

Department of Social Systems and Management

Discussion Paper Series

No. 1227

**Development of e-Marketing Contract Structure  
Based on Consumer-Generated Contents and Its  
Optimal Strategy**

by

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February 2009

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## Abstract

A new e-Marketing contract structure is proposed, where the contract would be exchanged between an e-Marketing company, named Company X, and a manufacturing company, named Company Y, which is to promote Product Z within a group of SNSs (Social Network Services). Company X promises Company Y to generate  $Q$  positive comments about Product Z within SNSs, and then hires  $K$  bloggers, asking them to experience Product Z and write comments about it. They are expected to play a role of pumping water so as to fulfill the contract. Each of such bloggers is paid by  $\$c$ . For this service, Company Y pays Company X by the amount of  $\$ \alpha Q$ . Should the actual number of positive comments exceed  $Q$ , the additional payment of  $\$ \beta$  for each positive comment beyond  $Q$  would be paid to Company X by Company Y. For controlling the risk of the actual number of positive comments to appear within SNSs falling below  $Q$ , Company X pays the penalty of  $\$ \gamma$  for each shortage below  $Q$ . The problem for Company X is then how to determine  $Q$  and  $K$  so as to optimize its objective. Two types of problems are considered, where the first problem is to maximize the expected profit of Company X while the second problem is based on the VaR (Value at Risk) approach to minimize the probability of the profit of Company X falling below  $v_0$  subject to having the expected profit above  $v_1$ . Although this problem has a flavor of the classical news vendor problem, it is more difficult because of its two dimensional nature. For the first problem, the exact optimal solution is derived apart from the integer constraints and an algorithmic procedure is developed for computing the optimal solution. For the second problem, it is shown that the distribution function of the profit of Company X can be given explicitly, thereby providing a computational foundation for solving the problem. Numerical examples are given, illustrating the stochastic structure of the e-Marketing contract and the differences of the optimal strategies for the two problems.

# 1 Introduction

In its infancy, the Internet provided the worldwide network infrastructure through which corporations could propagate information of their choice with speed and cost-performance efficiency. The broadcasting approach for one way communication to the market could be combined with the one-on-one approach for two way communication by utilizing open websites equipped with two way mail services and the like. On the other hand, the use of the Internet by consumers was somewhat limited to information search and purchasing. Since the beginning of this century, however, the Internet has entered a new era, which is often referred to as Web 2.0 and after, where a new generation of digital devices, mobile devices and common software would play a critical role. As Werbach [16] pointed out, “ The basics of the new today are that powerful digital devices are becoming pervasive and inexpensive; they’re becoming commodities. Services are available networked across the Internet and use common software. The world is heterogeneous, complex and decentralized.” Supported by such devices and services, Web 2.0 has impacted the way businesses are conducted in every industrial segment, and perhaps, has affected the area of marketing most significantly where Consumer Generated Media (CGM) has become the focal attention.

In traditional marketing, the classic AIDMA model by Hall [10] has been playing a central role in describing the psychological process of a consumer toward purchasing a product. Here, a product would first attract “ Attention” of a consumer, and then the consumer would become “ Interested”. Soon the consumer might “ Desire” to have the product and keep it in his “ Memory”. Finally, upon coming across the product at a store, the consumer would take an “ Action” to purchase it. In response to the era of Web 2.0, Dentsu Incorporated, the largest advertising agency in Japan, proposed a new model called AISAS by modifying the AIDMA model in the following manner. As for the AIDMA model, a product would first attract “ Attention” of a consumer, and then the consumer would become “ Interested”. The next action of the consumer in the AISAS model would be to “ Search” information through the Internet, followed by taking an “ Action” to purchase it. In this new model, the consumer behavior does not stop here and the actions by consumers to “ Share” experiences and assessments through the Internet would be emphasized. In June 2005, Dentsu Incorporated registered AISAS as its trademark. This episode symbolizes the importance of CGM in the era of Web 2.0.

As the market became heterogeneous, complex and decentralized, CGM became extremely important in marketing even outside the Internet. Since the middle of 1990’s, the study of WOM (Word of Mouth) has become prevalent in the literature. Ellison and Fudenberg [5], for example, proposed a WOM communication model

and analyzed its implications. Bone [3] examined the effects of WOM on decision of consumers for purchasing a product. Goldenberg, Libai and Muller [9] claimed that the effects of WOM would depend on the level of closeness of those involved in WOM. More recently, in Banerjee and Fudenberg [2], a new model was developed for measuring the effects of WOM. Along with this line of research on offline WOM, WOM through the Internet, which is often called eWOM, has begun to attract attention of more researchers. Shardanand and Maes [13] offered an information filtering algorithm so as to identify preferences of consumers from WOM and then to provide personalized recommendations. Stauss [14, 15] discussed potential threats and opportunities resulting from online articulations by consumers. Balasubramanian and Mahajan [1] provided a conceptual framework for describing three types of social interaction utility within a virtual community: focus related utility, consumption utility, and approval utility. Based on this framework, Henning-Thurau, Gwinner, Walsh, and Gremler examined an online sample of some 2000 consumers, and derived key elements for motivating consumers to participate in eWOM. Dellarocas [4] discussed potentials and difficulties of online feedback mechanisms for the digitization of eWOM.

In practice, several e-Marketing companies in Japan seem to be well ahead of the literature by taking advantage of CGM and eWOM. Companies such as Forefront Systems [7], Global Insight [8], and FC2 [6] offer points to bloggers for writing comments on selected products and services. These points may be accumulated to a certain level at which they can be cashed. Mobile Agent [12] openly arranges advertising clients and bloggers through its website where similar points would be given to bloggers for writing their blogs. This kind of marketing is sometimes called “stealth marketing” and is despised by certain marketing organizations. Word of Mouth Marketing Association [17], for example, strongly opposes to stealth marketing. Google is believed to lower the ranking of any website if the website becomes known to be involved in stealth marketing supported by payment.

While the practices of the above Japanese corporations may face the danger of committing to unethical business conducts, e-Marketing based on hired bloggers itself may not be necessarily evil as long as the following conditions are met: 1) the relationship between an advertising client, an e-Marketing company, and bloggers, who may be hired for writing comments about products and services selected by the advertising client, would be managed through contracts in a manner transparent between the involved parties; 2) the hired bloggers would actually experience the selected products and services, and then write their honest comments about them; 3) such a comment should state the fact that the comment is written for payment and 4) should such comments be screened by the advertising client and/or the e-Marketing

company, this fact should become publicly known. When a TV advertisement shows a next-door housewife praising a detergent, the TV audience would know that she would be paid but still might be affected. Similar marketing efforts through eWOM should be tolerated as long as the above conditions are met, thereby making such marketing efforts no longer be stealthy.

To the best knowledge of the authors, no research has been done in the literature concerning possible contracts for e-Marketing as discussed above. The purpose of this paper is to develop a new e-Marketing contract structure based on consumer-generated contents from a point of view of an e-Marketing company, and analyze its stochastic properties for managing risks involved. More specifically, we consider a contract model between Company X specializing in e-Marketing and Company Y trying to promote Product Z. Through the contract, Company X promises Company Y to generate  $Q$  positive comments about Product Z written by consumers in their blogs within specific SNSs (Social Network Services) during a certain period. In return, Company Y pays Company X by the amount of  $\alpha Q$ . Company X hires  $K$  bloggers, asking them to experience Product Z and write comments about it. They are expected to play a role of pumping water so as to fulfill the contract. Each of such bloggers is paid by  $c$ . If the actual number of positive comments exceeds  $Q$ , Company Y compensates Company X by paying  $\beta$  for each comment above  $Q$ . Should it fall below  $Q$ , Company X pays the penalty of  $\gamma$  to Company Y for each shortage below  $Q$ . The problem for Company X is then how to determine  $Q$  and  $K$  so as to optimize its objective. Two types of problems are considered, where the first problem is to maximize the expected profit of Company X while the second problem is based on the VaR (Value at Risk) approach which minimizes the probability of the profit of Company X falling below  $v_0$  subject to having the expected profit above  $v_1$ . Although this problem has a flavor of the classical news vendor problem, it is more difficult because of its two dimensional nature.

The structure of this paper is as follows. In Section 2, a new e-Marketing contract model is introduced and the two types of decision problems are formulated formally. Section 3 is devoted to analysis of the optimal strategy for maximizing the expected profit of Company X, where the exact optimal solution is derived apart from the integer constraints and an algorithmic procedure is developed for computing the optimal solution. In Section 4, the optimal strategy based on the VaR approach is discussed. It is shown that the distribution function of the profit of Company X can be given explicitly, thereby providing a computational foundation for solving the problem. Numerical examples are given in Section 5, illustrating the stochastic structure of the e-Marketing contract and the differences of the optimal strategies for the two problems. Finally in Section 6, some concluding remarks are given.

Throughout the paper, we use “increasing” and “decreasing” instead of “non-decreasing” and “non-increasing” respectively for notational simplicity. Strictness would be indicated explicitly whenever necessary, e.g. strictly increasing and strictly concave.

## 2 Model Description

We consider an e-Marketing company, named Company X, which proposes to generate positive comments about products and services for its clients within specific SNSs in a given time period  $[0, \tau]$ . More specifically, let  $SN$  be a group of social network services and suppose Company Y is interested in promoting Product Z within  $SN$ . Company X may then propose the following e-Marketing contract to Company Y.

“Company X promises Company Y to generate  $Q$  positive comments about Product Z within  $SN$  in  $[0, \tau]$ . For this service, Company Y pays Company X by the amount of  $\alpha Q$ . Should the actual number of positive comments exceed  $Q$ , the additional payment of  $\beta$  for each positive comment beyond  $Q$  would be paid to Company X by Company Y. For controlling the risk of the actual number of positive comments to appear within  $SN$  in  $[0, \tau]$  falling below  $Q$ , Company X pays the penalty of  $\gamma$  for each shortage below  $Q$  to Company Y.”

Throughout the paper, we assume that the unit penalty is larger than the original unit payment and the additional unit compensation is smaller than the original unit payment, i.e.

$$(2.1) \quad \gamma > \alpha > \beta .$$

In order to achieve the promised goal of generating  $Q$  positive comments within  $SN$  in  $[0, \tau]$ , Company X organizes a group of  $K$  bloggers, asking each of them to use Product Z and to write a positive comment about it, if they agree, through his/her blog with compensation of  $c$ . The organized group of the bloggers would play a role of pumping water for generating positive comments about Product Z. The problem for Company X is then how to determine  $Q$  and  $K$  so as to optimize its objective.

For formulating the decision problem of Company X more formally, let  $D$  be a non-negative random variable which describes the number of positive comments about Product Z to appear naturally within  $SN$  in  $[0, \tau]$  without any e-Marketing efforts. Throughout the paper, it is assumed that  $\tau$  is short enough so that any

interest can be ignored. Accordingly, the problem is considered as one-term problem and  $\tau$  is ignored in what follows. In addition, it is assumed that the mean of  $D$  is finite but large enough for its distribution function to be approximated by an absolutely continuous distribution function for analytical simplicity. We write

$$(2.2) \quad F_D(x) \stackrel{\text{def}}{=} P[D \leq x] = \int_0^x f_D(y)dy ; \mu_D \stackrel{\text{def}}{=} E[D] .$$

The corresponding survival function is given by

$$(2.3) \quad \bar{F}_D(x) \stackrel{\text{def}}{=} P[D > x] = 1 - F_D(x) = \int_x^\infty f_D(y)dy .$$

Because of the existence of concavity built into the model, this assumption of absolute continuity may be well approximated by employing rounding off or rounding up non-integer numbers in solving the optimization problems of our interest, as we will see.

Given  $K$  bloggers hired by Company X, let  $N(K)$  be the actual number of positive comments about Product Z that appear within  $SN$ . We assume that

$$(2.4) \quad N(K) = v(K)D ,$$

where  $v(K)$  is monotonically increasing and concave in  $K$  with  $v(0) = 1$ , i.e. there would be no e-Marketing effect if  $K = 0$  and therefore  $N(0) = D$ . In parallel with (2.2) and (2.3), let the distribution function and the survival function of  $N(K)$  be defined by

$$(2.5) \quad F_{N(K)}(x) = P[N(K) \leq x] ; \bar{F}_{N(K)}(x) = P[N(K) > x] .$$

From (2.4), it then follows that

$$(2.6) \quad F_{N(K)}(x) = F_D\left(\frac{x}{v(K)}\right) ; \bar{F}_{N(K)}(x) = \bar{F}_D\left(\frac{x}{v(K)}\right) .$$

Let  $L(Q, K)$  denote the portion of the contract limited to the additional compensation and the penalty for Company X. More specifically, we define

$$(2.7) \quad L(Q, K) = \beta[N(K) - Q]^+ - \gamma[Q - N(K)]^+$$

where

$$(2.8) \quad [a]^+ \stackrel{\text{def}}{=} \max\{0, a\} .$$

The distribution function and the survival function of  $L(Q, K)$  are defined as

$$(2.9) \quad H_{L(Q, K)}(x) = P[L(Q, K) \leq x] ; \bar{H}_{L(Q, K)}(x) = P[L(Q, K) > x]$$

with its expectation denoted by

$$(2.10) \quad \mu_L(Q, K) \stackrel{\text{def}}{=} E[L(Q, K)] .$$

Let  $PR_X(Q, K)$  be the random variable describing the profit of Company X. One finds that

$$(2.11) \quad PR_X(Q, K) = \alpha Q - cK + L(Q, K) .$$

In parallel with (2.9) and (2.10), we define

$$(2.12) \quad \begin{aligned} H_{PR:X(Q,K)}(x) &= P[PR_X(Q, K) \leq x] ; \\ \bar{H}_{PR:X(Q,K)}(x) &= P[PR_X(Q, K) > x] \end{aligned}$$

and

$$(2.13) \quad \begin{aligned} \pi_X(Q, K) &\stackrel{\text{def}}{=} E[PR_X(Q, K)] \\ &= \alpha Q - cK + \mu_L(Q, K) , \end{aligned}$$

where the last equality holds from (2.10) and (2.11).

The first problem we consider for Company X is to determine the optimal contract quantity  $Q^*$  and the optimal number of bloggers  $K^*$  to be hired so as to maximize  $\pi_X(Q, K)$ . Let  $\mathbb{N}$  be the set of non-negative integers. This problem can then be described formally as

$$\textbf{Problem 2.1} \quad \max_{Q,K} \pi_X(Q, K) \quad \text{subject to } Q, K \in \mathbb{N} .$$

For notational convenience, we write

$$(2.14) \quad (Q^*, K^*) = \arg \max_{Q,K} \pi_X(Q, K) \quad \text{subject to } Q, K \in \mathbb{N} .$$

Alternatively, the underlying risk may be controlled by the VaR approach where the probability of the profit of Company X falling below  $v_0$  is minimized subject to the expected profit staying above  $v_1$ . That is,

$$\textbf{Problem 2.2} \quad \begin{aligned} &\min_{Q,K} \eta \\ &\text{subject to } H_{PR:X(Q,K)}(v_0) \leq \eta, \pi_X(Q, K) \geq v_1, Q, K \in \mathbb{N} . \end{aligned}$$



An optimal solution of Problem 2.2 is denoted by  $(Q^{**}, K^{**})$  where

$$(2.15) \quad (Q^{**}, K^{**}) = \arg \min_{Q, K} \eta$$

$$\text{subject to } H_{PR:X}(Q, K)(v_0) \leq \eta, \pi_X(Q, K) \geq v_1, Q, K \in \mathbb{N} .$$

In the next two sections, Problems 2.1 and 2.2 are analyzed in detail and computational procedures are developed for finding optimal solutions.

### 3 Optimal Strategy for Maximizing Expected Profit

In order to study the basic properties of the expected profit  $\pi_X(Q, K)$  given in (2.13), it is necessary to understand those of  $\mu_L(Q, K)$  in (2.10), which is the expected return of the portion of the contract limited to the additional compensation and the penalty for Company X. We first express  $\mu_L(Q, K)$  in terms of the distribution function and the survival function of  $D$ .

**Theorem 3.1** *Let  $\mu_L(Q, K)$  be as in (2.10). One then has*

$$\mu_L(Q, K) = v(K) \left[ \beta \int_{\frac{Q}{v(K)}}^{\infty} \bar{F}_D(x) dx - \gamma \int_0^{\frac{Q}{v(K)}} F_D(x) dx \right] .$$

**Proof** From (2.5) and (2.7), it can be seen that

$$\mu_L(Q, K) = \beta \int_Q^{\infty} (x - Q) dF_{N(K)}(x) - \gamma \int_0^Q (Q - x) dF_{N(K)}(x) ,$$

which in turn leads to

$$(3.1) \quad \mu_L(Q, K) = \beta \left\{ \int_Q^{\infty} x dF_{N(K)}(x) - Q \bar{F}_{N(K)}(Q) \right\}$$

$$- \gamma \left\{ Q F_{N(K)}(Q) - \int_0^Q x dF_{N(K)}(x) \right\} .$$

By integration by parts, one sees that

$$\int_Q^{\infty} x dF_{N(K)}(x) = Q \bar{F}_{N(K)}(Q) + \int_Q^{\infty} \bar{F}_{N(K)}(x) dx$$

and

$$\int_0^Q x dF_{N(K)}(x) = Q F_{N(K)}(Q) - \int_0^Q F_{N(K)}(x) dx .$$

Substituting these into (3.1) then yields

$$(3.2) \quad \mu_L(Q, K) = \beta \int_Q^\infty \bar{F}_{N(K)}(x)dx - \gamma \int_0^Q F_{N(K)}(x)dx .$$

The theorem now follows from (2.6).  $\square$

Strict monotonicity and concavity of  $\mu_L(Q, K)$  would be shown next.

**Theorem 3.2** *Let  $\mu_L(Q, K)$  be as in (2.10). Given  $K \geq 0$ , the following statements hold true.*

- (a)  $\mu_L(Q, K)$  is strictly decreasing and concave in  $Q$ .
- (b)  $\mu_L(Q, K)$  achieves 0 uniquely at  $Q_0$  satisfying  $\int_{\frac{Q_0}{v(K)}}^\infty \bar{F}_D(x)dx = \frac{\gamma}{\beta} \int_0^{\frac{Q_0}{v(K)}} F_D(x)dx$ .

**Proof** By differentiating (3.2) with respect to  $Q$ , one finds that

$$(3.3) \quad \frac{\partial}{\partial Q} \mu_L(Q, K) = -\{\beta \bar{F}_{N(K)}(Q) + \gamma F_{N(K)}(Q)\} < 0 ,$$

and the strict monotonicity of  $\mu_L(Q, K)$  in  $Q$  follows. To prove concavity, we differentiate (3.3) one more time with respect to  $Q$ , yielding

$$(3.4) \quad \left(\frac{\partial}{\partial Q}\right)^2 \mu_L(Q, K) = -(\gamma - \beta) f_{N(K)}(Q) .$$

From (2.1), it can be seen that  $\gamma - \beta > 0$  and hence  $\left(\frac{\partial}{\partial Q}\right)^2 \mu_L(Q, K) < 0$  from (3.4), proving part (a).

For part (b), we note from Theorem 3.1 that

$$(3.5) \quad \lim_{Q \rightarrow 0} \mu_L(Q, K) = v(K)\beta\mu_D > 0 ; \quad \lim_{Q \rightarrow \infty} \mu_L(Q, K) = -\infty .$$

The strict monotonicity of  $\mu_L(Q, K)$  proven in part (a) then implies the unique existence of  $Q_0$  such that  $\mu_L(Q_0, K) = 0$ , and  $Q_0$  should satisfy  $\int_{\frac{Q_0}{v(K)}}^\infty \bar{F}_D(x)dx = \frac{\gamma}{\beta} \int_0^{\frac{Q_0}{v(K)}} F_D(x)dx$  from Theorem 3.1, completing the proof.  $\square$

We next turn our attention to the basic properties of the expected profit  $\pi_X(Q, K)$ , which can be written from (2.13) and Theorem 3.1 as

$$(3.6) \quad \pi_X(Q, K) = \alpha Q - cK + v(K) \left[ \beta \int_{\frac{Q}{v(K)}}^\infty \bar{F}_D(x)dx - \gamma \int_0^{\frac{Q}{v(K)}} F_D(x)dx \right] .$$

For notational convenience, the inverse function of  $F_D$  is denoted by  $F_D^{-1}$  so that  $y = F_D(x)$  implies  $x = F_D^{-1}(y)$ . We also define

$$(3.7) \quad \xi(\alpha, \beta, \gamma) \stackrel{\text{def}}{=} F_D^{-1}\left(\frac{\alpha - \beta}{\gamma - \beta}\right),$$

where  $0 < \frac{\alpha - \beta}{\gamma - \beta} < 1$  is assured from (2.1).

**Theorem 3.3** *Let  $\pi_X(Q, K)$  be as in (2.13). Given  $K \geq 0$ , the following statements hold true.*

- (a)  $\pi_X(Q, K)$  is strictly concave in  $Q$ .
- (b)  $\pi_X(Q, K)$  achieves the global maximum uniquely at  $Q_{\max}(K) = v(K)\xi(\alpha, \beta, \gamma)$ , where  $\xi(\alpha, \beta, \gamma)$  is as given in (3.7).

**Proof** From (3.3), one sees that

$$(3.8) \quad \frac{\partial}{\partial Q} \mu_L(Q, K) = -\{\beta + (\gamma - \beta)F_{N(K)}(Q)\}.$$

Since  $\frac{\partial}{\partial Q} \pi_X(Q, K) = \alpha + \frac{\partial}{\partial Q} \mu_L(Q, K)$  from (2.13), substitution of (3.8) into this equation along with (2.6) implies that

$$(3.9) \quad \frac{\partial}{\partial Q} \pi_X(Q, K) = (\alpha - \beta) \left\{ 1 - \frac{\gamma - \beta}{\alpha - \beta} F_D\left(\frac{Q}{v(K)}\right) \right\}.$$

By differentiating (3.9) with respect to  $Q$  once again, one finds that

$$(3.10) \quad \left(\frac{\partial}{\partial Q}\right)^2 \pi_X(Q, K) = -(\gamma - \beta) \frac{1}{v(K)} f_D\left(\frac{Q}{v(K)}\right) < 0,$$

where the last inequality results from (2.1) together with  $v(K) > 0$  and  $f_D(x) > 0$ , proving part (a).

For part (b), the strict concavity of  $\pi_X(Q, K)$  from part (a) implies that  $\pi_X(Q, K)$  achieves its global maximum uniquely at  $Q_{\max}(K)$  at which  $\frac{\partial}{\partial Q} \pi_X(Q, K)$  vanishes. From (3.9), this observation leads to

$$1 - \frac{\gamma - \beta}{\alpha - \beta} F_D\left(\frac{Q_{\max}(K)}{v(K)}\right) = 0,$$

or equivalently from (3.7),

$$(3.11) \quad Q_{\max}(K) = v(K)\xi(\alpha, \beta, \gamma),$$

proving the theorem. □

Given  $K \geq 0$ , Theorem 3.3 states that  $\pi_X(Q, K)$  achieves its global maximum at  $Q_{max}(K) = v(K)\xi(\alpha, \beta, \gamma)$ . In order to solve Problem 2.1, apart from the integer constraints  $Q, K \in \mathbb{N}$ , it then suffices to maximize  $\pi_X(Q_{max}(K), K)$  with respect to  $K \geq 0$ . In this regard, let

$$(3.12) \quad V_X(K) \stackrel{\text{def}}{=} \pi_X(Q_{max}(K), K) .$$

The next theorem enables one to evaluate the unique maximum of  $V_X(K)$  at  $K_{max}$ . For notational convenience, let  $G(z)$  be defined by

$$(3.13) \quad G(z) = \alpha z + \beta \int_z^\infty \bar{F}_D(x) dx - \gamma \int_0^z F_D(x) dx .$$

**Theorem 3.4** *Let  $\xi(\alpha, \beta, \gamma)$ ,  $V_X(K)$  and  $G(z)$  be as in (3.7), (3.12) and (3.13) respectively. The following statements then hold true.*

- (a)  $V_X(K)$  is strictly concave in  $K$ .
- (b)  $V_X(K)$  achieves the global maximum uniquely at  $K_{max} = \left(\frac{d}{dK}v\right)^{-1}\left(\frac{c}{G(\xi(\alpha, \beta, \gamma))}\right)$ , where  $\left(\frac{d}{dK}v\right)^{-1}$  is the inverse function of  $\frac{d}{dK}v$  so that  $y = \frac{d}{dK}v(K)$  implies  $K = \left(\frac{d}{dK}v\right)^{-1}(y)$ .

**Proof** From Theorems 3.2 and 3.3 together with (2.13) and (3.12), one finds, after a little algebra, that

$$(3.14) \quad V_X(K) = v(K)G(\xi(\alpha, \beta, \gamma)) - cK .$$

It should be noted that  $G(\xi(\alpha, \beta, \gamma))$  is independent of  $K$ . From (3.13), it can be seen that

$$(3.15) \quad \frac{d}{dz}G(z) = (\alpha - \beta)\left[1 - \frac{\gamma - \beta}{\alpha - \beta}F_D(z)\right] .$$

Clearly,  $\frac{d}{dz}G(z)$  is strictly decreasing in  $z$  with  $\lim_{z \rightarrow 0} \frac{d}{dz}G(z) = (\alpha - \beta) > 0$  and  $\lim_{z \rightarrow \infty} \frac{d}{dz}G(z) = -(\gamma - \alpha) < 0$ . This observation along with (3.7) and (3.15) then implies that  $\frac{d}{dz}G(z)$  takes the value of 0 uniquely at  $z = \xi(\alpha, \beta, \gamma)$ , which in turn proves that  $G(z)$  is strictly concave in  $z$  having the unique global maximum at  $z = \xi(\alpha, \beta, \gamma)$ . From (3.13), it can be seen that  $\lim_{z \rightarrow 0} G(z) = \beta\mu_D > 0$ . One then concludes that  $G(\xi(\alpha, \beta, \gamma)) > 0$ . Since  $v(K)$  is assumed to be strictly increasing and concave in  $K$ , part (a) now follows by differentiating  $V_X(K)$  in (3.14) twice with respect to  $K$ .

From part (a),  $V_X(K)$  achieves its global maximum at  $K_{max}$  at which the first derivative vanishes. From (3.14), it can be readily seen that

$$(3.16) \quad \frac{d}{dK}V_X(K) = \frac{d}{dK}v(K)G(\xi(\alpha, \beta, \gamma)) - c .$$

Since  $G(\xi(\alpha, \beta, \gamma)) > 0$  as shown above, part (b) can be proven by letting  $\frac{d}{dK} V_X(K) = 0$  in (3.16), completing the proof.  $\square$

Theorems 3.3 and 3.4 assure that

$$(3.17) \quad (Q_{max}(K_{max}), K_{max}) = \arg \max_{Q, K} \pi_X(Q, K) .$$

In order to cope with the integer constraints  $Q, K \in \mathbb{N}$ , we adopt the following approximation. For a real number  $x$ , let  $\lceil x \rceil$  and  $\lfloor x \rfloor$  be the ceiling and the flooring of  $x$ , describing the smallest integer which is greater than or equal to  $x$  and the largest integer which is less than or equal to  $x$  respectively. We then approximate  $(Q^*, K^*)$  given in (2.14) by

$$(3.18) \quad \pi_X(Q^*, K^*) = \max \begin{cases} \pi_X(\lceil Q_{max}(K_{max}) \rceil, \lceil K_{max} \rceil) \\ \pi_X(\lceil Q_{max}(K_{max}) \rceil, \lfloor K_{max} \rfloor) \\ \pi_X(\lfloor Q_{max}(K_{max}) \rfloor, \lceil K_{max} \rceil) \\ \pi_X(\lfloor Q_{max}(K_{max}) \rfloor, \lfloor K_{max} \rfloor) \end{cases} .$$

An algorithmic procedure for solving Problem 2.1 can now be summarized as follows.

### Algorithm 3.1

- [1] Calculate  $\xi(\alpha, \beta, \gamma)$  based on (3.7).
- [2] Evaluate  $G(\xi(\alpha, \beta, \gamma))$  from (3.13).
- [3] Find  $K_{max}$  from Theorem 3.4 (b).
- [4] Compute  $Q_{max}(K_{max})$  from (3.11).
- [5] Determine  $(Q^*, K^*)$  through (3.18) by computing  $\pi_X(Q, K)$  from (3.6).

## 4 Optimal Strategy Based on Value at Risk Approach

When the distribution function of  $D$  has a long tail, the optimal strategy for maximizing the expected profit could result in the probability of having the profit below  $v_0$  too high. One may contain this risk by trying to make the probability as low as possible with compromise of keeping the expected profit above  $v_1$  instead of pursuing its maximum value. This approach, called the VaR approach, is formulated as Problem 2.2 in Section 2.

In this section, we establish a computational foundation for finding the optimal strategy  $(Q^{**}, K^{**})$  for Problem 2.2. For this purpose, the first step would be to evaluate the distribution function  $H_{L(Q, K)}(x)$  of  $L(Q, K)$  in terms of the distribution

function of  $D$ . We define  $\delta_{\{Statement\}} = 1$  if Statement is true, and  $\delta_{\{Statement\}} = 0$  otherwise.

**Theorem 4.1** *Let  $H_{L(Q,K)}(x)$  be as in (2.9). One then has*

$$(4.1) \quad \begin{aligned} H_{L(Q,K)}(x) &= \delta_{\{-\gamma Q \leq x \leq 0\}} F_D\left(\frac{1}{v(K)}\left(\frac{x}{\gamma} + Q\right)\right) \\ &\quad + \delta_{\{x \geq 0\}} F_D\left(\frac{1}{v(K)}\left(\frac{x}{\beta} + Q\right)\right). \end{aligned}$$

**Proof** By the law of total probability, we first observe that

$$\begin{aligned} H_{L(Q,K)}(x) &= P[L(Q,K) \leq x] \\ &= P[L(Q,K) \leq x, 0 \leq N(K) < Q] + P[L(Q,K) \leq x, Q \leq N(K)]. \end{aligned}$$

From (2.7), the above equation then leads to

$$H_{L(Q,K)}(x) = P[-\gamma(Q - N(K)) \leq x, 0 \leq N(K) < Q] + P[\beta(N(K) - Q) \leq x, Q \leq N(K)].$$

Arranging the expressions inside the probabilities with focus on  $N(K)$  and using (2.4), one concludes that

$$(4.2) \quad \begin{aligned} H_{L(Q,K)}(x) &= P[D \leq \frac{1}{v(K)} \min\left\{\frac{x}{\gamma} + Q, Q\right\}] \\ &\quad + P[\frac{Q}{v(K)} \leq D \leq \frac{1}{v(K)}(\frac{x}{\beta} + Q)]. \end{aligned}$$

Since  $0 \leq \frac{x}{\gamma} + Q < Q$  if and only if  $-\gamma Q \leq x < 0$ , the first probability on the right hand side of (4.2) can be written as

$$(4.3) \quad \begin{aligned} P[D \leq \frac{1}{v(K)} \min\left\{\frac{x}{\gamma} + Q, Q\right\}] &= \delta_{\{-\gamma Q \leq x < 0\}} F_D\left(\frac{1}{v(k)}\left(\frac{x}{\gamma} + Q\right)\right) \\ &\quad + \delta_{\{x \geq 0\}} F_D\left(\frac{Q}{v(K)}\right). \end{aligned}$$

Similarly, for the second probability on the right hand side of (4.2), one sees that

$$(4.4) \quad P[\frac{Q}{v(K)} \leq D \leq \frac{1}{v(K)}(\frac{x}{\beta} + Q)] = \delta_{\{x \geq 0\}} [F_D(\frac{1}{v(k)}(\frac{x}{\beta} + Q)) - F_D(\frac{Q}{v(K)})].$$

The theorem now follows from (4.2), (4.3) and (4.4).  $\square$

The next corollary is a direct consequence of Theorem 4.1.

**Corollary 4.2** *Let  $H_{L(Q,K)}(x)$  be as in (2.9). Then  $H_{L(Q,K)}(x)$  is increasing in  $Q$  and decreasing in  $K$  for any given real number  $x$ .*

We are now in a position to evaluate the distribution function of the profit of Company X, i.e.  $H_{PR:X(Q,K)}(x) = P[PR_X(Q, K) \leq x]$  as in (2.12), which is of our main concern in this section. From (2.11), it can be seen that

$$(4.5) \quad H_{PR:X(Q,K)}(x) = H_{L(Q,K)}(x - \alpha Q + cK) .$$

The next theorem is then immediate from Theorem 4.1 and (4.5). For notational convenience, we define

$$(4.6) \quad T(x, \rho, Q, K) = \frac{x + cK + (\rho - \alpha)Q}{\rho v(K)}$$

and

$$(4.7) \quad LE(Q, K) = -(\gamma - \alpha)Q - cK ; \quad RE(Q, K) = \alpha Q - cK .$$

**Theorem 4.3**

$$\begin{aligned} H_{PR:X(Q,K)}(x) &= \delta_{\{LE(Q,K) \leq x < RE(Q,K)\}} F_D(T(x, \gamma, Q, K)) \\ &\quad + \delta_{\{RE(Q,K) \leq x\}} F_D(T(x, \beta, Q, K)) . \end{aligned}$$

Theorem 4.3 provides a computational foundation for solving Problem 2.2 numerically. Namely, one should find the feasible region  $FR$  defined by

$$(4.8) \quad FR = \{(Q, K); \pi_X(Q, K) \geq v_I\} ,$$

which is guaranteed to be compact, i.e. closed and bounded. By computing  $H_{PR:X(Q,K)}(v_0)$  for  $(Q, K) \in FR$  based on Theorem 4.3, the optimal solution  $(Q^{**}, K^{**})$  achieving the minimum value in  $FR$  could be found numerically.

## 5 Numerical Examples

In this section, the optimal strategy  $(Q^*, K^*)$  for Problem 2.1 and the optimal strategy  $(Q^{**}, K^{**})$  for Problem 2.2 are explored numerically. Throughout the section,

we assume that the function  $v(K)$  introduced in (2.4) is given by

$$(5.1) \quad v(K) = \frac{1 + rK}{1 + wK} ,$$

where  $r > w$ . It then follows that

$$(5.2) \quad \frac{d}{dK}v(K) = \frac{r - w}{(1 + wK)^2} > 0 ; \quad \left(\frac{d}{dK}\right)^2 v(K) = -\frac{2(r - w)w}{(1 + wK)^3} < 0 ,$$

satisfying  $v(0) = 1$  and  $v(K)$  is strictly increasing and concave in  $K$ , as assumed. The inverse function of  $\frac{d}{dK}v$  can be obtained from (5.2) explicitly. More specifically, with  $\varsigma = \frac{d}{dK}v(K)$ , one has

$$(5.3) \quad K = \left(\frac{d}{dK}v\right)^{-1}(\varsigma) = \frac{1}{w} \left[ \sqrt{\frac{r - w}{\varsigma}} - 1 \right] .$$

We also assume that the non-negative random variable  $D$ , describing the number of positive comments about Product Z to appear naturally within  $SN$  without any e-Marketing efforts, is exponentially distributed with parameter  $\lambda$ , that is

$$(5.4) \quad F_D(x) = 1 - e^{-\lambda x} ; \quad \bar{F}_D(x) = e^{-\lambda x} ; \quad \mu_D = \frac{1}{\lambda} .$$

The inverse function  $F_D^{-1}$  can then be expressed explicitly. With  $y = F_D(x)$ , one has

$$(5.5) \quad x = F_D^{-1}(y) = -\frac{1}{\lambda} \log(1 - y) .$$

From Theorem 3.1, it can be seen, after a little algebra, that

$$(5.6) \quad \mu_L(Q, K) = v(K) \left[ \frac{\beta - \gamma}{\lambda} e^{-\lambda \frac{Q}{v(K)}} + \gamma \left( \frac{1}{\lambda} - \frac{Q}{v(K)} \right) \right] .$$

From (2.13), the expected profit  $\pi_X(Q, K)$  is obtained accordingly as

$$(5.7) \quad \pi_X(Q, K) = \alpha Q - cK + v(K) \left[ \frac{\beta - \gamma}{\lambda} e^{-\lambda \frac{Q}{v(K)}} + \gamma \left( \frac{1}{\lambda} - \frac{Q}{v(K)} \right) \right] .$$

Furthermore,  $\xi(\alpha, \beta, \gamma)$  in (3.7) can also be evaluated in a closed form, i.e.

$$(5.8) \quad \xi(\alpha, \beta, \gamma) = -\frac{1}{\lambda} \log \left( \frac{\gamma - \alpha}{\gamma - \beta} \right) .$$

The function  $G(z)$  in (3.13) and its value at  $z = \xi(\alpha, \beta, \gamma)$  are obtained as

$$(5.9) \quad G(z) = \frac{\gamma}{\lambda} - (\gamma - \alpha)z - \frac{\gamma - \beta}{\lambda} e^{-\lambda z}$$

and

$$(5.10) \quad G(\xi(\alpha, \beta, \gamma)) = \frac{\alpha}{\lambda} - (\gamma - \alpha)\xi(\alpha, \beta, \gamma) .$$

Combining these results, Algorithm 3.1 can now be rewritten as follows.



**Algorithm 5.1**

[1] Calculate  $\xi(\alpha, \beta, \gamma) = -\frac{1}{\lambda} \log(\frac{\gamma-\alpha}{\gamma-\beta})$  .

[2] Evaluate  $G(\xi(\alpha, \beta, \gamma)) = \frac{\alpha}{\lambda} - (\gamma - \alpha)\xi(\alpha, \beta, \gamma)$  .

[3] Obtain  $K_{max}$  by

$$K_{max} = (\frac{d}{dK}v)^{-1}(\frac{c}{G(\xi(\alpha, \beta, \gamma))}) = \frac{1}{w} \left[ \sqrt{\frac{(r-w)G(\xi(\alpha, \beta, \gamma))}{c}} - 1 \right] .$$

[4] Compute  $Q_{max}(K_{max})$  by

$$Q_{max}(K_{max}) = \frac{1+rK_{max}}{1+wK_{max}} \xi(\alpha, \beta, \gamma) .$$

[5] Determine  $(Q^*, K^*)$  through (3.18) by computing  $\pi_X(Q, K)$  from (5.7).

For numerical examples to follow, the underlying parameter values employed are as summarized in Table 1, unless specified otherwise. In Figure 5.1, the expected profit  $\pi_X(Q, K)$  is plotted as a function of  $Q$  and  $K$ , where the optimal solution for Problem 2.1 is found to be  $(Q^*, K^*) = (66028, 20)$  with  $\pi_X(Q^*, K^*) = 11859$ . In Figures 5.2 and 5.3, the marginal functions  $\pi_X(Q, K^*)$  and  $\pi_X(Q^*, K)$  are depicted respectively. We observe that  $\pi_X(Q, K^*)$  is strictly concave in  $Q$  as proven in Theorem 3.3. For the basic set of the parameter values, the model seems to be insensitive with respect to  $K$  for  $K > 10$ . Figure 5.4 illustrates  $V_X(K) = \pi_X(Q_{max}(K), K)$ , where the strict concavity in  $K$  can be observed as expected from Theorem 3.4 (a).

Table 1: Basic Set of Parameter Values

$\alpha$	0.2
$\beta$	0.1
$\gamma$	0.3
$c$	30
$r$	50
$w$	1
$\lambda$	0.0005
$v_0$	5000
$v_1$	10000

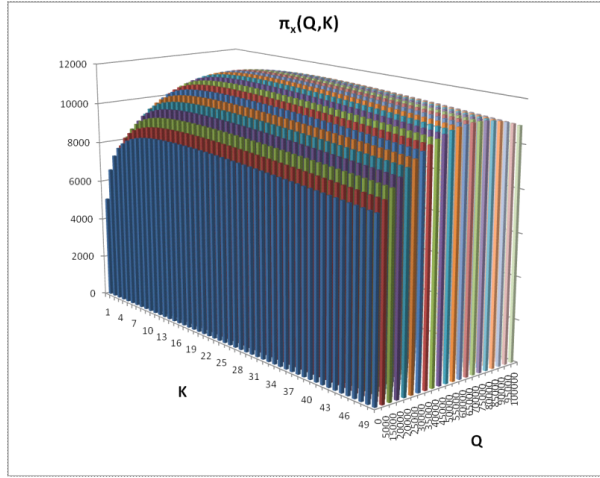


Figure 5.1: Expected Profit  $\pi_X(Q, K)$

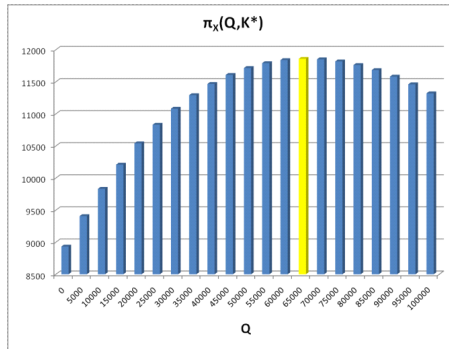


Figure 5.2: Marginal Function  $\pi_X(Q, K^*)$  of Q

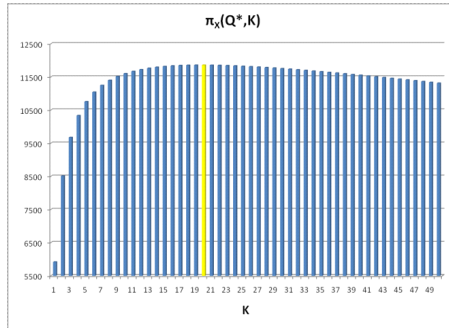


Figure 5.3: Marginal Function  $\pi_X(Q^*, K)$  of K

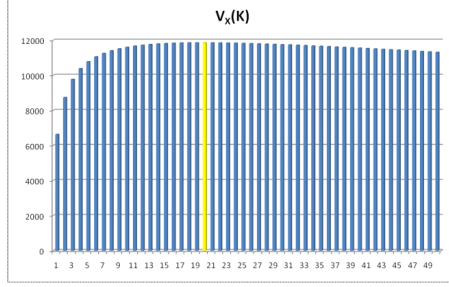


Figure 5.4:  $V_X(K)$

In our model, the two key parameters would be  $\alpha$  and  $c$ , where the former is the unit revenue per promised positive comment for Company X to receive from Company Y, while the latter is the cost paid to each blogger by Company X. In Figures 5.5, 5.6 and 5.7, the optimal solutions  $K^*$ ,  $Q^*$  and the associated optimal expected profit  $\pi_X(Q^*, K^*)$  are plotted on the  $\alpha - c$  plane. As one may expect, we observe that  $K^*$  is increasing in  $\alpha$ , that is, the higher the unit revenue is, the more bloggers Company X can afford to hire. In the opposite way, the higher the cost for each blogger is, the less number of bloggers Company X can afford to hire, and  $K^*$  is decreasing in  $c$ . If  $c$  becomes larger than 20 or so, Company X reaches the affordable level and  $K^*$  becomes almost constant. We find similar monotonicity properties for  $Q^*$ . However,  $Q^*$  is very sensitive to  $\alpha$  but quite insensitive to  $c$ , while  $K^*$  is quite sensitive to  $c$  for  $0 < c \leq 10$  but rather insensitive to  $\alpha$ . The monotonicity properties of  $\pi_X(Q^*, K^*)$  are also in parallel with those of  $K^*$  and  $Q^*$ . However, it is surprisingly insensitive to  $c$ , while it increases fairly rapidly as  $\alpha$  increases.

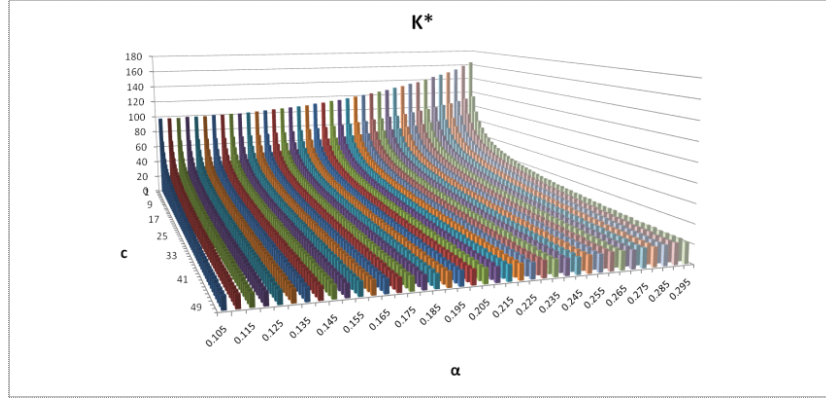


Figure 5.5:  $K^*$

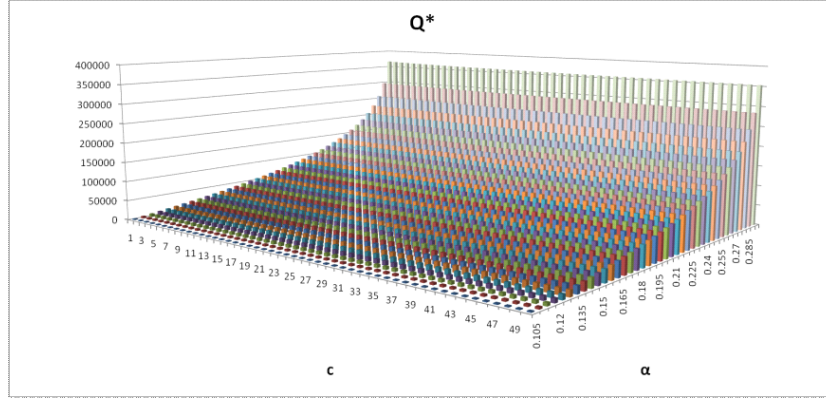


Figure 5.6:  $Q^*$

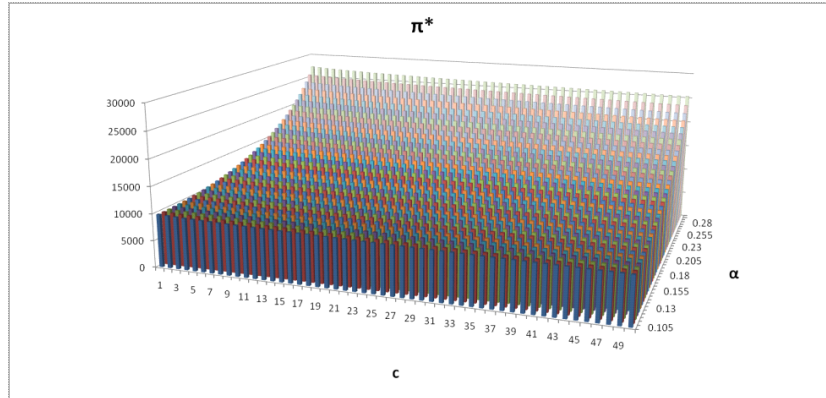


Figure 5.7:  $\pi_X(Q^*, K^*)$

We next demonstrate how Problem 2.2 can be solved numerically. Given  $v_0 = 5000$  and  $v_1 = 10000$ , Figures 5.8 and 5.9 depict  $\pi_X(Q, K)$  and  $H_{PR:X(Q,K)}(v_0)$  respectively for  $(Q, K) \in FR = \{(Q, K); \pi_X(Q, K) \geq v_1\}$ . In order to see the structure better, these graphs are projected onto the  $Q - K$  plane with contour lines in Figures 5.10 and 5.11. Here, the contour lines in the former figure indicate climbing up the hill to the top, whereas those in the latter figure represent going down the valley to the bottom. By reducing the mesh size at the bottom, one finds the optimal solution  $(Q^{**}, K^{**}) = (27000, 13)$  with step size of 1000 for  $Q$ . The corresponding values for  $H_{PR:X(Q^{**}, K^{**})}(v_0)$  and  $\pi_X(Q^{**}, K^{**})$  are 0.25 and 10897 respectively. We note that the maximum value  $\pi_X(Q^*, K^*)$  is 11859 at  $(Q^*, K^*) = (66028, 20)$ , where  $H_{PR:X(Q^*, K^*)}(v_0) = 0.55$ .

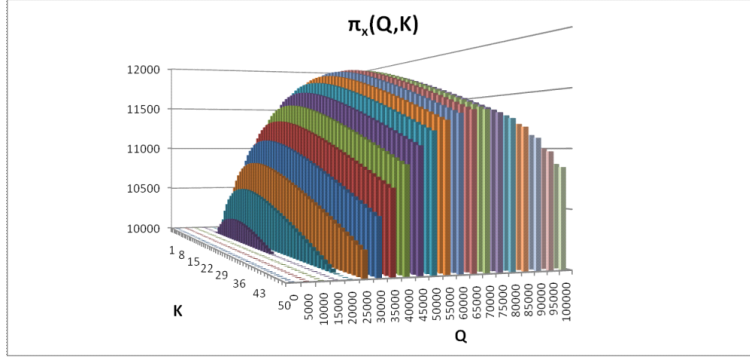


Figure 5.8:  $\pi_X(Q, K)$  for  $(Q, K) \in FR$

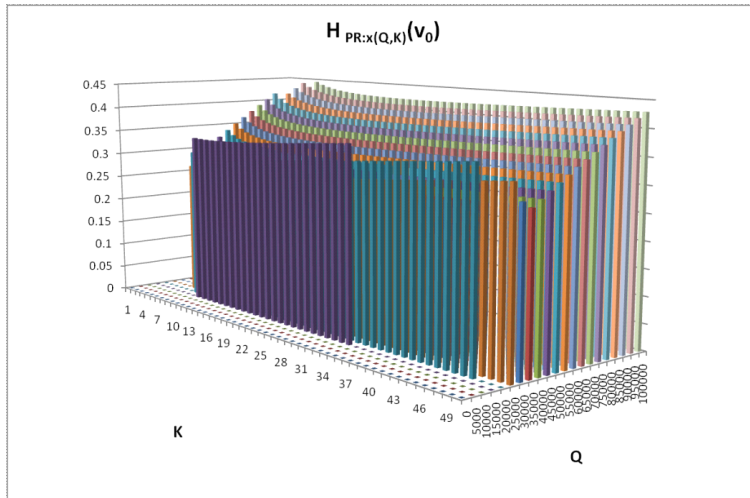


Figure 5.9:  $H_{PR:X(Q,K)}(v_0)$  for  $(Q, K) \in FR$

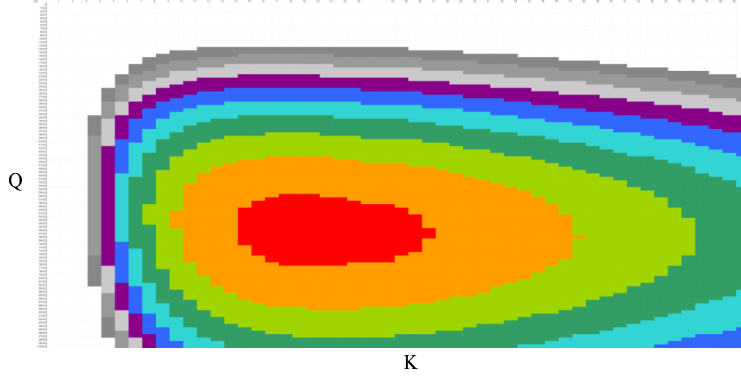


Figure 5.10:  $\pi_X(Q, K)$  for  $(Q, K) \in FR$

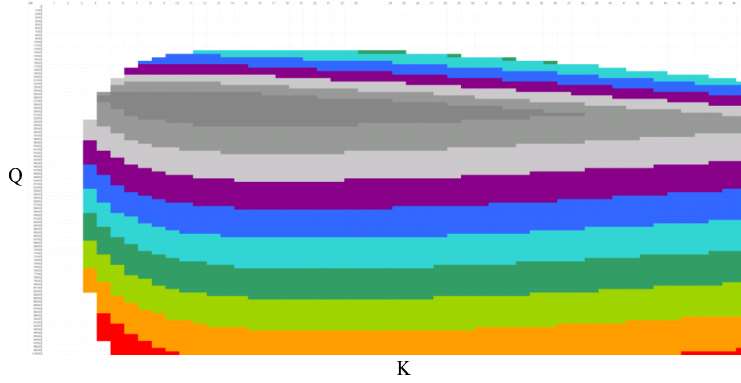


Figure 5.11:  $H_{PR:X(Q,K)}(v_0)$  for  $(Q, K) \in FR$

## 6 Concluding Remarks

In this paper, a new e-Marketing contract structure is proposed, where the contract would be exchanged between an e-Marketing company, named Company X, and a manufacturing company, named Company Y, which is to promote Product Z within  $SN$ . The structure of the contract is as follows.

“Company X promises Company Y to generate  $Q$  positive comments about Product Z within  $SN$ . For this service, Company Y pays Company X by the amount of  $\$ \alpha Q$ . Should the actual number of positive comments exceed  $Q$ , the additional payment of  $\$ \beta$  for each positive comment beyond  $Q$  would be paid to Company X by Company Y. For controlling the risk of the actual number of positive comments to appear within  $SN$  falling below  $Q$ , Company X pays the penalty of  $\$ \gamma$  for each

shortage below  $Q$  to Company Y.”

In order to achieve the promised goal of generating  $Q$  positive comments within  $SN$ , Company X organizes a group of  $K$  bloggers, asking each of them to use Product Z and to write a positive comment about it, if they agree, through his/her blog with compensation of  $\$c$ . The organized group of the bloggers would play a role of pumping water for generating positive comments about Product Z. The problem for Company X is then how to determine  $Q$  and  $K$  so as to optimize its objective.

Two types of problems are considered, where the first problem is to maximize the expected profit of Company X while the second problem is based on the VaR approach to minimize the probability of the profit of Company X falling below  $v_0$  subject to having the expected profit above  $v_1$ . Although this problem has a flavor of the classical news vendor problem, it is more difficult because of its two dimensional nature. For the first problem, the exact optimal solution is derived apart from the integer constraints and an algorithmic procedure is developed for computing the optimal solution. For the second problem, it is shown that the distribution function of the profit of Company X can be given explicitly, thereby providing a computational foundation for solving the problem. Numerical examples are given, illustrating the stochastic structure of the e-Marketing contract and the differences of the optimal strategies for the two problems.

This research is new and many key issues remain unaddressed. Some of such key issues include:

- 1) Effects of the distribution function of  $D$  which is non-exponential
- 2) Exact analysis of the integer approximation of the optimal solutions
- 3) Properties of the optimal solutions as a function of the underlying parameters
- 4) Development of computational algorithms for finding the optimal solution of

Problem 2.2 based on the VaR approach

- 5) Economic merits of Company Y to be incorporated explicitly in the model
- 6) Broad issues concerning the proposed e-Marketing contract structure and  
stealth marketing.

These issues will be addressed in due course and will be reported elsewhere.

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