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by

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Abstract

A new approach for customer segmentation in the CD market is proposed, where the customers are classified into twelve segments along two axes: "Artist Loyalty" and "Market Sensitivity". The purchasing behavior of individual customers is formulated as a Markov chain in discrete time. The transition structure of the customers across the twelve segments is common. However, the market segmentation is reflected by altering transition probabilities in terms of eight different parameters. The entire market for an artist is then expressed as an independent sum of all such customers. The eight parameters involved are estimated so as to minimize the Euclidian distance between the expected number of CDs sold over 20 weeks calculated from the Markov model and the actual POS data obtained from ten CD shops in urban areas in Japan. The resulting parameter values are then used to understand the brand characteristics of the leading Japanese music artists, suggesting new effective marketing strategies. While the paper is written by focusing on the Japanese music CD market, the proposed approach can be applied to any market in which products are likely to be purchased only once, provided that the data necessary for estimating the underlying parameter values could be obtained.

keywords: CD market; customer segmentation; customer behavior; Markov chain; spectral analysis; brand characteristics

1 Introduction

Application of Markov chains to marketing analysis dates back to 1950s originated by the work of Anderson and Goodman (1957). For the next decade, many researchers developed various Markov models represented by the brand switching models of Harary and Lipstein (1962), Ehrenberg (1965), Hartung and Fisher (1965) and Massy and Morrison (1968). However, such Markov models could not explain real data too well because of the assumption that all customers share a common transition probability matrix. For analyzing customer behavior in marketing, the diffusion models then emerged, see e.g. Bass (1969), Horsky and Simon (1983) and Horsky (1990). The unique feature of the diffusion process approach could be found in that the whole market would be expressed as the aggregated outcome of individual behaviors of infinitely many customers. While microscopic analysis of individual customer behaviors would be lost, rather simple diffusion equations would enable one to conduct a variety of analyses concerning the customer behavior as a whole.

In 1990s, as the market economy matured, many researchers in marketing began to pay attention to market segmentation, see e.g. Moorthy and Png (1992), Raaij and Verhallen (1994), Jedidi et al (1997), Moschis et al (1997), Kucukemiroglu (1999), Peter and Olson (1999), and Wedel and Kamakura (1999), among others. In this context, Markov models revived to some extent, where the pitfall of being unable to fit for real data was resolved by applying different transition probability matrices to different market segments. A Markov model by Poulsen (1990), for example, successfully analyzed brand switching phenomena with excellent data fit by mixing transition probability matrices across different market segments. Other examples of Markov models in marketing can be found in Danaher (1992), Ching et al (2002), Ha et al (2002), Rust et al (2004) and Sumita et al (2006). Sumita et al (2006) analyzed a whole market from the strategic perspective of corporations involved. Markov models have been also applied to capture the customer behavior not only after purchasing but also before purchasing, see e.g. Montgomery et al (2004) and Netzer et al (2008). This paper follows this line of research by developing a Markov chain model for understanding brand characteristics of music artists in the CD market.

A typical CD has an anomalous product life cycle in that as soon as the CD is released, its sales would increase sharply and then converge to 0 in about twenty weeks. This peculiarity concerning the product life cycle of CDs makes it difficult to develop a Markov model that can fit real data. The purpose of this paper is to overcome this difficulty by introducing a new approach for customer segmentation in the CD market, enabling one to obtain the excellent fitness for real data and to comprehend the brand characteristics of music artists in the CD market. While the study relies upon the specific data of the Japanese CD market, the proposed approach could be applied to any market in which products are likely to be purchased only once, provided that the data set necessary for estimating the underlying parameter values could be obtained.

For the music CD market, the customers may be classified into twelve segments along two axes: "Artist Loyalty" consisting of three different levels (Addict, Fan, and Neutral), and "Market Sensitivity" with four different levels (Ultra Sensitive, Sensitive, Normal, and Insensitive). The artist loyalty of a customer to a music artist is measured by the number of CDs of the artist purchased by the customer, while the market sensitivity of the customer concerning the music artist is defined by ranking how soon the CD is purchased by the customer since the release of the CD.

Given a music artist, his/her CD and the segment to which a customer belongs, the purchasing behavior of that customer for that CD is formulated as a Markov chain in discrete time defined on the following five states: Unknown, Known, Interested, Purchased, and Not Purchased. Since a music CD is not likely to be purchased repeatedly, the two states, Purchased and Not Purchased, are absorbing. The transition structure of the customers across the twelve segments is common. However, the market segmentation is reflected by altering transition probabilities in terms of eight different parameters. The entire market for the artist is then expressed as an independent sum of all such customers. The eight parameters are estimated so as to minimize the Euclidian distance between the expected number of CDs sold over 20 weeks calculated from the Markov model and the actual POS data obtained from ten CD shops in urban areas in Japan. The resulting parameter values are then used to understand the brand characteristics of the leading Japanese music artists, thereby providing a basis to suggest new effective marketing strategies.

The structure of this paper is as follows. The data set used in this study is first described in Section 2. A new approach for customer segmentation in the CD market is proposed in Section 3, where the customers are classified into twelve segments along two axes: "Artist Loyalty" and "Market Sensitivity". A Markov chain model in discrete time is developed in Section 4 to describe the purchasing behavior of individual customers in each market segment. The transition structure of the customers across the twelve segments is common. However, the market segmentation is reflected by altering transition probabilities in terms of eight different parameters. Section 5 is devoted to development of algorithmic procedures for estimating the eight parameter values based on spectral analysis of the underlying transition probability matrices. The resulting parameter values are then utilized in Section 6 for analyzing the brand characteristics of the leading Japanese music artists under the study. Based on this analysis, new effective marketing strategies are suggested. Finally, some concluding remarks are given in Section 7.

In this paper, matrices and vectors are indicated by double-underlines ($\underline{\underline{a}}$, $\underline{\underline{b}}$, etc.) and single-underlines ($\underline{x}$, $\underline{y}$, etc.) respectively. The identity matrix is denoted by $\underline{\underline{I}}$ and the zero matrix by $\underline{\underline{0}}$. The vector with all components equal to 1 is written as $\underline{1}$.

2 Data Description

The set of POS data used in this study is provided by a company having ten retail stores for CD sales in urban areas in Japan. The data set consists of purchasing records of the customers who are house-card members of the company, covering the period of 24 months between September 2003 and August 2005. From the original data set, a subset is constructed for the study by extracting the purchasing records of the customers who purchased at least one of POP CDs by Japanese artists during the data period. The top 5 Japanese music artists are then selected based on the total sales monetary amount in the data period for understanding their brand characteristics. Also chosen are their CDs which were released within the data period with more than 20 weeks remained before the end of the period. The selected artists and their respective CDs are listed in Table 1.

Mr.Children, Ketsumeishi and EXILE are music groups. For simplicity, however, each of them is referred to as an artist in this paper.

Table 1: Selected Artists and Their Respective CDs

Artist Name	CD Title
Mr.Children	Palm(S), Sign(S), Sound of Beatitude(A)
Ketsumeishi	Ketsuno Plice 3(A), Tear(S), Bump into You(S), Cherry-Blossom(S)
Ayumi Hamasaki	Memorial Address(A), INSPIRE(S), MY STORY(A), STEP You(S)
Ken Hirai	Ken's Bar(A), Close Your Eyes(S), You Are My Friend(S), Before Affection Deepens (S), SENTIMENTALovers(A)
EXILE	EXILE ENTERTAINMENT(A), PERFECT BEST(A)

S:signle A:album

3 Customer Segmentation Matrix

In this paper, the Japanese music CD market is decomposed into twelve segments along two axes: "Artist Loyalty" and "Market Sensitivity". With respect to "Artist Loyalty", customers are classified into 3 segments: Addict, Fan and Neutral. We define $\mathcal{I} \stackrel{\text{def}}{=} \{Add, Fan, Neu\}$. A customer is defined to be an Addict of a music artist if he/she has purchased three or more CDs of the artist. A customer is a Fan of a music artist if he/she has purchased at least one CD but at most two CDs of the artist. Finally, a customer is a Neutral of a music artist if he/she has not purchased any CD of the artist.

Another axis for market segmentation is "Market Sensitivity" consisting of four categories: Ultra Sensitive, Sensitive, Normal and Insensitive. We define $\mathcal{J} \stackrel{\text{def}}{=} \{Usen, Sen, Norm, Ins\}$. A customer is defined to be Ultra Sensitive if he/she has purchased a CD within 3 days since the release of the CD. He/She is Sensitive (Normal) if a CD is purchased between 4 to 14 days (15 and 28 days) since the release of the CD. Finally, a customer is Insensitive if he/she has not purchased any CD of a music artist or the purchase is done after 29 days or more since the release of the CD. These definitions of

$|\mathcal{I}| \times |\mathcal{J}| = 12$ segments are summarized in Table 2.

Table 2: "Artist Loyalty" and "Market Sensitivity"

Artist Loyalty	Definition
Addict	a customer who purchased 3 or more CDs of an artist
Fan	a customer who purchased 1 CD or 2 CDs of an artist
Neutral	a customer who has not purchased any CD of an artist
Market Sensitivity	**Definition**
Ultra Sensitive	a customer who purchased a CD within 3 days since the release of the CD
Sensitive	a customer who purchased a CD between 4 and 14 days since the release of the CD
Normal	a customer who purchased a CD between 15 and 28 days since the release of the CD
Insensitive	a customer who has not purchased any CD of a music artist or purchased a CD since more than 28 days later since the release of the CD

For each selected artist, the relative sizes of the twelve segments can be estimated from the POS data based on the definitions in Table 2. The resulting table is called the customer segmentation matrix. More specifically, let $N_{IJ}(a)$ be the number of customers of artist a in segment $(I, J) \in \mathcal{I} \times \mathcal{J}$. The customer segmentation matrix for artist a, denoted by $\underline{\underline{Q}}(a)$, is then defined by

$$\underline{\underline{Q}}(a) \stackrel{\text{def}}{=} [Q_{IJ}(a)] \ ; \ Q_{IJ}(a) \stackrel{\text{def}}{=} \frac{N_{IJ}(a)}{N_{POS}} \ , \tag{1}$$

where N_{POS} is the number of customers who have purchased at least one Japanese POP CD during the data period. For the data employed, one has $N_{POS} = 230,933$. The customer segmentation matrices of the top five artists selected in Section 2 are shown in Table 3.

Table 3: Customer Segmentation Matrices of the Five Artists ($N_{POS} = 230,933$)

Mr.Children	$N(Mr.Children) = 16,885$ $N_{POS} - N(Mr.Children) = 214,048$			
$I \setminus J$	$Usen$	Sen	$Norm$	Ins
Add	0.00510	0.00029	0.00004	0.00011
Fan	0.04339	0.01591	0.00390	0.00438
Neu	0.39603	0.26152	0.08946	0.17987

Ketsumeishi	$N(Ketsumeishi) = 13,408$ $N_{POS} - N(Ketsumeishi) = 217,525$			
$I \setminus J$	$Usen$	Sen	$Norm$	Ins
Add	0.00220	0.00036	0.00007	0.00014
Fan	0.02936	0.01634	0.00450	0.00508
Neu	0.41296	0.26102	0.08882	0.17914

Ken Hirai	$N(Ken\ Hirai) = 11,816$ $N_{POS} - N(Ken\ Hirai) = 219,117$			
$I \setminus J$	$Usen$	Sen	$Norm$	Ins
Add	0.00133	0.00014	0.00003	0.00005
Fan	0.02566	0.01436	0.00440	0.00520
Neu	0.41753	0.26322	0.08897	0.17912

Ayumi Hamasaki	$N(Ayumi\ Hamasaki) = 8,828$ $N_{POS} - N(Ayumi\ Hamasaki) = 222,105$			
$I \setminus J$	$Usen$	Sen	$Norm$	Ins
Add	0.00379	0.00032	0.00003	0.00005
Fan	0.02101	0.00913	0.00227	0.00163
Neu	0.41972	0.26827	0.09110	0.18269

EXILE	$N(EXILE) = 8,338$ $N_{POS} - N(EXILE) = 222,595$			
$I \setminus J$	$Usen$	Sen	$Norm$	Ins
Add	0.00168	0.00012	0.00002	0.00001
Fan	0.01960	0.01048	0.00209	0.00210
Neu	0.42324	0.26712	0.09129	0.18225

Note. $N(a)$ denotes the number of customers who purchased at least one CD of artist a on the POS data

4 Development of a Markov Chain Model for Describing Customer Behaviors in the Music CD Market

In this section, a stochastic model is developed for describing the purchasing behavior of individual customers for a CD of a music artist. More specifically, let $\{J(k) : k = 0, 1, 2, \cdots\}$ be a Markov chain in discrete time defined on the following five states: (1) Unknown, (2) Known, (3) Interested, (4) Purchased, and (5) Not Purchased. The transition structure of the Markov chain is depicted in Figure 1, where the time unit is taken to be a week. A customer in segment $(I, J) \in \mathcal{I} \times \mathcal{J}$ may not know about the release of the CD under consideration in the beginning. With probability $\alpha_{1:IJ}$, the CD remains to be unknown to the customer, and the customer comes to know the CD with probability $\beta_{1:IJ} = 1 - \alpha_{1:IJ}$. The self-transition probabilities $\alpha_{i:IJ} (i = 2, 3)$ represent a degree of hesitation of the customer for advancing his/her interest toward the decision of purchasing or not purchasing. The probabilities $\beta_{i:IJ}(i = 2, 3)$ describe a degree of inclination of the customer toward purchasing and the customer decides not to buy the CD with probabilities $\gamma_{i:IJ}(i = 2, 3)$. Since a music CD is not likely to be purchased repeatedly, the two states, Purchased and Not Purchased, are absorbing.

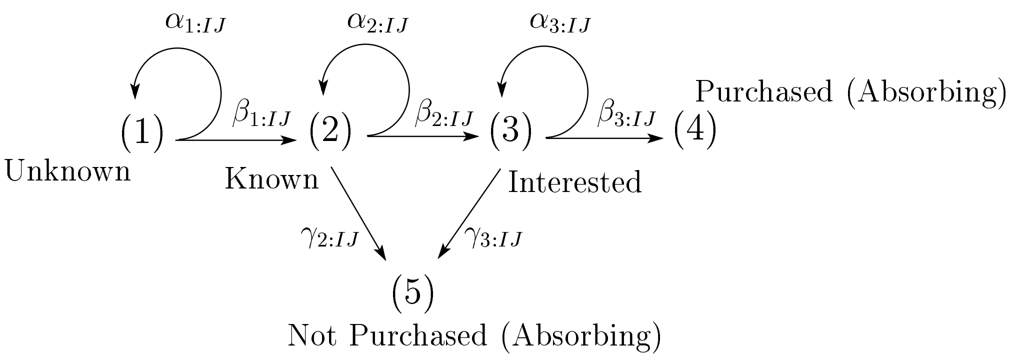

Figure 1: State Transition Diagram

The transition structure of the customers across the twelve segments is assumed to be common. However, the market segmentation is reflected by altering transition probabilities in terms of eight different parameters and the initial state in the following manner. For those customers who belong to the customer segment $(I, J) \in \mathcal{I} \times \mathcal{J}$, we first define $\beta_{1:IJ} = p_{I:J}$, which represents a chance for a customer to come to know the CD under consideration during a week. It is assumed that this probability is equal to the probability of encountering an opportunity to deepen the interest of the customer so that $\beta_{2:IJ} = \beta_{1:IJ}$. However, making the final decision of purchasing it may demand more careful consideration and we reduce $\beta_{3:IJ}$ by $\beta_{3:IJ} = \beta_{2:IJ} \cdot \zeta$ where $0 \leqslant \zeta \leqslant 1$. For state 1, one has $\alpha_{1:IJ} = 1 - \beta_{1:IJ}$. For states 2 and 3, the remaining probabilities $1 - \beta_{i:IJ}$ for $i = 2, 3$ are split between hesitation with probability r_I and making the decision of not purchasing it with probability $1 - r_I$.

We assume that $p_{I:J} = p_I$ for $I \in \mathcal{I} = \{Add, Fan, Neu\}$ and $J \in \mathcal{J} \setminus \{Usen\}$. For $J = Usen$, this probability reflecting the inclination toward purchasing is increased as $p_{I:Usen} = p_I + \xi$ where $0 \leqslant \xi \leqslant 1 - p_I$. In addition, those customers in (I, Ins) segment for $I \in \mathcal{I}$ are assumed to start in state 1. The initial state of those customers in $(I, Norm)$ segment for $I \in \mathcal{I}$ are set to be state 2, and those in $(I, \{Usen\} \cup \{Sen\})$ for $I \in \mathcal{I}$ are assumed to start at state 3. Consequently, through the eight parameters $p_{Add}, p_{Fan}, p_{Neu}, r_{Add}, r_{Fan}, r_{Neu}, \zeta$ and ξ together with the way the initial state is set, the stochastic behavior of those customers in $(I, J) \in \mathcal{I} \times \mathcal{J}$ can be set differently. This structure is summarized in Tables 4 and 5.

Table 4: $\alpha_{i:IJ}$, $\beta_{i:IJ}$ and $\gamma_{i:IJ}$

	$\alpha_{1:IJ}$	$\alpha_{2:IJ}$	$\alpha_{3:IJ}$	$\beta_{1:IJ}$	$\beta_{2:IJ}$	$\beta_{3:IJ}$
Addict	$1 - p_{Add:J}$	$r_{Add}(1 - p_{Add:J})$	$r_{Add}(1 - p_{Add:J} \cdot \zeta)$	$p_{Add:J}$	$p_{Add:J}$	$p_{Add:J} \cdot \zeta$
Fan	$1 - p_{Fan:J}$	$r_{Fan}(1 - p_{Fan:J})$	$r_{Fan}(1 - p_{Fan:J} \cdot \zeta)$	$p_{Fan:J}$	$p_{Fan:J}$	$p_{Fan:J} \cdot \zeta$
Neutral	$1 - p_{Neu:J}$	$r_{Neu}(1 - p_{Neu:J})$	$r_{Neu}(1 - p_{Neu:J} \cdot \zeta)$	$p_{Neu:J}$	$p_{Neu:J}$	$p_{Neu:J} \cdot \zeta$

	$\gamma_{2:IJ}$	$\gamma_{3:IJ}$
Addict	$(1 - r_{Add})(1 - p_{Add:J})$	$(1 - r_{Add})(1 - p_{Add:J} \cdot \zeta)$
Fan	$(1 - r_{Fan})(1 - p_{Fan:J})$	$(1 - r_{Fan})(1 - p_{Fan:J} \cdot \zeta)$
Neutral	$(1 - r_{Neu})(1 - p_{Neu:J})$	$(1 - r_{Neu})(1 - p_{Neu:J} \cdot \zeta)$

Note 1. $p_{I:J} = p_I$ for $J \in \mathcal{J} \setminus \{Usen\}$ and $p_{I:Usen} = p_I + \xi$
Note 2. $p_{Neu:J} \leqslant p_{Fan:J} \leqslant p_{Add:J}$ and $r_{Neu} \leqslant r_{Fan} \leqslant r_{Add}$

Table 5: The Initial State of the Customer for $J \in \mathcal{J}$

J	$Usen$	Sen	Nor	Ins
Initial State	3(Interested)	3(Interested)	2(Known)	1(Unknown)

Given a music artist, his/her CD and the segment to which a customer belongs, the Markov chain model described above enables one to capture the state of the customer at the beginning of the τ-th week. The entire market for the artist is then expressed as an independent sum of the Markov chains corresponding to all the CDs of that artist over all such customers under consideration. In the next section, we derive the time dependent state probability vector of the resulting Markov chain via spectral analysis of the underlying transition probability matrix depicted in Figure 1. This in turn enables one to estimate the eight parameters so as to minimize the Euclidian distance between the expected number of CDs sold over 20 weeks calculated from the Markov model and the actual POS data. The resulting parameter values are then used to understand the brand characteristics of the leading Japanese music artists, thereby providing a basis to suggest new effective marketing strategies.

5 Estimation of Eight Parameter Values via Spectral Analysis of Transition Probability Matrix

The Markov chain in discrete time described in Section 4 is governed by the one step transition probability matrix $\underline{\underline{a}}_{IJ}$ given by

$$\underline{\underline{a}}_{IJ} = \begin{bmatrix} \alpha_{1:IJ} & \beta_{1:IJ} & 0 & 0 & 0 \\ 0 & \alpha_{2:IJ} & \beta_{2:IJ} & 0 & \gamma_{2:IJ} \\ 0 & 0 & \alpha_{3:IJ} & \beta_{3:IJ} & \gamma_{3:IJ} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{2}$$

Since this matrix is upper triangular, its eigenvalues are given by $\alpha_{1:IJ}$, $\alpha_{2:IJ}$, and $\alpha_{3:IJ}$ which are simple, and 1 with multiplicity two. For $M \stackrel{\text{def}}{=} \{1, 2, 3, 4, 5\}$, based on the spectral analysis outlined in Appendix, one has

$$\underline{\underline{a}}_{IJ} = \sum_{m \in M} \eta_{IJ-m} \underline{\underline{\widehat{J}}}_{IJ-m} \ ; \ \eta_{IJ-m} \stackrel{\text{def}}{=} \begin{cases} \alpha_{m:IJ} & \text{if } \ m = 1, 2 \text{ or } 3 \\ 1 & \text{if } \ m = 4 \text{ or } 5 \end{cases} , \tag{3}$$

where $\underline{\underline{\widehat{J}}}_{IJ-m}$ are as specified in Appendix. Since $\underline{\underline{\widehat{J}}}_{IJ-m}$ are idempotent and matrix orthogonal to each other, it follows that

$$\underline{\underline{a}}_{IJ}^{\tau} = \sum_{m \in M} \eta_{IJ-m}^{\tau} \underline{\underline{\widehat{J}}}_{IJ-m} \quad (\tau = 0, 1, 2, \cdots) \ , \tag{4}$$

where $\underline{\underline{a}}^0_{IJ} = \sum_{m\in M} \underline{\underline{\widehat{J}}}_{IJ-m} = \underline{\underline{I}}$.

Given $(I, J) \in \mathcal{I} \times \mathcal{J}$, the matrix $\underline{\underline{a}}_{IJ}$ can be determined in terms of the eight parameters in Table 4. Furthermore, based on Table 5, the initial state probability vector of a customer in the (I, J) segment is given by

$$\underline{p}^{\top}_{IJ}(0) = \begin{cases} (1,0,0,0,0) & \text{if } J = Ins \\ (0,1,0,0,0) & \text{if } J = Nor \\ (0,0,1,0,0) & \text{if } J = Sen \text{ or } Usen \end{cases} . \tag{5}$$

Consequently, the corresponding state probability vector at week τ, denoted by $\underline{p}_{IJ}(\tau) \stackrel{\text{def}}{=} [p_{IJ:1}(\tau), \cdots, p_{IJ:5}(\tau)]^{\top}$, can be computed from (4) and (5) as

$$\underline{p}^{\top}_{IJ}(\tau) = \underline{p}^{\top}_{IJ}(0)\underline{\underline{a}}^{\tau}_{IJ} = \sum_{m\in M} \eta^{\tau}_{IJ-m}\, \underline{p}^{\top}_{IJ}(0)\underline{\underline{\widehat{J}}}_{IJ-m}\ (\tau = 0, 1, \cdots, 20) . \tag{6}$$

Let $S_{IJ:a}(\tau)$ be the expected number of CDs of artist a purchased by those customers in the (I, J) segment during week τ, and define

$$S_a(\tau) \stackrel{\text{def}}{=} \sum_{(I,J)\in\mathcal{I}\times\mathcal{J}} S_{IJ:a}(\tau) . \tag{7}$$

Since state 4 corresponding to the decision of purchasing is absorbing as shown in Figure 1, it follows that

$$S_{IJ:a}(\tau) = N_{POS} \cdot Q_{IJ}(a)\{p_{IJ:4}(\tau) - p_{IJ:4}(\tau - 1)\} ,\ \tau = 1, \cdots, 20 . \tag{8}$$

From (6), (7) and (8), one finds that

$$\begin{aligned} & S_a(\tau + 1) \\ = \;& N_{POS} \sum_{I\in\mathcal{I}} \Big[Q_{I,Usen}(a)[r_I\{1 - (p_I + \xi)\zeta\}]^{\tau}(p_I + \xi)\zeta + Q_{I,Sen}(a)\{r_I(1 - p_I\zeta)\}^{\tau} p_I\zeta \\ & + Q_{I,Norm}(a)\frac{p_I\zeta r_I^{\tau-1}}{1-\zeta}\{(1 - p_I\zeta)^{\tau} - (1 - p_I)^{\tau}\} + Q_{I,Ins}(a)\frac{p_I^2\zeta}{(1 - r_I)(1 - \zeta)\chi_I} \\ & \cdot\Big[p_I(1 - \zeta)(1 - p_I)^{\tau-1} + \chi_I\{r_I(1 - p_I)\}^{\tau-1} - (1 - r_I)r_I^{\tau-1}(1 - p_I\zeta)^{\tau}\Big]\Big] , \end{aligned} \tag{9}$$

where

$$\chi_I \stackrel{\text{def}}{=} (1 - r_I) - p_I(1 - r_I\zeta) \ . \tag{10}$$

We are now in a position to estimate the values of the eight parameters so as to minimize the Euclidian distance between the expected number of CDs sold over 20 weeks calculated from the Markov model and the actual POS data. More specifically, let $d_a(\tau)$ be the actual sales of CDs per music piece by artist a in week τ $(\tau = 1, \cdots, 20)$ obtained from the POS data and define

$$D_a(20) \stackrel{\text{def}}{=} \sum_{\tau=1}^{20} d_a(\tau) \ . \tag{11}$$

Of interest is to estimate the values of the eight parameters by minimizing

$$\varepsilon(a) \stackrel{\text{def}}{=} \sqrt{\frac{\sum_{M=1}^{20}\{\sum_{\tau=1}^{M} S_a(\tau) - \sum_{\tau=1}^{M} d_a(\tau)\}^2}{\{D_a(20)\}^2}} \ . \tag{12}$$

By differentiating $\varepsilon^2(a)$ with respect to $x \in \{p_{Add}, p_{Fan}, p_{Neu}, r_{Add}, r_{Fan}, r_{Neu}, \zeta, \xi\}$, it can be readily seen that

$$\frac{\partial}{\partial x}\varepsilon(a)^2 = \frac{2}{\{D_a(20)\}^2} \sum_{M=1}^{20} \Big[\sum_{\tau=1}^{M} \{S_a(\tau) - d_a(\tau)\} \Big] \Big\{ \sum_{\tau=1}^{M} \frac{\partial}{\partial x} S_a(\tau) \Big\} \ , \tag{13}$$

where $\frac{\partial}{\partial x} S_a(\tau)$ for $x \in \{p_{Add}, p_{Fan}, p_{Neu}, r_{Add}, r_{Fan}, r_{Neu}, \zeta, \xi\}$ are given in Appendix. Using MATLAB as a programming language, $(p^*_{Add}, p^*_{Fan}, p^*_{Neu}, r^*_{Add}, r^*_{Fan}, r^*_{Neu}, \zeta^*, \xi^*)$ can be determined by solving $\frac{\partial}{\partial x}\varepsilon(a)^2 = 0$ for $x \in \{p_{Add}, p_{Fan}, p_{Neu}, r_{Add}, r_{Fan}, r_{Neu}, \zeta, \xi\}$. Summarized in Tables 6 and 7 are the estimated eight parameter values for each of the five artists. Figure 2 demonstrates the excellent fit between the expected number of CDs sold over 20 weeks calculated from the Markov model and the actual POS data.

Table 6: Estimated Values of Eight Parameters and Resulting Transition Probabilities Based on the POS Data for $J = Usen$

Artist Name (# of CDs Sold per Music Piece)											
Mr.Children	Parameter		p^*_{Add}	p^*_{Fan}	p^*_{Neu}	r^*_{Add}	r^*_{Fan}	r^*_{Neu}	ζ^*	ξ^*	ε^*
(4780)	Values		0.8921	0.4212	0.4202	0.9877	0.8135	0.0653	0.0410	0.0918	0.0030
	Population		$Q_{I.}$	$\alpha^*_{1:IJ}$	$\alpha^*_{2:IJ}$	$\alpha^*_{3:IJ}$	$\beta^*_{1:IJ}$	$\beta^*_{2:IJ}$	$\beta^*_{3:IJ}$	$\gamma^*_{2:IJ}$	$\gamma^*_{3:IJ}$
	1280	Add	0.00554	0.0161	0.0159	0.9479	0.9839	0.9839	0.0403	0.0002	0.0118
	15605	Fan	0.06757	0.4870	0.3962	0.7964	0.5130	0.5130	0.0210	0.0908	0.1826
	214048	Neu	0.92688	0.4880	0.0319	0.0639	0.5120	0.5120	0.0210	0.4561	0.9151
	230933	$W.A.$	1	0.4853	0.0564	0.1183	0.5147	0.5147	0.0211	0.4289	0.8606
Ketsumeishi	Parameter		p^*_{Add}	p^*_{Fan}	p^*_{Neu}	r^*_{Add}	r^*_{Fan}	r^*_{Neu}	ζ^*	ξ^*	ε^*
(1642)	Values		0.4443	0.4433	0.0853	0.9245	0.8337	0.2691	0.0258	0.0687	0.0131
	Population		$Q_{I.}$	$\alpha^*_{1:IJ}$	$\alpha^*_{2:IJ}$	$\alpha^*_{3:IJ}$	$\beta^*_{1:IJ}$	$\beta^*_{2:IJ}$	$\beta^*_{3:IJ}$	$\gamma^*_{2:IJ}$	$\gamma^*_{3:IJ}$
	642	Add	0.00278	0.4870	0.4502	0.9123	0.5130	0.5130	0.0132	0.0368	0.0745
	12766	Fan	0.05528	0.4880	0.4068	0.8227	0.5120	0.5120	0.0132	0.0812	0.1641
	217525	Neu	0.94194	0.8460	0.2277	0.2680	0.1540	0.1540	0.0040	0.6183	0.7280
	230933	$W.A.$	1	0.8252	0.2382	0.3005	0.1748	0.1748	0.0045	0.5870	0.6950
Ken Hirai	Parameter		p^*_{Add}	p^*_{Fan}	p^*_{Neu}	r^*_{Add}	r^*_{Fan}	r^*_{Neu}	ζ^*	ξ^*	ε^*
(2497)	Values		0.9900	0.9890	0.3485	0.8379	0.8369	0.2992	0.0159	0.0010	0.0180
	Population		$Q_{I.}$	$\alpha^*_{1:IJ}$	$\alpha^*_{2:IJ}$	$\alpha^*_{3:IJ}$	$\beta^*_{1:IJ}$	$\beta^*_{2:IJ}$	$\beta^*_{3:IJ}$	$\gamma^*_{2:IJ}$	$\gamma^*_{3:IJ}$
	359	Add	0.00155	0.0090	0.0075	0.8247	0.9910	0.9910	0.0158	0.0015	0.1595
	11457	Fan	0.04961	0.0100	0.0084	0.8237	0.9900	0.9900	0.0157	0.0016	0.1605
	219117	Neu	0.94883	0.6505	0.1946	0.2975	0.3495	0.3495	0.0056	0.4559	0.6969
	230933	$W.A.$	1	0.6177	0.1851	0.3245	0.3823	0.3823	0.0061	0.4326	0.6695
Ayumi Hamasaki	Parameter		p^*_{Add}	p^*_{Fan}	p^*_{Neu}	r^*_{Add}	r^*_{Fan}	r^*_{Neu}	ζ^*	ξ^*	ε^*
(2059)	Values		0.9900	0.4341	0.4331	0.9355	0.2144	0.2134	0.0174	0.0010	0.0147
	Population		$Q_{I.}$	$\alpha^*_{1:IJ}$	$\alpha^*_{2:IJ}$	$\alpha^*_{3:IJ}$	$\beta^*_{1:IJ}$	$\beta^*_{2:IJ}$	$\beta^*_{3:IJ}$	$\gamma^*_{2:IJ}$	$\gamma^*_{3:IJ}$
	966	Add	0.00418	0.0090	0.0084	0.9194	0.9910	0.9910	0.0172	0.0006	0.0634
	7862	Fan	0.03404	0.5649	0.1211	0.2128	0.4351	0.4351	0.0076	0.4438	0.7797
	222105	Neu	0.96177	0.5659	0.1208	0.2118	0.4341	0.4341	0.0076	0.4451	0.7807
	230933	$W.A.$	1	0.5635	0.1203	0.2148	0.4365	0.4365	0.0076	0.4432	0.7776
EXILE	Parameter		p^*_{Add}	p^*_{Fan}	p^*_{Neu}	r^*_{Add}	r^*_{Fan}	r^*_{Neu}	ζ^*	ξ^*	ε^*
(2736)	Values		0.7630	0.4265	0.4255	0.9871	0.8188	0.0649	0.0402	0.0990	0.0202
	Population		$Q_{I.}$	$\alpha^*_{1:IJ}$	$\alpha^*_{2:IJ}$	$\alpha^*_{3:IJ}$	$\beta^*_{1:IJ}$	$\beta^*_{2:IJ}$	$\beta^*_{3:IJ}$	$\gamma^*_{2:IJ}$	$\gamma^*_{3:IJ}$
	424	Add	0.00184	0.1380	0.1362	0.9529	0.8620	0.8620	0.0347	0.0018	0.0125
	7914	Fan	0.03427	0.4745	0.3885	0.8015	0.5255	0.5255	0.0211	0.0860	0.1774
	222595	Neu	0.96389	0.4755	0.0309	0.0635	0.5245	0.5245	0.0211	0.4446	0.9154
	230933	$W.A.$	1	0.4748	0.0433	0.0905	0.5252	0.5252	0.0211	0.4315	0.8884

Note. $Q_{I.}$ describes the portion of the supporters of an artist in the entire population with $I = Add, Fan, Neu$. $W.A.$ means the weighted average over $I = Add, Fan, Neu$.

Table 7: Estimated Values of Eight Parameters and Resulting Transition Probabilities Based on the POS Data for $J \neq Usen$

Artist Name (# of CDs Sold per Music Piece)											
Mr.Children	Parameter		p^*_{Add}	p^*_{Fan}	p^*_{Neu}	r^*_{Add}	r^*_{Fan}	r^*_{Neu}	ζ^*	ξ^*	ε^*
(4780)	Values		0.8921	0.4212	0.4202	0.9877	0.8135	0.0653	0.0410	0.0918	0.0030
	Population		$Q_{I\cdot}$	$\alpha^*_{1:IJ}$	$\alpha^*_{2:IJ}$	$\alpha^*_{3:IJ}$	$\beta^*_{1:IJ}$	$\beta^*_{2:IJ}$	$\beta^*_{3:IJ}$	$\gamma^*_{2:IJ}$	$\gamma^*_{3:IJ}$
	1280	Add	0.00554	0.1079	0.1066	0.9516	0.8921	0.8921	0.0366	0.0013	0.0119
	15605	Fan	0.06757	0.5788	0.4709	0.7995	0.4212	0.4212	0.0173	0.1079	0.1833
	214048	Neu	0.92688	0.5798	0.0379	0.0642	0.4202	0.4202	0.0172	0.5419	0.9186
	230933	$W.A.$	1	0.5771	0.0675	0.1188	0.4229	0.4229	0.0173	0.5096	0.8639
Ketsumeishi	Parameter		p^*_{Add}	p^*_{Fan}	p^*_{Neu}	r^*_{Add}	r^*_{Fan}	r^*_{Neu}	ζ^*	ξ^*	ε^*
(1642)	Values		0.4443	0.4433	0.0853	0.9245	0.8337	0.2691	0.0258	0.0687	0.0131
	Population		$Q_{I\cdot}$	$\alpha^*_{1:IJ}$	$\alpha^*_{2:IJ}$	$\alpha^*_{3:IJ}$	$\beta^*_{1:IJ}$	$\beta^*_{2:IJ}$	$\beta^*_{3:IJ}$	$\gamma^*_{2:IJ}$	$\gamma^*_{3:IJ}$
	642	Add	0.00278	0.5557	0.5137	0.9139	0.4443	0.4443	0.0115	0.0420	0.0746
	12766	Fan	0.05528	0.5567	0.4641	0.8242	0.4433	0.4433	0.0114	0.0926	0.1644
	217525	Neu	0.94194	0.9147	0.2461	0.2685	0.0853	0.0853	0.0022	0.6686	0.7293
	230933	$W.A.$	1	0.8939	0.2589	0.3010	0.1061	0.1061	0.0027	0.6350	0.6962
Ken Hirai	Parameter		p^*_{Add}	p^*_{Fan}	p^*_{Neu}	r^*_{Add}	r^*_{Fan}	r^*_{Neu}	ζ^*	ξ^*	ε^*
(2497)	Values		0.9900	0.9890	0.3485	0.8379	0.8369	0.2992	0.0159	0.0010	0.0180
	Population		$Q_{I\cdot}$	$\alpha^*_{1:IJ}$	$\alpha^*_{2:IJ}$	$\alpha^*_{3:IJ}$	$\beta^*_{1:IJ}$	$\beta^*_{2:IJ}$	$\beta^*_{3:IJ}$	$\gamma^*_{2:IJ}$	$\gamma^*_{3:IJ}$
	359	Add	0.00155	0.0100	0.0084	0.8247	0.9900	0.9900	0.0157	0.0016	0.1595
	11457	Fan	0.04961	0.0110	0.0092	0.8237	0.9890	0.9890	0.0157	0.0018	0.1605
	219117	Neu	0.94883	0.6515	0.1949	0.2975	0.3485	0.3485	0.0055	0.4566	0.6969
	230933	$W.A.$	1	0.6187	0.1854	0.3245	0.3813	0.3813	0.0061	0.4333	0.6695
Ayumi Hamasaki	Parameter		p^*_{Add}	p^*_{Fan}	p^*_{Neu}	r^*_{Add}	r^*_{Fan}	r^*_{Neu}	ζ^*	ξ^*	ε^*
(2059)	Values		0.9900	0.4341	0.4331	0.9355	0.2144	0.2134	0.0174	0.0010	0.0147
	Population		$Q_{I\cdot}$	$\alpha^*_{1:IJ}$	$\alpha^*_{2:IJ}$	$\alpha^*_{3:IJ}$	$\beta^*_{1:IJ}$	$\beta^*_{2:IJ}$	$\beta^*_{3:IJ}$	$\gamma^*_{2:IJ}$	$\gamma^*_{3:IJ}$
	966	Add	0.00418	0.0100	0.0094	0.9194	0.9900	0.9900	0.0172	0.0006	0.0634
	7862	Fan	0.03404	0.5659	0.1213	0.2128	0.4341	0.4341	0.0076	0.4446	0.7797
	222105	Neu	0.96177	0.5669	0.1210	0.2118	0.4331	0.4331	0.0075	0.4459	0.7807
	230933	$W.A.$	1	0.5645	0.1205	0.2148	0.4355	0.4355	0.0076	0.4440	0.7776
EXILE	Parameter		p^*_{Add}	p^*_{Fan}	p^*_{Neu}	r^*_{Add}	r^*_{Fan}	r^*_{Neu}	ζ^*	ξ^*	ε^*
(2736)	Values		0.7630	0.4265	0.4255	0.9871	0.8188	0.0649	0.0402	0.0990	0.0202
	Population		$Q_{I\cdot}$	$\alpha^*_{1:IJ}$	$\alpha^*_{2:IJ}$	$\alpha^*_{3:IJ}$	$\beta^*_{1:IJ}$	$\beta^*_{2:IJ}$	$\beta^*_{3:IJ}$	$\gamma^*_{2:IJ}$	$\gamma^*_{3:IJ}$
	424	Add	0.00184	0.2370	0.2339	0.9568	0.7630	0.7630	0.0307	0.0031	0.0125
	7914	Fan	0.03427	0.5735	0.4696	0.8048	0.4265	0.4265	0.0171	0.1039	0.1781
	222595	Neu	0.96389	0.5745	0.0373	0.0638	0.4255	0.4255	0.0171	0.5372	0.9191
	230933	$W.A.$	1	0.5738	0.0525	0.0908	0.4262	0.4262	0.0171	0.5214	0.8920

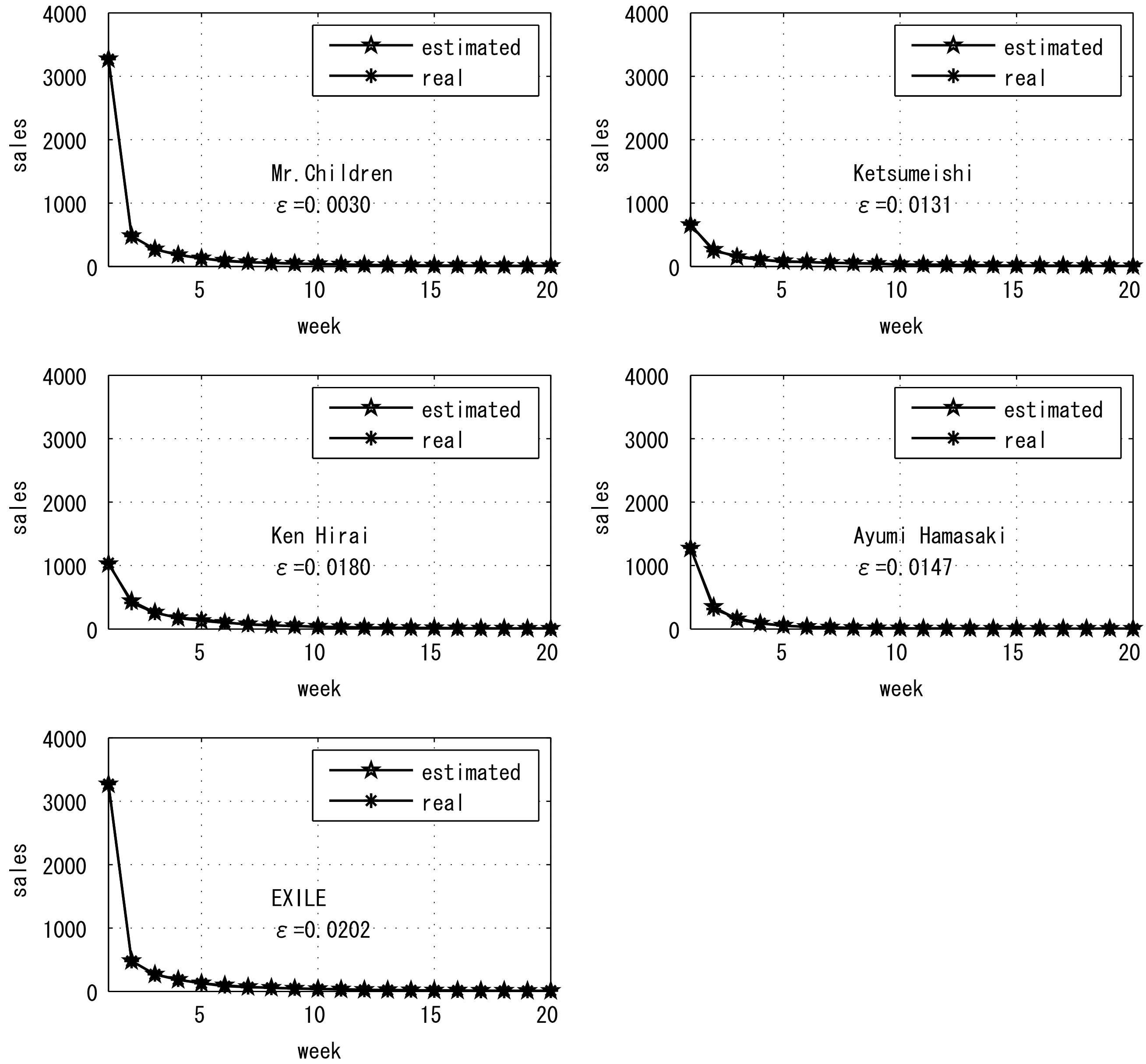

Figure 2: The Expected Number of CDs Sold over 20 Weeks Based on the Markov Model and the Actual POS Data

6 Assessment of Brand Characteristics of the Five Artists

In this section, we evaluate the brand characteristics of the five Japanese music artists selected in Section 2. Their strengths and weaknesses can be captured based on the eight estimated parameter values obtained in Section 5. For this purpose, it should be noted that the strong presence of a music artist in the CD market may be characterized by the following three factors:

1) the overall brand power in terms of the total number of CDs sold ;

2) the rapid sales growth upon release of a CD ;

3) the fat tail of the sales records over a long time period, i.e. the CD continues to sell for a long time although the sales speed may not be necessarily high.

In order to establish certain performance indicators for capturing these factors, we first consider a customer whose purchasing behavior can be characterized by the Markov chain in discrete time on $\mathcal{N} \stackrel{\text{def}}{=} \{1, 2, 3, 4, 5\}$ governed by $\underline{\underline{a}}$ as introduced in Section 4, where states 4 and 5 are absorbing with the former representing the decision of purchasing and the latter describing that of not purchasing. We recall that the state probability vector $\underline{p}^{\top}(\tau)$ at time τ is available in closed form through the spectral analysis of $\underline{\underline{a}}$ discussed in Appendix. The entire market consists of N such customers who behave independently of each other.

Let S be the total number of CDs sold within its product lifecycle. In order to find the probability distribution of S, it is necessary to evaluate the absorption probability $\rho_{j:4}$ that a customer starting at state j eventually reaches state 4 for $j = 1, 2, 3$. Let

$G \stackrel{\text{def}}{=} \{1, 2, 3\}$ and define the submatrix $\underline{\underline{a}}_{GG} \stackrel{\text{def}}{=} [a_{ij}]_{i,\ j \in G}$. One finds that

$$\begin{bmatrix} \rho_{1:4} \\ \rho_{2:4} \\ \rho_{3:4} \end{bmatrix} = [\underline{\underline{I}}_{GG} - \underline{\underline{a}}_{GG}]^{-1} \begin{bmatrix} 0 \\ 0 \\ \beta_3 \end{bmatrix} . \tag{14}$$

Given that the initial state of the customers under consideration is state j, it can be seen that S has the binomial distribution characterized by the probability generating function

$$\pi(u) \stackrel{\text{def}}{=} \mathrm{E}[u^S] = \{1 - \rho_{j:4} + \rho_{j:4} u\}^N . \tag{15}$$

For capturing the overall brand power, of interest is then to evaluate

$$\psi \stackrel{\text{def}}{=} \frac{\mathrm{E}[S]}{N} \quad ; \quad \kappa \stackrel{\text{def}}{=} \mathrm{P}[S \geqslant K] . \tag{16}$$

The growth speed of sales of a CD upon its release can be described by the number of CDs purchased by time τ. This random variable, denoted by $S_H(\tau)$ where H stands for "Head", clearly has the binomial distribution with the associated probability generating function given by

$$\pi_H(u, \tau) \stackrel{\text{def}}{=} \mathrm{E}[u^{S_H(\tau)}] = \{1 - p_4(\tau) + p_4(\tau) u\}^N . \tag{17}$$

In parallel with (16), we define

$$\psi_H(\tau) \stackrel{\text{def}}{=} \frac{\mathrm{E}[S_H(\tau)]}{N} \quad ; \quad \kappa_H(\tau) \stackrel{\text{def}}{=} \mathrm{P}[S_H(\tau) \geqslant K_H] . \tag{18}$$

As for expressing the strength of having a fat tail of sales, let $\underline{V}^\top(\tau) \stackrel{\text{def}}{=} [V_1(\tau),\ V_2(\tau),\ V_3(\tau)]$ be a stochastic vector describing the purchasing behavior of a customer defined by, for $j = 1, 2, 3$,

$$V_j(\tau) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if the customer is in state } j \text{ at time } \tau \text{ and would eventually} \\ & \text{purchase the CD after } \tau \\ 0 & \text{else} \end{cases} . \tag{19}$$

Let $\underline{u}^\top \stackrel{\text{def}}{=} [u_1,\ u_2,\ u_3]$ and define the associated multivariate probability generating function by

$$\varphi(\underline{u},\tau) \stackrel{\text{def}}{=} \mathrm{E}\big[\underline{u}^{\underline{V}(\tau)}\big] = \mathrm{E}\Big[\prod_{j=1}^{3} u_j^{V_j(\tau)}\Big] \ . \tag{20}$$

Clearly, $\underline{V}^\top(\tau)$ takes one of the following four vectors $[0,\ 0,\ 0]$, $[1,\ 0,\ 0]$, $[0,\ 1,\ 0]$ and $[0,\ 0,\ 1]$ with probabilities $1-\eta(\tau)$, $p_1(\tau)\rho_{1:4}$, $p_2(\tau)\rho_{2:4}$ and $p_3(\tau)\rho_{3:4}$, where

$$\eta(\tau) \stackrel{\text{def}}{=} p_1(\tau)\rho_{1:4} + p_2(\tau)\rho_{2:4} + p_3(\tau)\rho_{3:4} \ . \tag{21}$$

It then follows that

$$\varphi(\underline{u},\tau) = 1-\eta(\tau) + p_1(\tau)\rho_{1:4}u_1 + p_2(\tau)\rho_{2:4}u_2 + p_3(\tau)\rho_{3:4}u_3 \ . \tag{22}$$

Since we are interested in knowing whether or not a customer eventually reaches state 4 starting from one of the three states in $G = \{1,2,3\}$, we set $u_1 = u_2 = u_3 = u$ so that the resulting probability generating function becomes

$$\widehat{\varphi}(u,\tau) \stackrel{\text{def}}{=} \varphi(u\underline{1},\tau) = 1-\eta(\tau)+\eta(\tau)u \ . \tag{23}$$

Let $S_T(\tau)$ be the number of CDs purchased after time τ, where T denotes "Tail". It can be readily seen that the probability generating function of $S_T(\tau)$ is obtained as

$$\varphi_T(u,\tau) \stackrel{\text{def}}{=} \mathrm{E}[u^{S_T(\tau)}] = \{\widehat{\varphi}(u,\tau)\}^N = \{1-\eta(\tau)+\eta(\tau)u\}^N \ . \tag{24}$$

As for (16) and (18), the key performance indicators for capturing the strength of having a fat tail of CD sales can then be defined by

$$\psi_T(\tau) \stackrel{\text{def}}{=} \frac{\mathrm{E}[S_T(\tau)]}{N} \quad ; \quad \kappa_T(\tau) \stackrel{\text{def}}{=} \mathrm{P}[S_T(\tau) \geqslant K_T] \ . \tag{25}$$

Using the key performance indicators in (16), (18) and (25), we now assess the brand characteristics of the five Japanese music artists. It should be noted that, given an artist a and the market segment $(I, J) \in \mathcal{I} \times \mathcal{J}$, one has $N_{IJ}(a)$ and the eight estimated parameter values, as well as the initial state of the individual customers in this market segment. In order to emphasize the specification of an artist a and the market segment $(I, J) \in \mathcal{I} \times \mathcal{J}$, the random variables S, $S_H(\tau)$ and $S_T(\tau)$ in the above discussion are denoted by $S_{IJ:a}$ $S_{H:IJ:a}(\tau)$ and $S_{T:IJ:a}(\tau)$. The key performance indicators $\psi_{IJ:a}$, $\kappa_{IJ:a}(\tau)$, etc. are written similarly. We now define

$$S_{I:a} \stackrel{\text{def}}{=} \sum_{J \in \mathcal{J}} S_{IJ:a} \ , \ S_{H:I:a}(\tau) \stackrel{\text{def}}{=} \sum_{J \in \mathcal{J}} S_{H:IJ:a}(\tau) \ , \ S_{T:I:a}(\tau) \stackrel{\text{def}}{=} \sum_{J \in \mathcal{J}} S_{T:IJ:a}(\tau) \ , \tag{26}$$

and

$$S_a \stackrel{\text{def}}{=} \sum_{I \in \mathcal{I}} S_{I:a} \ , \ S_{H:a}(\tau) \stackrel{\text{def}}{=} \sum_{I \in \mathcal{I}} S_{H:I:a}(\tau) \ , \ S_{T:a}(\tau) \stackrel{\text{def}}{=} \sum_{I \in \mathcal{I}} S_{T:I:a}(\tau) \ . \tag{27}$$

The corresponding probability generating functions can then be obtained as

$$\pi_{I:a}(u) \stackrel{\text{def}}{=} \mathrm{E}[u^{S_{I:a}}] = \prod_{J \in \mathcal{J}} \pi_{IJ:a}(u) \ , \tag{28}$$

$$\pi_{H:I:a}(u, \tau) \stackrel{\text{def}}{=} \mathrm{E}[u^{S_{H:I:a}(\tau)}] = \prod_{J \in \mathcal{J}} \pi_{H:IJ:a}(u, \tau) \ , \tag{29}$$

$$\pi_{T:I:a}(u, \tau) \stackrel{\text{def}}{=} \mathrm{E}[u^{S_{T:I:a}(\tau)}] = \prod_{J \in \mathcal{J}} \pi_{T:IJ:a}(u, \tau) \ , \tag{30}$$

and

$$\pi_a(u) \stackrel{\text{def}}{=} \mathrm{E}[u^{S_a}] = \prod_{I \in \mathcal{I}} \pi_{I:a}(u) \ , \tag{31}$$

$$\pi_{H:a}(u, \tau) \stackrel{\text{def}}{=} \mathrm{E}[u^{S_{H:a}(\tau)}] = \prod_{I \in \mathcal{I}} \pi_{H:I:a}(u, \tau) \ , \tag{32}$$

$$\pi_{T:a}(u, \tau) \stackrel{\text{def}}{=} \mathrm{E}[u^{S_{T:a}(\tau)}] = \prod_{I \in \mathcal{I}} \pi_{T:I:a}(u, \tau) \ . \tag{33}$$

Given an artist a, by employing the eight parameter values estimated from the POS data, the key performance indicators in (16), (18) and (25) can be computed for each $I \in \mathcal{I}$ and for the entire market via discrete convolutions based on (28) through (33). Concerning K_I, $K_{H:I}$ and $K_{T:I}$, the values adopted for computing κ_I, $\kappa_{H:I}$ and $\kappa_{T:I}$ are summarized in Table 8. For examining the speed of sales growth upon release, we focus on the first five weeks. In order to investigate the strength of a fat tail of sales records, we consider the period after ten weeks. The computational results based on the POS data are summarized in Tables 9 through 11 for $I \in \mathcal{I} = \{Add, Fan, Neu\}$ for each artist.

Table 8: K_I, $K_{H:I}$ and $K_{T:I}$

	K_I	$K_{H:I}$	$K_{T:I}$
Add	200	50	25
Fan	800	450	50
Neu	1000	750	125
Total	2000	1250	200

Table 9: $\mathrm{E}[S_{I:a}]$, $\psi_{I:a}$ and $\kappa_{I:a}$ Based on the POS Data

Mr.Children	N_I	$\mathrm{E}[S_{I:a}]$	$\psi_{I:a}$	$\kappa_{I:a}$
Add	1280	988	0.7719	1.0000
Fan	15605	1482	0.0950	1.0000
Neu	214048	3657	0.0171	1.0000
Total	230933	6127	0.0265	1.0000

Ketsumeishi	N_I	$\mathrm{E}[S_{I:a}]$	$\psi_{I:a}$	$\kappa_{I:a}$
Add	642	94	0.1464	0.0000
Fan	12766	869	0.0681	0.0000
Neu	217525	714	0.0033	0.0000
Total	230933	1677	0.0073	0.0000

Ken Hirai	N_I	$\mathrm{E}[S_{I:a}]$	$\psi_{I:a}$	$\kappa_{I:a}$
Add	359	32	0.0891	0.0000
Fan	11457	1022	0.0892	1.0000
Neu	219117	1452	0.0066	1.0000
Total	230933	2506	0.0109	0.5913

Ayumi Hamasaki	N_I	$\mathrm{E}[S_{I:a}]$	$\psi_{I:a}$	$\kappa_{I:a}$
Add	966	207	0.2143	0.4834
Fan	7862	71	0.0090	0.0000
Neu	222105	1822	0.0082	1.0000
Total	230933	2100	0.0091	0.5617

EXILE	N_I	$\mathrm{E}[S_{I:a}]$	$\psi_{I:a}$	$\kappa_{I:a}$
Add	424	311	0.7335	0.9679
Fan	7914	762	0.0963	0.0640
Neu	222595	3840	0.0173	1.0000
Total	230933	4913	0.0213	0.9679

Table 10: $\mathrm{E}[S_{H:I:a}(5)]$, $\psi_{H:I:a}(5)$ and $\kappa_{H:I:a}(5)$ Based on the POS Data

Mr.Children	N_I	$\mathrm{E}[S_{H:I:a}(5)]$	$\psi_{H:I:a}(5)$	$\kappa_{H:I:a}(5)$
Add	1280	229	0.1789	0.6687
Fan	15605	963	0.0617	1.0000
Neu	214048	3587	0.0168	1.0000
Total	230933	4779	0.0207	0.6687

Ketsumeishi	N_I	$\mathrm{E}[S_{H:I:a}(5)]$	$\psi_{H:I:a}(5)$	$\kappa_{H:I:a}(5)$
Add	642	33	0.0514	0.0002
Fan	12766	505	0.0396	0.9924
Neu	217525	706	0.0032	0.0406
Total	230933	1244	0.0054	0.0437

Ken Hirai	N_I	$\mathrm{E}[S_{H:I:a}(5)]$	$\psi_{H:I:a}(5)$	$\kappa_{H:I:a}(5)$
Add	359	20	0.0557	0.0000
Fan	11457	608	0.0531	1.0000
Neu	219117	1399	0.0064	1.0000
Total	230933	2027	0.0088	0.0973

Ayumi Hamasaki	N_I	$\mathrm{E}[S_{H:I:a}(5)]$	$\psi_{H:I:a}(5)$	$\kappa_{H:I:a}(5)$
Add	966	71	0.0735	0.1454
Fan	7862	71	0.0090	0.0000
Neu	222105	1772	0.0080	1.0000
Total	230933	1914	0.0083	0.0884

EXILE	N_I	$\mathrm{E}[S_{H:I:a}(5)]$	$\psi_{H:I:a}(5)$	$\kappa_{H:I:a}(5)$
Add	424	66	0.1557	0.0692
Fan	7914	487	0.0615	0.9493
Neu	222595	3768	0.0169	1.0000
Total	230933	4321	0.0187	0.4237

Table 11: $\mathrm{E}[S_{T:I:a}(10)]$, $\psi_{T:I:a}(10)$ and $\kappa_{T:I:a}(10)$ Based on the POS Data

Mr.Children	N_I	$\mathrm{E}[S_{T:I:a}(10)]$	$\psi_{T:I:a}(10)$	$\kappa_{T:I:a}(10)$
Add	1280	582	0.4547	0.9975
Fan	15605	154	0.0099	1.0000
Neu	214048	5	0.0000	0.0000
Total	230933	741	0.0032	0.8854

Ketsumeishi	N_I	$\mathrm{E}[S_{T:I:a}(10)]$	$\psi_{T:I:a}(10)$	$\kappa_{T:I:a}(10)$
Add	642	39	0.0607	0.5826
Fan	12766	141	0.0110	1.0000
Neu	217525	7	0.0000	0.0000
Total	230933	187	0.0008	0.3265

Ken Hirai	N_I	$\mathrm{E}[S_{T:I:a}(10)]$	$\psi_{T:I:a}(10)$	$\kappa_{T:I:a}(10)$
Add	359	5	0.0139	0.0000
Fan	11457	157	0.0137	1.0000
Neu	219117	7	0.0000	0.0000
Total	230933	169	0.0007	0.0018

Ayumi Hamasaki	N_I	$\mathrm{E}[S_{T:I:a}(10)]$	$\psi_{T:I:a}(10)$	$\kappa_{T:I:a}(10)$
Add	966	89	0.0921	0.3760
Fan	7862	0	0.0000	0.0000
Neu	222105	3	0.0000	0.0000
Total	230933	92	0.0004	0.0000

EXILE	N_I	$\mathrm{E}[S_{T:I:a}(10)]$	$\psi_{T:I:a}(10)$	$\kappa_{T:I:a}(10)$
Add	424	193	0.4552	0.8198
Fan	7914	94	0.0119	0.9979
Neu	222595	4	0.0000	0.0000
Total	230933	291	0.0013	0.7171

From the computational results presented in Tables 9 through 11, the following observations can be made.

Mr.Children:

- In the urban areas represented by the POS data, Mr.Children has the strongest overall brand power among the five music artists with $\mathrm{E}[S] = 6127$ and $\kappa = \mathrm{P}[S \geqslant 2000] = 1.0000$.

- This strength comes from excelling in the rapid sales growth upon release of a CD with $\mathrm{E}[S_H(5)] = 4779$ and $\kappa_H(5) = \mathrm{P}[S_H(5) \geqslant 1250] = 0.6687$, as well as the fat tail of the sales records over a long time period with $\mathrm{E}[S_T(10)] = 741$ and $\kappa_T(10) = \mathrm{P}[S_T(10) \geqslant 200] = 0.8854$.

- Mr.Children dominates the other four artists in all of the above categories.

- It is worth noting that Mr.Children has the core of serious supporters who always pay attention to the release of new CDs by the artist with $\psi_{Add} = \mathrm{E}[S_{Add}]/N_{Add} = 0.7719$ and $\kappa_{Add} = \mathrm{P}[S_{Add} \geqslant 200] = 1.0000$.

- Somewhat surprisingly, such serious supporters contribute more to the fat tail strength with $\psi_{T:Add}(10) = 0.4547$ and $\kappa_{T:Add}(10) = 0.9975$ than to the rapid sales growth with only $\psi_{H:Add}(5) = 0.1788$ and $\kappa_{H:Add}(5) = 0.6687$.

- As far as the support of Fan is concerned, one sees that $\psi_{Fan} = \mathrm{E}[S_{Fan}]/N_{Fan} = 0.0950$ and $\kappa_{Fan} = \mathrm{P}[S_{Fan} \geqslant 200] = 1.0000$, i.e. 10% of Fan on the average would purchase a CD and the total number of CDs sold would exceed 200 with probability one.

- Such fans are split into three groups: 65% of them responding soon after the release of a CD ($\mathrm{E}[S_{H:Fan}]/\mathrm{E}[S_{Fan}] = 0.6498$), 10% of them contributing to the fat tail strength ($\mathrm{E}[S_{T:Fan}]/\mathrm{E}[S_{Fan}] = 0.1039$), and the remaining 25% purchasing in between.

- In terms of the total number of CDs sold, the contribution from Neu is larger than that of Add or Fan with $\mathrm{E}[S_{Neu}]/\mathrm{E}[S] = 0.5968$. This support is more or less limited to rather quick response to the release of a CD with $\mathrm{E}[S_{H:Neu}]/\mathrm{E}[S_{Neu}] = 0.9809$. This phenomenon seems to be common also for the other four music artists.

< **Suggested Marketing Strategy for Mr.Children** >

Along with EXILE, those supporters of Mr.Children in Fan and Neu seem to have a strong potential in that, on the average, 9.5% of Fan and 1.7% of Neu would purchase a CD, in comparison with 7% and 0.3% for Ketsumeishi, 9% and 0.7% for Ken Hirai, 0.9% and 0.8% for Ayumi Hamasaki and 9.6% and 1.7% for EXILE. The strong brand power of Mr.Children may be further strengthened by concentrating on conversion of Fan into Add, and Neu into Fan.

Ketsumeishi:

- The overall brand power of Ketsumeishi is the weakest among the five music artists with $\mathrm{E}[S] = 1677$ and $\kappa = \mathrm{P}[S \geqslant 2000] = 0.0000$.

- This weakness is present in both the slow sales growth with $\mathrm{E}[S_H(5)] = 1244$ and $\kappa_H(5) = \mathrm{P}[S_H(5) \geqslant 1250] = 0.0437$, and the thin tail of the sales records with $\mathrm{E}[S_T(10)] = 187$ and $\kappa_T(10) = \mathrm{P}[S_T(10) \geqslant 200] = 0.3265$.

- In addition, the support from *Add* is quite low in that, on the average, only 14.6% of *Add* would purchase a CD with $\mathrm{E}[S_{Add}] = 94$, $\psi_{Add} = 0.1464$ and $\kappa_{Add} = 0.0000$, only above Ken Hirai.

< **Suggested Marketing Strategy for Ketsumeishi** >

Since the overall brand power of Ketsumeishi is rather weak and the support from *Add* is not strong either, it may be wise to begin to strengthen its presence in the CD market by focusing on development of the strong core of supporters in *Add*. This may be achieved by providing special services to those who purchase three CDs or more of Ketsumeishi.

Ken Hirai:

- Ken Hirai is unique in that the support from *Add* is extremely weak with $\mathrm{E}[S_{Add}] = 32$, $\psi_{Add} = 0.0891$ and $\kappa_{Add} = 0.0000$, being the lowest among the five music artists in these categories.

- However, he has a fairly strong presence in the CD market with $\mathrm{E}[S] = 2506$ and $\kappa = \mathrm{P}[S \geqslant 2000] = 0.5913$ only next to Mr.Children and Ketsumeishi.

- This hidden strength relies upon the support from Fan with $N_{Fan} = 11457$ ranked the third among the five music artists and $\mathrm{E}[S_{Fan}] = 1022$ ranked the second after Mr.Children.

- Such fans are split into three groups: 59% of them responding soon after the release of a CD ($\mathrm{E}[S_{H:Fan}]/\mathrm{E}[S_{Fan}] = 0.5949$), 15% of them contributing to the fat tail strength ($\mathrm{E}[S_{T:Fan}]/\mathrm{E}[S_{Fan}] = 0.1536$), and the remaining 26% purchasing in between.

< **Suggested Marketing Strategy for Ken Hirai** >

The hidden strength of Fan along with comparable support from Neu may suggest that Ken Hirai has not developed the core of serious supporters and his music CDs are bought not because they are composed by Ken Hirai but because their musical contexts are appreciated. For enforcing the presence of Ken Hirai in the CD market, it would be necessary to develop a core of serious supporters who would be attracted to Ken Hirai as a musician, rather than to the musical contexts of his CDs.

Ayumi Hamasaki:

- The brand characteristics of Ayumi Hamasaki would be opposite of Ken Hirai in that she has a small group of rather strong supporters in Add with $\mathrm{E}[S_{Add}] = 207$, $\psi_{Add} = 0.2143$ and $\kappa_{Add} = 0.4834$, while the support from Fan is the weakest among the five music artists with $\mathrm{E}[S_{Fan}] = 71$, $\psi_{Fan} = 0.0090$ and $\kappa_{Fan} = 0.0000$.
- In particular, the weakness of Fan can be found in its extremely thin tail with $\mathrm{E}[S_{T:Fan}(10)] = 0$, $\psi_{T:Fan}(10) = 0.0000$ and $\kappa_{T:Fan}(10) = 0.0000$.

< **Suggested Marketing Strategy for Ayumi Hamasaki** >

The top priority for promoting Ayumi Hamasaki in the CD market would be to convert those customers who have purchased one of her CDs into repeated customers, establishing the foundation for Fan. Relying upon this foundation, it would be then necessary to expand the existing group of Add further.

EXILE:

- The overall brand power of EXILE is the second among the five music artists, only next to Mr.Children, with $\mathrm{E}[S] = 4913$ and $\kappa = \mathrm{P}[S \geqslant 2000] = 0.9679$.

- As for Mr.Children, EXILE is rather strong in both the rapid sales growth upon release of a CD with $\mathrm{E}[S_H(5)] = 4321$ and $\kappa_H(5) = \mathrm{P}[S_H(5) \geqslant 1250] = 0.4237$ and the fat tail of the sales records over a long time period with $\mathrm{E}[S_T(10)] = 291$ and $\kappa_T(10) = \mathrm{P}[S_T(10) \geqslant 200] = 0.7171$.

- The strong support from *Add* of EXILE can be found in that $\psi_{Add} = \mathrm{E}[S_{Add}]/N_{Add} = 0.7335$, comparable with Mr.Children having the value of 0.7719.

- Like Mr.Children, such serious supporters contribute more to the fat tail strength with $\psi_{T:Add}(10) = 0.4552$ and $\kappa_{T:Add}(10) = 0.8198$ than to the rapid sales growth with only $\psi_{H:Add}(5) = 0.1557$ and $\kappa_{H:Add}(5) = 0.0692$.

- For the support of Fan, one has $\psi_{Fan} = \mathrm{E}[S_{Fan}]/N_{Fan} = 0.0963$ and $\kappa_{Fan} = \mathrm{P}[S_{Fan} \geqslant 200] = 0.0640$, i.e. 10% of Fan on the average would purchase a CD as Mr.Children, but the total number of CDs sold would exceed 200 only with probability 0.0640, while this probability for Mr.Children is one.

$<$ **Suggested Marketing Strategy for EXILE** $>$

The presence of EXILE in the CD market may be strengthened by following the strategy in parallel with that of Mr.Children, i.e., to concentrate on conversion of Fan into *Add*, and Neu into Fan.

7 Concluding Remarks

In this paper, a new approach for customer segmentation in the music CD market is proposed, where the customers are classified into twelve segments along two axes: "Artist Loyalty" consisting of three different levels (Addict, Fan, and Neutral), and "Market Sensitivity" with four different levels (Ultra Sensitive, Sensitive, Normal, and Insensitive). The artist loyalty of a customer to a music artist is measured by the number of CDs of the artist purchased by the customer, while the market sensitivity of the customer concerning the music artist is defined by ranking how soon the CD is purchased by the customer since the release of the CD. Given a music artist and his/her CD, the purchasing behavior of individual customers for that CD is formulated as a Markov chain in discrete time defined on the following five states: Unknown, Known, Interested, Purchased, and Not Purchased. Because a music CD is not likely to be purchased repeatedly, the two states, Purchased and Not Purchased, are absorbing. The transition structure of the customers across the twelve segments is common. However, the market segmentation is reflected by altering transition probabilities in terms of eight different parameters. The entire market for the artist is then expressed as an independent sum of all such customers.

The eight parameters are estimated so as to minimize the Euclidian distance between the expected number of CDs sold over 20 weeks calculated from the Markov model and the actual POS data obtained from ten CD shops in urban areas in Japan. The resulting parameter values are then used to understand the brand characteristics of the leading Japanese music artists, suggesting new effective marketing strategies.

It should be noted that, while the paper is written by focusing on the Japanese CD market, the proposed approach can be applied to any market in which products are likely to be purchased only once, provided that the data set necessary for estimating the underlying parameter values could be accessible.

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A Spectral Analysis of $\underline{\underline{a}}_{IJ}$

In this appendix, the spectral analysis of matrix $\underline{\underline{a}}_{IJ}$ of (2) in Section 5 is conducted. Since $\underline{\underline{a}}_{IJ}$ is upper triangular, its eigenvalues are given by $\alpha_{1:IJ}$, $\alpha_{2:IJ}$, and $\alpha_{3:IJ}$ which are simple, and 1 with multiplicity two. The right eigenvectors and left eigenvectors are obtained as

$$
\begin{aligned}
\underline{x}_{IJ-1} &= [1,\ 0,\ 0,\ 0,\ 0]^\top\ , \\
\underline{x}_{IJ-2} &= [\beta_{1:IJ},\ \alpha_{2:IJ} - \alpha_{1:IJ},\ 0,\ 0,\ 0]^\top\ , \\
\underline{x}_{IJ-3} &= [\beta_{1:IJ}\beta_{2:IJ},\ -(\alpha_{1:IJ} - \alpha_{3:IJ})\beta_{2:IJ},\ (\alpha_{1:IJ} - \alpha_{3:IJ})(\alpha_{2:IJ} - \alpha_{3:IJ}),\ 0,\ 0]^\top\ , \\
\underline{x}_{IJ-4} &= [\beta_{2:IJ}\beta_{3:IJ},\ \beta_{2:IJ}\beta_{3:IJ},\ (1 - \alpha_{2:IJ})\beta_{3:IJ},\ (1 - \alpha_{2:IJ})(1 - \alpha_{3:IJ}),\ 0]^\top\ , \\
\underline{x}_{IJ-5} &= [\beta_{2:IJ}\gamma_{3:IJ} + (1 - \alpha_{3:IJ})\gamma_{2:IJ},\ \beta_{2:IJ}\gamma_{3:IJ} + (1 - \alpha_{3:IJ})\gamma_{2:IJ},\ (1 - \alpha_{2:IJ})\gamma_{3:IJ}, \\
&\qquad 0,\ (1 - \alpha_{2:IJ})(1 - \alpha_{3:IJ})]^\top\ , \\
\underline{y}_{IJ-1} &= [(\alpha_{1:IJ} - \alpha_{2:IJ})(\alpha_{1:IJ} - \alpha_{3:IJ}),\ -(\alpha_{3:IJ} - \alpha_{1:IJ})\beta_{1:IJ},\ \beta_{1:IJ}\beta_{2:IJ},\ -\beta_{2:IJ}\beta_{3:IJ}, \\
&\qquad -\Delta_{1:IJ}]^\top\ , \\
\underline{y}_{IJ-2} &= [0,\ (\alpha_{3:IJ} - \alpha_{2:IJ})(1 - \alpha_{2:IJ}),\ -(1 - \alpha_{2:IJ})\beta_{2:IJ},\ \beta_{2:IJ}\beta_{3:IJ},\ \Delta_{2:IJ}]^\top\ , \\
\underline{y}_{IJ-3} &= [0,\ 0,\ 1 - \alpha_{3:IJ},\ -\beta_{3:IJ},\ -\gamma_{3:IJ}]^\top\ , \\
\underline{y}_{IJ-4} &= [0,\ 0,\ 0,\ 0,\ 1]^\top\ , \\
\underline{y}_{IJ-5} &= [0,\ 0,\ 0,\ 1,\ 0]^\top\ ,
\end{aligned}
$$

where $\Delta_{i:IJ} \overset{\text{def}}{=} \beta_{2:IJ}\gamma_{3:IJ} - (\alpha_{3:IJ} - \alpha_{i:IJ})\gamma_{2:IJ}$ $(i = 1, 2)$. We define

$$
\widehat{\underline{x}}_{IJ-m} \overset{\text{def}}{=} k_{IJ-m}\underline{x}_{IJ-m}\ ,\ \widehat{\underline{y}}_{IJ-m} \overset{\text{def}}{=} k_{IJ-m}\underline{y}_{IJ-m}\ ,\ m \in M \overset{\text{def}}{=} \{1, 2, 3, 4, 5\}\ ,
$$

$$
\begin{aligned}
k_{IJ-1} &\stackrel{\text{def}}{=} \sqrt{\frac{1}{(\alpha_{2:IJ}-\alpha_{1:IJ})(\alpha_{3:IJ}-\alpha_{1:IJ})}} \ , \\
k_{IJ-2} &\stackrel{\text{def}}{=} \sqrt{\frac{1}{(\alpha_{2:IJ}-\alpha_{1:IJ})(\alpha_{3:IJ}-\alpha_{2:IJ})(1-\alpha_{2:IJ})}} \ , \\
k_{IJ-3} &\stackrel{\text{def}}{=} \sqrt{\frac{1}{(\alpha_{1:IJ}-\alpha_{3:IJ})(\alpha_{2:IJ}-\alpha_{3:IJ})(1-\alpha_{3:IJ})}} \ , \\
k_{IJ-4} &= k_{IJ-5} \stackrel{\text{def}}{=} \sqrt{\frac{1}{(1-\alpha_{2:IJ})(1-\alpha_{3:IJ})}} \ .
\end{aligned}
$$

One then has

$$
\underline{\widehat{y}}_{IJ-m_1}^{\top} \underline{\widehat{x}}_{IJ-m_2} = \delta_{\{m_1=m_2\}} \ , \quad m_1, m_2 \in M \ . \tag{34}
$$

We also define $\underline{\underline{\widehat{J}}}_{IJ-m} \stackrel{\text{def}}{=} \underline{\widehat{x}}_{IJ-m} \underline{\widehat{y}}_{IJ-m}^{\top}$. It then follows that

$$
\underline{\underline{a}}_{IJ} = \sum_{m \in M} \eta_{IJ-m} \underline{\underline{\widehat{J}}}_{IJ-m} \ ; \ \eta_{IJ-m} \stackrel{\text{def}}{=} \begin{cases} \alpha_{m:IJ} & \text{if} \ \ m = 1, 2 \text{ or } 3 \\ 1 & \text{if} \ \ m = 4 \text{ or } 5 \end{cases} \ . \tag{35}
$$

Since $\underline{\underline{\widehat{J}}}_{IJ-m_1} \underline{\underline{\widehat{J}}}_{IJ-m_2} = \underline{\underline{0}}$ for m_1 and m_2 $(m_1 \neq m_2, \ m_1, m_2 \in M)$ from (34), $\underline{\underline{a}}_{IJ}^{\tau}$ is expressed as

$$
\underline{\underline{a}}_{IJ}^{\tau} = \sum_{m \in M} \eta_{IJ-m}^{\tau} \underline{\underline{\widehat{J}}}_{IJ-m} \quad (\tau = 0, 1, \cdots) \ , \tag{36}
$$

where $\underline{\underline{a}}_{IJ}^{0} = \sum_{m \in M} \underline{\underline{\widehat{J}}}_{IJ-m} = \underline{\underline{I}}$.

B Derivation of $\frac{\partial}{\partial x} S_a(\tau)$ for $x \in \{p_{Add}, p_{Fan}, p_{Neu}, r_{Add}, r_{Fan}, r_{Neu}, \zeta, \xi\}$

In this section, $\frac{\partial}{\partial x} S_a(\tau)$ for $x \in \{p_{Add}, p_{Fan}, p_{Neu}, r_{Add}, r_{Fan}, r_{Neu}, \zeta, \xi\}$ in (13) is provided. By differentiating $S_a(\tau)$ with respect to $x \in \{p_{Add}, p_{Fan}, p_{Neu}, r_{Add}, r_{Fan}, r_{Neu}, \zeta, \xi\}$, it can be seen that

$$\frac{\partial}{\partial p_I}S_a(\tau) = N_{POS}\sum_{I\in\mathcal{I}}\Bigg[r_I^{\tau-1}\zeta\Big[Q_{I,Usen}(a)\big\{1-(p_I+\xi)\zeta\big\}^{\tau-2}\Big]$$

$$+Q_{I,Sen}(a)(1-p_I)^{\tau-2}(1-\tau p_I\zeta)$$

$$+\frac{\zeta}{1-\zeta}\Big[Q_{I,Norm}(a)r_I^{\tau-2}\{(1-p_I\zeta)^{\tau-2}(1-\tau p_I\zeta)-(1-p_I)^{\tau-2}(1-\tau p_I)\}\Big]$$

$$+Q_{I,Ins}(a)\cdot\frac{p_I}{(1-r_I)\chi_I}\Big[\frac{2\chi_I+p_I(1-r_I\zeta)}{\chi_I}\big[(1-p_I)^{\tau-2}\{p_I(1-\zeta)+\chi_I r_I^{\tau-2}\}$$

$$-(1-r_I)^{\tau-2}(1-p_I\zeta)^{\tau-1}\big]+p_I\big[(1-p_I)^{\tau-2}\{(1-\zeta)-r_I^{\tau-2}(1-r_I\zeta)\}$$

$$-(\tau-2)(1-p_I)^{\tau-3}\{(1-\zeta)p_I(1-p_I)-r_I^{\tau-2}\chi_I\}\big]\Big]\Bigg], \tag{37}$$

$$\frac{\partial}{\partial r_I}S_a(\tau) = N_{POS}\sum_{I\in\mathcal{I}}\Bigg[(\tau-1)r_I^{\tau-2}\zeta\Big[Q_{I,Usen}(a)\big\{1-(p_I+\xi)\zeta\big\}^{\tau-1}(p_I+\xi)$$

$$+Q_{I,Sen}(a)(1-p_I\zeta)^{\tau-1}p_I\Big]+\frac{p_I\zeta}{1-\zeta}\Big[Q_{I,Norm}(a)(\tau-2)\big\{(1-p_I\zeta)^{\tau-1}$$

$$-(1-p_I)^{\tau-1}\big\}r_I^{\tau-3}+Q_{I,Ins}(a)\cdot\frac{p_I}{\chi_I(1-r_I)}\Big[\frac{(1-p_I\zeta)(1-r_I)+\chi_I}{\chi_I(1-r_I)}$$

$$\cdot\big[(1-p_I)^{\tau-2}\{p_I(1-\zeta)+\chi_I r_I^{\tau-2}\}-(1-p_I\zeta)^{\tau-1}(1-r_I)r_I^{\tau-2}\big]$$

$$+(1-p_I)^{\tau-2}r_I^{\tau-3}\big\{(\tau-2)\chi_I-(1-p_I\zeta)r_I\big\}$$

$$-(1-p_I\zeta)^{\tau-1}r_I^{\tau-3}\{(\tau-2)(1-r_I)-r_I\}\Big]\Big]\Bigg], \tag{38}$$

$$\frac{\partial}{\partial \zeta}S_a(\tau) = N_{POS}\sum_{I\in\mathcal{I}}\Bigg[r_I^{\tau-1}\Big[Q_{I,Usen}(a)(p_I+\xi)\big\{1-(p_I+\xi)\zeta\big\}^{\tau-2}\big\{1-\tau(p_I+\xi)\zeta\big\}$$

$$+Q_{I,Sen}(a)p_I(1-p_I\zeta)^{\tau-2}(1-\tau p_I\zeta)\Big]+\frac{p_I}{1-\zeta}\Big[Q_{I,Norm}(a)r_I^{\tau-2}\big[\frac{1}{1-\zeta}\{(1-p_I\zeta)^{\tau-1}\}$$

$$-(\tau-1)(1-p_I\zeta)^{\tau-2}\zeta\big]+Q_{I,Ins}(a)\cdot\frac{p_I}{\chi_I(1-r_I)}\Big[\frac{\chi_I-r_I p_I\zeta(1-\zeta)}{\chi_I(1-\zeta)}$$

$$\cdot\big[(1-p_I)^{\tau-2}\{p_I(1-\zeta)+\chi_I r_I^{\tau-2}\}-r_I(1-r_I)(1-p_I\zeta)^{\tau-1}\big]$$

$$-\zeta\big\{p_I(1-p_I)^{\tau-2}(1-r_I^{\tau-1})-(\tau-1)r_I(1-r_I)p_I(1-p_I\zeta)^{\tau-2}\big\}\Big]\Big]\Bigg], \tag{39}$$

and

$$\frac{\partial}{\partial \xi}S_a(\tau) = N_{POS}\sum_{I\in\mathcal{I}}Q_{I,Usen}(a)r_I^{\tau-1}\zeta\big\{1-(p_I+\xi)\zeta\big\}^{\tau-2}\big\{1-\tau(p_I+\xi)\zeta\big\}\ . \tag{40}$$