\}$ is, from $K^j=\overline{S}_D(j)$ and $K^i=\{i\}\cup(S_D(i)\setminus\overline{S}_D(j))\cup G$,

$$1\times c_j-\frac{\epsilon^0c_i+\sum_{h\in S_D(i)\setminus\overline{S}_D(j)}\epsilon^{\tau^i(h)}c_j+\sum_{g\in G}\epsilon^{\tau^i(g)}c_g}{\epsilon^0+\sum_{h\in S_D(i)\setminus\overline{S}_D(j)}\epsilon^{\tau^i(h)}+\sum_{g\in G}\epsilon^{\tau^i(g)}}=0,$$

which is equivalently

$$(\epsilon^0+\sum_{g\in G}\epsilon^{\tau^i(g)})c_j-\epsilon^0c_i-\sum_{g\in G}\epsilon^{\tau^i(g)}c_g=0. \tag{4.3}$$

Next let m be the parent of i and consider the equation for row $\{i,m\}$. Since $K^i=\{i\}\cup S_D(i)$ and $K^m=G$ in this case, we see that

$$\frac{\epsilon^0c_i+\sum_{h\in S_D(i)}\epsilon^{\tau^i(h)}c_j}{\epsilon^0+\sum_{h\in S_D(i)}\epsilon^{\tau^i(h)}}-\frac{\sum_{g\in G}\epsilon^{\tau^m(g)}c_g}{\sum_{g\in G}\epsilon^{\tau^m(g)}}=0.$$

Since $\epsilon^{\tau^m(g)}=\epsilon^{\tau^i(g)-1}$ for $g\in G$, it is equivalent to

$$(\sum_{g\in G}\epsilon^{\tau^i(g)-1})(\epsilon^0c_i+\sum_{h\in S_D(i)}\epsilon^{\tau^i(h)}c_j)-(\epsilon^0+\sum_{h\in S_D(i)}\epsilon^{\tau^i(h)})(\sum_{g\in G}\epsilon^{\tau^i(g)-1}c_g)=0. \tag{4.4}$$

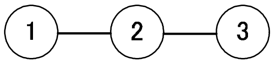

FIGURE 5.1. Example

By row operation $(\epsilon^0 + \sum_{h \in S_D(i)} \epsilon^{\tau^i(h)}) \times (4.3) - \epsilon \times (4.4)$, we have

$$\begin{aligned} &(\epsilon^0 + \sum_{h \in S_D(i)} \epsilon^{\tau^i(h)})((\epsilon^0 + \sum_{g \in G} \epsilon^{\tau^i(g)})c_j - \epsilon^0 c_i) - (\sum_{g \in G} \epsilon^{\tau^i(g)})(\epsilon^0 c_i + \sum_{h \in S_D(i)} \epsilon^{\tau^i(h)} c_j) \\ = \ &(\epsilon^0 + \sum_{h \in S_D(i)} \epsilon^{\tau^i(h)} + \sum_{g \in G} \epsilon^{\tau^i(g)})c_j - (\epsilon^0 + \sum_{h \in S_D(i)} \epsilon^{\tau^i(h)} + \sum_{g \in G} \epsilon^{\tau^i(g)})c_i = 0 \end{aligned}$$

and hence $c_i = c_j$. □

Proof of Theorem 4.9. By applying Lemma 4.4 and Lemma 4.5 alternately from the leaves to the root on the tree, we see that $c_i = c_j$ for all $i, j \in N$. Since the system $A\boldsymbol{c} = \boldsymbol{0}$ includes the equation $\sum_{i \in N} c_i = 0$ corresponding to Axiom 3.2, we have $\boldsymbol{c} = \boldsymbol{0}$ and conclude the proof. □

We call this unique solution the ϵ-parameterized solution. The whole set of the solutions consisting of those for all $\epsilon \geq 0$ incorporates the Myerson value and the average tree solution by Corollary 4.2. Note that, on the class of graph games containing cycles, i.e., $(v, L) \in \mathcal{V} \times (\mathcal{L} \setminus \mathcal{M})$, the uniqueness of a solution satisfying Axiom 3.2 and Axiom 3.3 is, if exists, is shown by Theorem 4.3. We choose, among all the equations consisting of the axioms, component efficiency equations and fairness ones for the edges forming an arbitrary tree. A unique solution exists for such system of equalities and the rest of them do not extend the feasible region.

5. EXAMPLES

We will give an example of graph games and compare the solutions in the preceding sections. The graph consists of $N = \{1, 2, 3\}$ and $L = \{\{1, 2\}, \{2, 3\}\}$, see Figure 5.1. The characteristic function is given by

$$\begin{aligned} &v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, \\ &v(\{1, 2\}) = 0.8, \\ &v(\{2, 3\}) = 0.9, \\ &v(\{1, 2, 3\}) = 1. \end{aligned}$$

Note that this example satisfies super-additivity. The core of this game is given by

$$\begin{aligned} &x_1, x_2, x_3 \geq 0, \\ &x_1 + x_2 \geq 0.8, \\ &x_2 + x_3 \geq 0.9, \\ &x_1 + x_2 + x_3 = 1. \end{aligned}$$

The tree solutions $\boldsymbol{x}^r$ for $r = 1, 2$ and 3, the Myerson value μ and the average tree solution

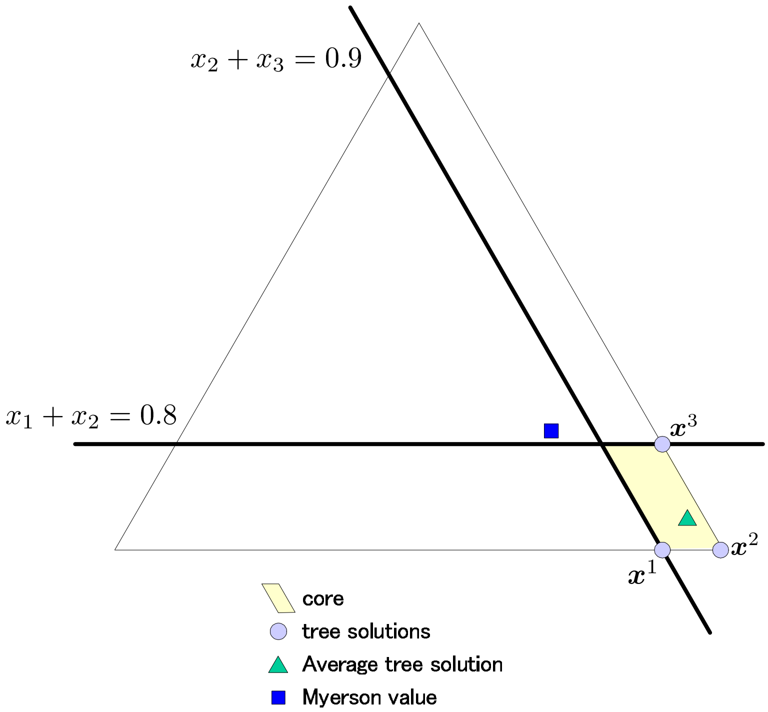

FIGURE 5.2. Core and existing solutions

AT are displayed in Table 5.1. For example, the tree solution $\boldsymbol{x}^3$ is derived from

$$x_1^3 = v(\varnothing) = 0,$$
$$x_2^3 = v(\{1,2\}) - x_1^3 = 0.8,$$
$$x_3^3 = v(\{1,2,3\}) - x_2^3 = 0.2.$$

We also plot the position of these solutions on the triangular graph, see Figure 5.2. We observe that the tree solutions are some extreme points of the core and the average tree solution is in it, while the Myerson value is out of it.

Table 5.2 and Figure 5.3 show the ϵ-parameterized solutions for various value of ϵ between 0 and 1. We obtain the Myerson value for $\epsilon = 0$ and the average tree solution for $\epsilon = 1$. We observe that the ϵ-parameterized solutions lie between the Myerson value and the average tree solution since the ϵ-parameterized solution a solution of linear algebraic equations and therefore is a continuous function of ϵ. In particular our solutions are in the core when $\epsilon \geq (1+\sqrt{237})/34 \approx 0.515$. The average tree solution is in the convex hull of the tree solutions by its nature on super-additive games. In contrast the ϵ-parameterized

TABLE 5.1. Existing solutions

	tree solution				
	$\boldsymbol{x}^1$	$\boldsymbol{x}^2$	$\boldsymbol{x}^3$	μ	AT
node 1	0.100	0.000	0.000	0.167	0.033
node 2	0.900	1.000	0.800	0.617	0.900
node 3	0.000	0.000	0.200	0.217	0.067

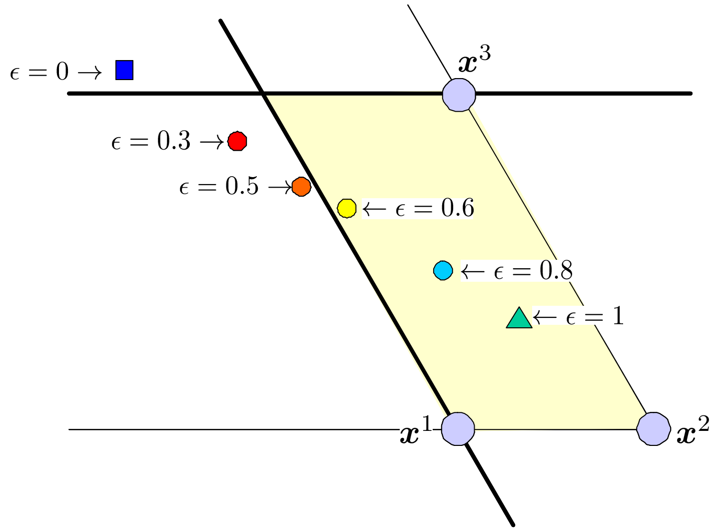

FIGURE 5.3. ϵ-parameterized solutions

solutions can be out of this convex hull but lie in the left half of the core when $(1 + \sqrt{237})/34 \leq \epsilon \leq (3 + 8\sqrt{2})/17 \approx 0.842$. Note that our solutions do not lie on the straight line connecting the Myerson value and the average tree solution. The solution for $\epsilon = 0.5$ is not the midpoint between $\epsilon = 0$ and $\epsilon = 1$.

6. Conclusion

In this paper a new solution for cycle-free graph games is presented in contrast to the Myerson value proposed by Myerson [6] and to the average tree solution by Herings et al. [5]. The Myerson value is characterized by component efficiency and fairness while the average tree solution by component efficiency and component fairness. We propose ϵ-parameterized fairness where ϵ is a parameter on the distribution of loss in payoff among players. It corresponds to fairness when $\epsilon = 0$ and component fairness when $\epsilon = 1$, hence our fairness axiom incorporates the preceding axioms. We show the existence and the uniqueness of a solution satisfying component efficiency and ϵ-parameterized fairness for any nonnegative ϵ. We call such a solution the ϵ-parameterized solution.

A numerical example of a three-person super-additive graph game is exhibited and the existing and our solutions are displayed visually. Starting from the Myerson value which is out of the core in the example, the ϵ-parameterized solution moves on the triangular graph as ϵ increases and reaches the average tree solution lying on the convex hull of some extreme points of the core. It intersects the core and the convex hull of the tree solutions on its way since the ϵ-parameterized solution is a continuous function on ϵ. Our solutions, however, do not lie on the straight line connecting the two preceding solutions.

TABLE 5.2. ϵ-parameterized solutions

ϵ	0.0	0.3	0.5	0.515	0.6	0.8	0.842	1.0
node 1	0.167	0.129	0.102	0.100	0.088	0.061	0.055	0.033
node 2	0.617	0.702	0.758	0.763	0.787	0.843	0.855	0.900
node 3	0.217	0.169	0.140	0.137	0.125	0.096	0.009	0.067

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