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Contentwise Complexity of Inferences: An Evaluation of Arrow's Impossibility Theorem

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Contentwise Complexity of Inferences: An Evaluation of Arrow's Impossibility Theorem*, †

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Abstract

We evaluate the required size of a possible proof of Arrow's impossibility theorem in a proof-theoretic manner. The primary purpose is the consideration of (contentwise) complexity required for some statements, and Arrow's theorem is taken as an instance in order to show the possibility that even in a finite case, the required proof is inevitably gigantic. We consider the simplest case with two individuals, three social alternatives, and linear orderings for individual and social preferences. We formulate Arrow's theorem in propositional classical logic in the Gentzen-style proof theory. The size of a proof is measured by the number of leaves of a proof tree. We show that a proof is necessarily gigantic; a lower bound is 6^{36} and an upper bound $6^{37} + 420$. These numbers exceed the limit of human manageability to construct such a proof, but we have a proof of Arrow's theorem, which appears contradictory. We discuss this result from various points of views such as deductive and inductive game theoretical points of view.

1. Introduction

1.1. General Motivation and Background

It is informative to start with describing the main result of the paper, since its motivation has some distance from the subject and result and needs a slightly long explanation. We formulate Arrow's [2] impossibility theorem in propositional classical logic in the Gentzen-style sequent calculus. The size of a proof is measured by its width, i.e., the

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number of leaves of a proof tree. We consider the simplest case with 2 individuals, 3 social alternatives, and linear orderings for individual preferences and social orderings. We show that a proof is necessarily gigantic; a lower bound is 6^{36} and an upper bound $6^{37} + 420$. This is interpreted as meaning that required deductive inferences to prove Arrow's theorem exceed practical manageability.

We aim to understand limitations on human inferential abilities and their behavioral consequences, and are motivated with inductive game theory and more specifically with epistemic logics with shallow depths. We choose Arrow's theorem as an example, since it has various interesting features and is fairly complicated. We start with explaining our general motivation from the view point of inductive game theory and epistemic logics of shallow depths. Since we are not motivated by social choice theory/welfare economics, we will give only a small consideration of the result from the viewpoint of social choice theory in Section 6.[1]

Our basic motivation is to study the origin/emergence of beliefs/knowledge, which was the target of *inductive game theory*. Kaneko-Kline [11] described a basic scenario for inductive game theory from experimentations to inductive derivations of personal views, to behavioral uses, and again to experimentations. In this scenario, inductive inferences are emphasized, but at various steps, deductive inferences also appear. A lot of limitations are and/or should be involved in both experiences and inferences at those steps. We need to evaluate such limitations. The other side of evaluating a limitation is to evaluate the complexity of the structure to which a limitation is applied. In this paper, we will discuss certain features of complexity of deductive inferences, which gives also some implications to inductive game theory.

The concept of "computational complexity" has been discussed a lot in computer science. Computational complexity is applied to a set consisting of *a countably infinite number of problem instances* (cf. Grey-Johnson [8], Urquhart [24]); this is natural in computer science, since its aim is to implement some computational method for it, ultimately, with a computer. In our context, an implementation with a computer is not a problem, but our problem is to measure inferences required in a particular problem. Thus, we would like to see "complexity" of inferences for *a particular problem instance.* Here, we consider a required size of a proof of a particular problem instance; we take Arrow's theorem as such a particular instance specified in the above mentioned form.

Now, let us see our subject from the viewpoint of epistemic logic (cf., Meyer-van der Hoek [19], Kaneko [13]). In the literature of epistemic logic, common knowledge has been focussed a lot, which is rather opposite to the limited case of inferences in that it is an ideal approximation in terms of an infinite nesting structure: "I know you know I know, and so on". Common knowledge has been often regarded as necessary in the game

[1] Ågotnes *et al* [1] formulated Arrow's theorem in a modal- and model-theoretic manner, and provided several model theoretic considerations such as model-checking.

theory literature, but only very shallow depths of interpersonal inferences are typically feasible. To capture such limitations, the concept of a finite epistemic structure was introduced in Kaneko-Suzuki [14], [15] and [16]. This is about *interpersonal* complexity required for a game theoretical problem.

A more basic problem than interpersonal complexity is of *intrapersonal* inferences. This is the direct and natural continuation of the introduction of limitations to epistemic logic. In this paper, we target this intrapersonal complexity of inferences, but do not discuss interpersonal complexity. That is, we do not touch epistemic logic in the strict sense, but the background should be kept in mind.

The result of lower bound 6^{36} and upper bound $6^{37} + 420$ for any proofs of Arrow's theorem is interpreted as meaning that a player cannot find a proof. Hence, if it is involved in his mind for his decision making, it cannot be used. Moreover, we, researchers, would meet the question why we can have Arrow's theorem. Some different structures must be hidden in our thinking (or method). This must have great implications to epistemic logic as well as game theory. To discuss such implications, we need a more specific description of our problem.

1.2. Specific Description of our Problem

Deductive inferences have been studied in mathematical logic, particularly, in proof theory. In mathematical logic, provability (equivalently, validity in model theory) is typically considered, which is characterized as a deductive closure of specified inferences: It describes the complete deductive ability relative to the specified inference rules. This concept does not distinguish between easy and difficult statements. One possibility to distinguish between them is to measure the required inferences for these statements.

Proof theory has various logical systems: The representatives are the Hilbert-style formulation and Gentzen-style (sequent) formulation (cf. Kleene [18] and Kaneko [13])[2]. The former system has a simpler representation and targets to express the provability of each statement. But it does not give a clear-cut description of a proof. On the other hand, the latter is a heavier system, but is constructed in order to discuss a structure of a proof. For this reason, the Hilbert-style is often used to discuss provability together with its semantic counterpart, but when we discuss a proof itself, the Gentzen-style is much more useful. For our problem, since we care about a proof structure of a statement, we will adopt the Gentzen-style sequent formulation.

This required size of a proof of each statement (theorem) was first introduced in Kaneko-Suzuki [17], which they called the contentwise complexity measure, in short, *contemplexity measure.* In [17], general properties of the contemplexity measure are studied and various relatively small examples are examined. In this paper, we apply

[2] Gentzen [7] gave also the system of natural deduction.

the measure to Arrow's theorem, and show that the contemplexity value is gigantic, as already stated in Section 1.1.

Arrow's theorem involves various higher-order concepts as variables: For examples, "a preference ordering" is already a 2nd-order concept, and also since a social welfare function is defined over the set of all preference orderings, it is even a higher-order concept. Nevertheless, we confine ourselves to a finite case. It is a thesis in mathematical logic that any finite mathematical problem can be expressed in a propositional logic. Following this thesis, Arrow's problem in a finite case should be formulated in propositional logic, though it includes higher-order concepts such as orderings and functions over them.

We adopt a nominalistic method, i.e., we give names to identify orderings and social welfare functions. This nominalistic method is possible, since we assume that the structure of the problem is all finite. By this method, we can formulate Arrow's theorem in a propositional logic. With giving names to each higher-order concepts, we will overcome the difficulty arising from them.

In fact, the nominalistic method to treat the higher-order concepts is one reason for our result, as stated already, that we have lower bound 6^{36} and upper bound $6^{37}+420$ for the possible proofs of Arrow's theorem. The nominalist method in propositional logic needs to enumerate all cases, but cannot abbreviate those cases by a "representative case". The concept of a "representative case" is a free variable, which is very often in our ordinary thinking. From the viewpoint of mathematical logic, "free variable" is a concept in predicate logic. We will not touch this problem in this paper, but we cannot avoid "free variables" at the meta-level.

The use of "free variables" at the meta-level enables us to prove our result. Our proof of an upper bound is to give a specific proof of Arrow's theorem, which appears contradictory in that we cannot practically give a proof of size $6^{37} + 420$. The key for this seeming contradiction is the distinction between the target problem and the meta-treatment of it. This will be found clearly in Section 8. This observation resolves the doubt caused by "these numbers exceeding the limit of human manageability to construct such a proof" stated in the beginning of Section 1.1.

An implication of our result to epistemic logic is also important. It is straightforward to embed our result to the system of epistemic logic with a shallow depth in Kaneko-Suzuki [14][3]. Our result can be interpreted from the viewpoint of a player rather than a logician: If a player has a limited logical ability and if a too long process of inferences is needed, then he cannot derive a conclusion. Our result for Arrow's theorem is one (extreme) example for a statement. Also, it implies that if a player' beliefs are (objectively) inconsistent but if they need a too long process to derive a contradiction, he

[3]In Kaneko-Suzuki [17], the base logic of the epistemic logic is taken as *intuitionistic logic*. The contemplexity measure behaves more regularly than the case of classical logic. Thus, it can give more precise treatments of the contemplexity measure.

would keep his beliefs without noticing the inconsistency of his beliefs.

It is also important to notice that it takes so many steps to reach a contradiction. Once a contradiction is reached, any statement can be derived; the axiomatic system effectively collapses. Before reaching a contradiction, the system works properly. This is interpreted as suggesting a possibility that a person has an inconsistent set of beliefs but does not find its inconsistency while keeping his beliefs. This will raise an interesting future research topic.

This paper is written in the following format. Section 2 summarizes the simplest version of Arrow's theorem. Section 3 gives a formalized language and the Gentzen-style sequent calculus of classical logic, and then formulates the contemplexity measure with some illustrative examples. Then, Section 4 formulates Arrow's theorem in the formalized language, and Section 5 present our main theorem. Section 6 summarizes our considerations and give some discussions on possible implications from our result. Then, Section 7 proves the lower bound assertion. Section 8 gives a proof of the upper bound assertion. In proof theory, there are several systems, from which we choose the Gentzen-style sequent calculus of classical logic.

2. The Simplest Version of Arrow's Impossibility Theorem

We have two individuals: $1, 2$, and three social alternatives: α, β, γ. Consider the set $SO := \{\succ^1, ..., \succ^6\}$ of all strict orderings over $\mathbf{A} := \{\alpha, \beta, \gamma\}$, where these are specified as follows:

$$\begin{array}{lll} \succ^1: \alpha, \beta, \gamma; & \succ^2: \beta, \gamma, \alpha; & \succ^3: \gamma, \alpha, \beta \\ \succ^4: \gamma, \beta, \alpha; & \succ^5: \beta, \alpha, \gamma; & \succ^6: \alpha, \gamma, \beta. \end{array} \tag{2.1}$$

In $\succ^2$, for example, β is the best, γ is the second, and α is the worst. For an ordering $\succ$ in SO, we write $u \succ v$ (also $v \prec u$) for $(u, v) \in \succ$. When an ordering in SO is an individual preference relation, we use the symbol $\succ$ for it, and when it a social ordering, we use the symbol $\succcurlyeq$.

Each pair of the product set $SO^2 := SO \times SO$ is called a *profile of individual preferences*. A profile is denoted by $\succ^t = (\succ_1^{t_1}, \succ_2^{t_2})$, where t_1, t_2 are from $\mathbf{6} := \{1, ..., 6\}$. A *social welfare function* φ in Arrow's sense takes the following form:

$$\varphi : SO^2 \to SO. \tag{2.2}$$

The value $\varphi(\succ^t) = \varphi(\succ_1^{t_1}, \succ_2^{t_2})$ is a social (strict) ordering $\succcurlyeq$ in SO.

In the literature of social choice theory, there are various formulations of Arrow's theorem (cf. Arrow *et al.* [3]). Here, we follow one simplest form, which is a simpler variant of Feldman's [4] formulation of Arrow's theorem. It is quite typical that four conditions are imposed in Arrow's theorem. Here, the domain of a social welfare function in (2.2) is already assumed to be SO^2. This assumption is called the *unrestricted domain UD*. In this paper, condition *UD* is always assumed, and thus we will not mention it in an explicit manner.

Now, we give other three conditions for Arrow's theorem. The first is the Pareto rule.

Pareto-Rule (*PR*): Let $u, v \in \mathbf{A}$ and $\succ^t = (\succ_1^{t_1}, \succ_2^{t_2}) \in SO^2$ with $\varphi(\succ^t) = \succcurlyeq^t$. If $u \succ_1^{t_1} v$ and $u \succ_2^{t_2} u$, then $u \succcurlyeq^t v$.

The following formulation of the IIA condition is slightly weaker than the standard formulation.

Independence of Irrelevant Alternatives (*IIA*): Let any $u, v \in \mathbf{A}$, $\succ^t, \succ^{t'} \in SO^2$ with $\varphi(\succ^t) = \succcurlyeq^t$, $\varphi(\succ^{t\prime}) = \succcurlyeq^{t\prime}$, and $\{i, j\} = \{1, 2\}$. If $u \succ_i^{t_i} v$, $v \succ_j^{t_j} u$ and $u \succcurlyeq^t v$, then $u \succ_i^{t'_i} v$ and $v \succ_j^{t'_j} u$ imply $u \succcurlyeq^{t'} v$.

The standard formulation of *IIA* for the case of strict orderings takes care of the two other cases:

(1): if $u \succ_i^{t_i} v$, $u \succ_j^{t_j} v$ and $u \succcurlyeq^t v$, then $u \succ_i^{t'_i} v$ and $u \succ_j^{t'_j} v$ imply $u \succcurlyeq^{t'} v$;

(2): if $u \succ_i^{t_i} v$, $u \succ_j^{t_j} v$ and $v \succcurlyeq^t u$, then $u \succ_i^{t'_i} v$ and $u \succ_j^{t'_j} v$ imply $v \succcurlyeq^{t'} u$.

But *PR* takes care of both cases, i.e., the above *IIA* is equivalent to the standard one under *PR*. In the proof-theoretical treatment, there is a large difference between the above two formulations; the form we adopt is much simpler than the standard one.

The third means that a social ordering should not be totally determined by one individual.

Non-Dictatorship (*ND*): For neither $i = 1, 2$, it holds that for any $u, v \in \mathbf{A}$ and $\succ^t = (\succ_1^{t_1}, \succ_2^{t_2}) \in SO^2$ with $\varphi(\succ^t) = \succcurlyeq^t$, if $u \succ_i^{t_i} v$, then $u \succcurlyeq^t v$.

Now, Arrow's theorem is stated as follows.

Theorem 2.1 (Arrow's Impossibility Theorem: the Simplest Version): There is no social welfare function in the form of (2.2) satisfying the three conditions, *PR,IIA* and *ND*. In other words, *PR,IIA* and *ND* (also *UD*) on social welfare functions are contradictory.

This is a simplification of Feldman's [4] version of Arrow's theorem in that social orderings are assumed here to be strict ones, while they may be complete preorderings in [4].

The number of possible candidates for a social welfare function in the sense of (2.2) is calculated as

$$6^{36} = 10314424798490535546171949056 > 10^{28} \doteqdot 2^{93}$$

Our question is whether these gigantic number of cases are essential or not. We will give an affirmative answer in the propositional formulation. For the later purpose, it would be convenient to name these social welfare functions:

$$\varphi_1,\ \varphi_2, ...,\ \varphi_{6^{36}}. \tag{2.3}$$

For example, φ_1 is fixed to be the social welfare function with dictator 1, i.e., for all profiles $\succ^t = (\succ_1^{t_1}, \succ_1^{t_1}) \in SO^2$ with $\varphi_1(\succ^t) = \succcurlyeq^t$ and for all $u, v \in \mathbf{A}$, $u \succ_1^{t_1} v$ implies $u \succcurlyeq^t v$. This naming will be used in Section 3.

In the literature of social choice theory, we have various impossibility theorems besides Arrow's theorem. Some are variants of Arrow's theorem, e.g., the Gibbart-Satterthwaite Theorem (cf. Peleg [21]), Sen's Libertarian Impossibility Theorem (cf., Sen [23], Chap.6). Since they are similar to Arrow's theorem, once Arrow's theorem is evaluated from the viewpoint of contemplexity, we could have similar estimates for those theorems.

However, we have some other simpler contradictory statements. In order to see how our theory distinguish between such statements and Arrow's theorem, we consider one very simple example. The comparison will be continued in the formalized language in Section 4. Consider a binary relation D over $\mathbf{A} = \{\alpha, \beta, \gamma\}$. We assume the following two conditions:

Transitivity (TR) : for any $u, v, w \in \mathbf{A}$, $D(u, v)$ and $D(v, w)$ imply $D(u, w)$;

Asymmetry (AS): for any $u, v \in \mathbf{A}$, $D(u, v)$ imply not $D(v, u)$.

Then, we state the following as a theorem.

Theorem 2.2.(Cyclical Impossibility): There is no binary relation D satisfying $\Phi = \{D(\alpha, \beta), D(\beta, \gamma), D(\gamma, \alpha)\}$, TR and AS. In other words, the set of conditions Φ, TR, AS is contradictory.

Proof. A proof is simple: By $D(\alpha, \beta), D(\beta, \gamma)$ and TR, we have $D(\alpha, \gamma)$. By AS, we have not $D(\gamma, \alpha)$. However, we have $D(\gamma, \alpha)$ as an assumption in Φ. Hence, Φ, TR, AS are contradictory. ■

Both theorems exhibit impossibility results. We may say that Arrow's theorem seems to require a much more complex proof than Theorem 2.2. Nevertheless, this statement is informal in the above formulations, and we cannot evaluate which is really more complex. Our contemplexity measure will separate these theorems, which will be discussed in the subsequent sections.

3. Classical Logic in the Gentzen-style and the Contemplexity Measure

3.1. Language and the Formal System

Here, we give a formal language in which Arrow's impossibility theorem is formalized. First, we prepare:

alternative constants: α, β, γ;

binary preference letters: $R_i^{t_i} : i = 1, 2$ and $t_i \in \mathbf{6}$;

binary social ordering predicate letters: $P_s^t : s = 1, ..., 6^{36}$ and $t = (t_1, t_2) \in \mathbf{6}^2$;

[for the formalization of Theorem 2.2, we add one more binary predicate D]

logical connectives: $\neg$ (not), $\supset$ (implies), $\wedge$ (and), $\vee$ (or)[4];

auxiliary symbols: (,) (parentheses), $\{, \}$ (the set-braces), , (comma).

Using those symbols, we define the formulae. The symbol $R_i^{t_i}$ is intended to express the t_i-th preference ordering of individual i, and P_s^t to express the social ordering given by the s-th social welfare function and the profile named $t = (t_1, t_2)$. These correspond to the nonformalized concepts $\succ_i^{t_i}$ and $\succcurlyeq_s^t$ given in Section 2. Here, social welfare functions are all named by $s = 1, ..., 6^{36.}$

To avoid some complications, we use a finite *multi-set* rather than a finite set. It counts the occurrences of each element, but not the order of the occurrences of elements. For example, $\{x, x, y\}$ differs from $\{x, y\}$, but is the same as $\{x, y, x\}$. More precisely, The mutli-sets $\{x_1, ..., x_m\}$ is characterized as the set of all sequences $(y_1, ..., y_m)$ obtained from $(x_1, ..., x_m)$ with permutations. This remark will be elaborated after our logical system is given.

First, the *atomic formulae* are defined as follows:

(i): for any preference letters $R_i^{t_i}$ and any u, v in $\mathbf{A}$, $R_i^{t_i}(u, v)$ is an atomic formula;

(ii): for any social ordering predicate P_i^t and any u, v in $\mathbf{A}$, $P_s^t(u, v)$ is an atomic formula;

[(iii): also for the formalization of Theorem 2.2, for any u, v in $\mathbf{A}$, $D(u, v)$ is an atomic formula].

Then, we define *formulae* by the following induction:

(0): any atomic formula is a formula;

(1): if A and B are formulae, so are $(\neg A)$ and $(A \supset B)$;

(2): if $\{A_1, ..., A_m\}$ is a nonempty multi-set of formulae, then $\wedge\{A_1, ..., A_m\}$ and $\vee\{A_1,$

[4]It is known that in classical logic, we could choose some subset of these connectives and can define the other connectives by the chosen ones. We have two reasons for the choice of four connectives. The conept of a proof depends upon this choice: Hence, it would be natural to use the four connectives. Another reason is that we would like to connect our consideration to the case of intuitionistic logic: In intuitionistic logic, these four connectives are independent.

$..., A_m\}$ are formulae;

(3): any formula is generated by a finite number of applications of (0) - (2).

We denote the set of all formulae by $\mathcal{P}$.

We abbreviate some parentheses as far as no confusions are expected. Also, we will write $\vee\{A_1, A_2\}, \wedge\{A_1, A_2, A_3\}$, etc., as $A_1 \vee A_2, A_1 \wedge A_2 \wedge A_3$.

In the above language, the identity of a "social welfare function" will be determined by the index s. For each s, a profile $R^t = (R_1^{t_1}, R_2^{t_2})$ and two social alternatives u, v, social preference relation $P_s^t(u, v)$ or $\neg P_s^t(u, v)$ is determined. In Section 4, we will formulate conditions, *PR, IIA* and *ND* in this language.

We adopt propositional classical logic in the Gentzen-style sequent formulation (cf. Gentzen [7] and Kleene [18]). Classical logic in the Gentzen style is governed by one axiom schema and twelve inference rules. First, it needs the concept of a sequent. We prepare another auxiliary symbol $\rightarrow$ for the system. Let Γ, Θ be finite (possibly empty) multi-sets of formulae. Then, we call the expression $\Gamma \rightarrow \Theta$ a *sequent*[5], and Γ, Θ the *antecedent* and *succedent* of the sequent, respectively. When $\Gamma = \emptyset$ or $\Theta = \emptyset$, we write $\rightarrow \Theta$ or $\Gamma \rightarrow$.

When $\Gamma = \{A_1, ..., A_m\}$ and $\Theta = \{B_1, ..., B_n\}$, the intended meaning of $\Gamma \rightarrow \Theta$ is $\wedge\{A_1, ..., A_m\} \supset \vee\{B_1, ..., B_n\}$: In particular, if $\Gamma = \emptyset$ or $\Theta = \emptyset$, then it is intended to means $(\neg A) \vee A \supset \vee\Theta$ or $\wedge\Gamma \supset (\neg A) \wedge A$.

We make the following abbreviations: $\Delta \cup \Gamma \rightarrow \Theta \cup \Lambda$ as $\Delta, \Gamma \rightarrow \Theta, \Lambda$, and also $\{A\}, \Gamma \rightarrow \Theta, \{B\}$ as $A, \Gamma \rightarrow \Theta, B$[6]. Here, we allow the sets Δ, Γ and Θ, Λ to have nonempty intersections. In particular, $A \rightarrow A$ means $\{A\} \rightarrow \{A\}$.

The sequent calculus consists of one axiomatic schema and twelve inference rules. Each inference is formulated as

$$\frac{\xi_1, ..., \xi_k}{\xi_0},$$

where each $\xi_0, \xi_1, ..., \xi_k$ $(1 \leq k)$ are sequents. This means that under the assumption that $\xi_1, ..., \xi_k$ are already proved, ξ_0 is inferred from those $\xi_1, ..., \xi_k$. The concept of a proof is defined by means of axioms and inference rules.

Now, we give the axiom schema and the twelve rules.

Axiom Schema (Initial Sequent): $A \rightarrow A$, where A is any formula.

Structural Rules: The following three types of inference rules are called the *thinning,*

[5]In the Gentzen's [7] original formulation, Γ and Θ are assumed to be finite sequences of formulae. Hence, the original system has the exchange rule.

[6]When $\Gamma = \{A_1, ..., A_m\}$ and $\Delta = \{B_1, ..., B_n\}$, the union of the multi-sets Γ and Δ is defined by $\Gamma \cup \Delta = \{A_1, ..., A_m, B_1, ..., B_n\}$. In this paper, we talk only about the emptiness or nonemptiness of the intersection of multi-sets Γ and Δ : It is empty if and only if Γ and Δ have a common element.

contraction and *cut rules*:

$$\frac{\Gamma \to \Theta}{\Delta, \Gamma \to \Theta, \Lambda}\ (\text{th})$$

$$\frac{A, A, \Gamma \to \Theta}{A, \Gamma \to \Theta}\ (\text{c} \to) \qquad \frac{\Gamma \to \Theta, A, A}{\Gamma \to \Theta, A}\ (\to \text{c})$$

$$\frac{\Gamma \to \Theta, A \qquad A, \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda}\ (\text{cut})$$

In these rules, the mutli-sets Γ, Θ, Δ and Λ may be empty. Hence, (th) is allowed to be a trivial inference having the same upper and lower sequents. In particular, we call formulae in Δ and Λ in (th) *thinning formulae*.

Operational Rules:

$$\frac{\Gamma \to \Theta, A}{\neg A, \Gamma \to \Theta}\ (\neg \to) \qquad \frac{A, \Gamma \to \Theta}{\Gamma \to \Theta, \neg A}\ (\to \neg)$$

$$\frac{\Gamma \to \Theta, A \qquad B, \Gamma \to \Theta}{A \supset B, \Gamma \to \Theta}\ (\supset\to) \qquad \frac{A, \Gamma \to \Theta, B}{\Gamma \to A \supset B, \Theta}\ (\to\supset)$$

$$\frac{A, \Gamma \to \Theta}{\wedge\Phi, \Gamma \to \Theta}\ (\wedge \to),\ \text{where } A \in \Phi \qquad \frac{\{\Gamma \to \Theta, A : A \in \Phi\}}{\Gamma \to \Theta, \wedge\Phi}\ (\to \wedge)$$

$$\frac{\{A, \Gamma \to \Theta : A \in \Phi\}}{\vee\Phi, \Gamma \to \Theta}\ (\vee \to) \qquad \frac{\Gamma \to \Theta, A}{\Gamma \to \Theta, \vee\Phi}\ (\to \vee),\ \text{where } A \in \Phi.$$

In those operational rules, we say that the formula(s) to be changes in the uppersequent(s) the *side formula(s)*, and that the formula formed in the lower sequent is the *principal formula*. For example, in $(\supset\to)$, (the designated occurrences of) A, B are side formulae and (the designated occurrence of) $A \supset B$ is the principal formula[7].

A *proof* P is defined to be a triple $(T, <, \psi)$ with the following properties:

(i): $(T, <)$ is a finite tree and its immediate successor relation is denoted by $<^I$;

(ii): ψ is a function associating a sequent $\psi(n) = \Delta \to \Lambda$ to each node $n \in T$;

(a): for any leaf (maximal node) n in $(T, <)$, $\psi(n)$ is an instance of the axiom;

(b): for any non-leaf $n \in T$,

$$\frac{\{\psi(n') : n <^I n'\}}{\psi(n)}$$

is an instance of one inference rule.

[7]Exactly speaking, these are attributes of occurrences, since Γ, Θ may contain the same formulae as well. But this slightly abused wording will cause no confusion in this paper.

Let $\Gamma \to \Theta$ be a sequent. We say that $P = (T, <, \psi)$ is *a proof of* $\Gamma \to \Theta$ iff P is a proof with $\psi(n_0) = \Gamma \to \Theta$, where n_0 is the root of $(T, <)$. We say that $\Gamma \to \Theta$ is *provable* iff there is a proof of $\Gamma \to \Theta$, in which case we denote $\vdash \Gamma \to \Theta$. When it is unprovable, we denote $\nvdash \Gamma \to \Theta$.

The reader may be more familiar to classical logic in the Hilbert-style (cf. Mendelson [20] and Kaneko [13]). The above Gentzen-style formulation is directly comparable to the system of classical logic given in [13] (cf. Kaneko-Nagashima [12]). These formal systems are equivalent *with respect to provability*, but have significant differences in other respects. The aim of the Gentzen-style formulation is to study logical inferences, while the emphasis of the Hilbert-style is to express provability. This will be clear presently.

Arrow's theorem is a contradictory statement. A contradictory statement in the above system is formulated as

$$\vdash \Gamma \to \quad . \tag{3.1}$$

This is equivalent to

$$\vdash \Gamma \to \neg A \wedge A \text{ for any formula } A. \tag{3.2}$$

Indeed, if (3.1) holds, $\Gamma \to \neg A \wedge A$ is inferred from $\Gamma \to$ by (th). Conversely, if (3.2) holds, then we have

$$\dfrac{\dfrac{\vdots}{\Gamma \to \neg A \wedge A} \qquad \dfrac{\vdots}{\neg A \wedge A \to}}{\Gamma \to} \text{ (cut)}.$$

The right-upper part of this proof is given as

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}(\neg \to)}{\neg A \wedge A, A \to}(\wedge \to)}{\neg A \wedge A, \neg A \wedge A \to}(\wedge \to)}{\neg A \wedge A \to}(\text{c}\!\to) \tag{3.3}$$

We will use the expression of the form (3.1) for our evaluation of Arrow's impossibility theorem, rather than (3.2).

In (3.3), the uses of mutli-sets and the contraction rule are observed. If sets of formulae are used rather than multi-sets, then we would not need the contraction rules (c$\to$) and ($\to$c), e.g., the 2nd line from the bottom would disappear in (3.3). This simplification might require unnecessary attention to avoid some mistakes. Therefore, we adopt mutli-sets for sequents and the contraction rules; accordingly $\wedge, \vee$ are applied to also mutli-sets of formulae.

One important theorem differentiating the Gentzen-style sequent calculus from the Hilbert-style is the cut-elimination theorem, proved by Gentzen [7].

Theorem 3.1. (Cut-Elimination): Suppose $\vdash \Gamma \to \Theta$. Then, there is a cut-free proof of $\Gamma \to \Theta$.

That is, for any proof, we can find a proof so that it has no (cut)'s but has the same endsequent. Although the Hilbert-style calculus is equivalent to the Gentzen-style calculus with respect to provability, the Hilbert-style has is no counterpart of Theorem 3.1.

Theorem 3.1 has an important implication, which will be used in Section 7.

Theorem 3.2. (Subformula Property): Let $P = (T, <, \psi)$ be any cut-free proof of $\Gamma \to \Theta$. Then, if a formula A occurs in any sequent $\psi(n)$ in P, then A must occur (as a subformula of some formula) in the endsequent $\Gamma \to \Theta$.

3.2. The Contemplexity Measure

Now, we define the *contemplexity measure* η for *a proof*. That is, for each proof $P = (T, <, \psi)$, we define

$$\eta(P) = \text{the number of the leaves of the tree } (T, <). \tag{3.4}$$

We count the number of leaves even if some leaves have the same initial sequents. Thus, η measures the width of P but does not count its depth.

The width and depth of a proof is affected by inference rules. Looking at the list of inference rules given in Section 3.2, we find that they are classified into

$$(\supset\to), (\to\wedge), (\vee\to), (\text{cut}); \text{ and} \tag{3.5}$$

$$(\text{th}), (\text{c}\to), (\to\text{c}), (\neg\to), (\to\neg), (\wedge\to), (\to\vee). \tag{3.6}$$

Each of (3.5) has two or more uppersequents, while each of (3.6) has only one uppersequent. Hence, an occurrence of an inference rule in the former makes a proof larger, but any in the latter affects only the depth. Some proofs may have just contemplexity 1, but their depths are quite large. For example, the proof of $\neg A \wedge A \to$ given in in (3.3) has contemplexity 1, but its depth is 5.

First, consider the inference rules in (3.5). In these inferences, the lower sequent is either a logical weakening or a logically equivalent of the upper sequent. These rules only change expressions. The contemplexity $\eta(P)$ ignores these inferences. On the other hand, each of (3.6) has two or more upper sequents. Each combines two or more essential cases into one lower sequent. The definition of (3.4) counts these branchings.

In fact, the contemplexity $\eta(P)$ can be written using the numbers of occurrences of these inferences, which is given in Kaneko-Suzuki [17].

Example 3.1. In the above logical system, it holds that

$$\vdash \Theta, R_1^{t_1}(\alpha,\beta) \wedge R_2^{t_2}(\alpha,\beta) \supset P_s^t(\alpha,\beta) \to P_s^t(\alpha,\beta), \tag{3.7}$$

where Θ is the (multi-) set $\{R_1^{t_1}(\alpha,\beta), R_2^{t_2}(\alpha,\beta)\}$. That is, when (partial) Pareto condition from α to β is assumed and both individuals prefer α to β, society also prefers α to β. Abbreviating $R_1^{t_1}(\alpha,\beta)$, $R_2^{t_2}(\alpha,\beta)$, $P_s^t(\alpha,\beta)$ as $r_{1\alpha\beta}, r_{2\alpha\beta}$ and $p_{s\alpha\beta}$, we have the following proof:

$$\dfrac{\dfrac{\dfrac{r_{1\alpha\beta} \to r_{1\alpha\beta}}{\Theta \to r_{1\alpha\beta}}(\mathrm{th}) \quad \dfrac{r_{2\alpha\beta} \to r_{2\alpha\beta}}{\Theta \to r_{2\alpha\beta}}(\mathrm{th})}{\Theta \to r_{1\alpha\beta} \wedge r_{2\alpha\beta}}(\to\wedge) \quad \dfrac{p_{s\alpha\beta} \to p_{s\alpha\beta}}{p_{s\alpha\beta}, \Theta \to p_{s\alpha\beta}}(\mathrm{th})}{\Theta, r_{1\alpha\beta} \wedge r_{2\alpha\beta} \supset p_{s\alpha\beta} \to p_{s\alpha\beta}}(\supset\to) \tag{3.8}$$

This proof P has three leaves and its contemplexity is 3.

Our ultimate goal is to study the contemplexity measure for sequents, rather than proofs. A provable sequent may have many proofs. Hence, we need the following definition.

Definition 3.1 (Contemplexity). We define the *contemplexity measure* η *for* a *sequent* $\Gamma \to \Theta$ as follows:

$$\eta(\Gamma \to \Theta) = \begin{cases} \min\{\eta(P) : P \text{ is a cut-free proof of } \sigma\} & \text{if } \vdash \Gamma \to \Theta \\ +\infty & \text{otherwise.} \end{cases} \tag{3.9}$$

Any proof of $\Gamma \to \Theta$ has at least this width $\eta(\Gamma \to \Theta)$ whenever it is provable. If it is unprovable, we set $\eta(\Gamma \to \Theta) = +\infty$. Sometimes, $\eta(\Gamma \to \Theta)$ is simply 1, which means that $\Gamma \to \Theta$ is obtained by changing and/or adding expressions. The depth of such a proof may be quite long, but we are not interested in this depth. For example, $\eta(\neg A \wedge A \to \ ) = 1$, but its (minimum) depth is 5. When $\eta(\Gamma \to \Theta)$ is large, a proof is necessarily wide and requires many essential cases. In this paper, we will evaluate this value for Arrow's theorem.

If we eliminate cut-freeness in the first case of (3.9), then the resulting value $\eta_{wc}(\Gamma \to \Theta)$ may differ from $\eta(\Gamma \to \Theta)$. Since a cut-free proof is a proof, we have

$$\eta_{wc}(\Gamma \to \Theta) \leq \eta(\Gamma \to \Theta). \tag{3.10}$$

The difference was discussed in [17]. In this paper, we will consider only the measure η.

Before going to the application of η to Arrow's theorem, perhaps, some small examples would help the reader understand the measure η. In Example 3.1, $\eta(P) = 3$, but this fact implies only $\eta(\Theta, r_{1\alpha\beta} \wedge r_{2\alpha\beta} \supset p_{s\alpha\beta} \to p_{s\alpha\beta}) \leq 3$, since we do not know its minimality. Actually, we can prove the equality for this, but we need some detailed argument.

Let us consider another example.

Example 3.2. Consider the sequent $\Theta, R_1^{t_1}(\alpha,\beta) \supset P_s^t(\alpha,\beta) \to P_s^t(\alpha,\beta)$, where Θ is the same as $\{R_1^{t_1}(\alpha,\beta), R_2^{t_2}(\alpha,\beta)\}$ in Example 3.1. It states that if the (local) dictatorship from α to β is assumed and both individuals prefer α to β, then society also prefers α to β. This has a simpler proof:

$$\dfrac{\dfrac{r_{1\alpha\beta} \to r_{1\alpha\beta}}{\Theta \to r_{1\alpha\beta}}(\text{th}) \quad \dfrac{p_{s\alpha\beta} \to p_{s\alpha\beta}}{p_{s\alpha\beta}, \Theta \to p_{s\alpha\beta}}(\text{th})}{\Theta, r_{1\alpha\beta} \supset p_{s\alpha\beta} \to p_{s\alpha\beta}}(\supset\to)$$

Hence, this proof has a contemplexity 2. Also, we can prove $\eta(\Theta, r_{1\alpha\beta} \supset p_{s\alpha\beta} \to p_{s\alpha\beta}) = 2$. Suppose, on the contrary, that this was 1, i.e., there is a proof P of this sequent which has no application of $(\supset\to)$ with the principal formula $r_{1\alpha\beta} \supset p_{s\alpha\beta}$. Tracing the occurrences of $r_{1\alpha\beta} \supset p_{s\alpha\beta}$ in P, we find that the uppermost occurrence is introduced by (th), because otherwise, the uppermost ancestor of P is an initial sequent having $r_{1\alpha\beta} \supset p_{s\alpha\beta}$ as the antecedent but the succedent has the descendant $r_{1\alpha\beta} \supset p_{s\alpha\beta}$ in the endsequent, which is not the case. Thus, $\eta(\Theta, r_{1\alpha\beta} \supset p_{s\alpha\beta} \to p_{s\alpha\beta})$ is not 1. Hence, the contemplexity is 2.

Finally, let us look at one example to have at least two proofs with different contemplexities. We combine the above example with Example 3.1: Let us denote

$$\Xi = \{R_1^{t_1}(\alpha,\beta) \wedge R_2^{t_2}(\alpha,\beta) \supset P_s^t(\alpha,\beta), R_1^{t_1}(\alpha,\beta) \supset P_s^t(\alpha,\beta)\}.$$

Then, the sequent $\Theta, \Xi \to P_s^t(\alpha,\beta)$ has two different proofs: One is obtained from the proof in Example 3.1 by adding (th):

$$\dfrac{\dfrac{\cdots}{\Theta, r_{1\alpha\beta} \wedge r_{2\alpha\beta} \supset p_{s\alpha\beta} \to p_{s\alpha\beta}}(\supset\to)}{\Theta, \Xi \to p_{s\alpha\beta}}(\text{th})$$

Thus, this has contemplexity 3. The other is obtained from the proof in Example 3.2 by adding (th) :

$$\dfrac{\dfrac{\cdots}{\Theta, r_{1\alpha\beta} \supset p_{s\alpha\beta} \to p_{s\alpha\beta}}(\supset\to)}{\Theta, \Xi \to p_{s\alpha\beta}}(\text{th})$$

This has contemplexity 2. Thus, $\eta(\Theta, \Xi \to p_{s\alpha\beta}) \leq 2$ by (3.9).

Actually, we can prove $\eta(\Theta, \Xi \to p_{s\alpha\beta}) = 2$ in the same way as in the proof of $\eta(\Theta, r_{1\alpha\beta} \supset p_{s\alpha\beta} \to p_{s\alpha\beta}) = 2$.

These two examples show that it would not be easy to calculate the exact contemplexity value $\eta(\Gamma \to \Theta)$, though the existence of it is apparent. Therefore, Kaneko-Suzuki [17] gave a certain method of determine the value $\eta(\Gamma \to \Theta)$, which is not straightforward, too. In this paper, we will calculate certain lower bound and upper bound for Arrow's theorem.

4. Formalization of Arrow's Theorem and its Contemplexity

In this section, first, we formulate individual and social preferences in our formal language. Then, we give the axioms for a social welfare function. For this formulation, we adopt the nominalistic method: In Section 3, we gave names to each individual preferences, social orderings and social welfare functions but did not fix their intended meanings. Now, we need to determine the intended meanings of these symbols. Then, we state the three axioms, Arrow's theorem.

4.1. Individual Preferences and Social Orderings

We express the six orderings over α, β, γ listed in (2.1) using the preference symbols R_i^k ($k \in \mathbf{6}$ and $i = 1, 2$). Here, we repeat the orderings listed in (2.1):

$$\begin{array}{lll} (1) : \alpha, \beta, \gamma; & (2) : \beta, \gamma, \alpha; & (3) : \gamma, \alpha, \beta \\ (4) : \gamma, \beta, \alpha; & (5) : \beta, \alpha, \gamma; & (6) : \alpha, \gamma, \beta. \end{array}$$

For example, (1) means that the three alternatives are ordered as α, β, γ. Suppose that this ordering is a preference relation of individual $i = 1, 2$. Then this ordering is expressed by R_i^1 as the (multi-) set $\Pi_i^1 =$:

$$\{R_i^1(\alpha, \beta), \neg R_i^1(\beta, \alpha), R_i^1(\beta, \gamma), \neg R_i^1(\gamma, \beta), R_i^1(\alpha, \gamma), \neg R_i^1(\gamma, \alpha)\}. \tag{4.1}$$

Similarly, we have the set Π_i^k by using R_i^k describing the k-th preference relation of individual i. Then, we define

$$\Pi_i = \Pi_i^1 \cup ... \cup \Pi_i^6 \text{ for } i = 1, 2; \text{ and } \Pi = \Pi_1 \cup \Pi_2. \tag{4.2}$$

The first is the set of possible preference relations for individual i, and Π is the set of possible preference relations for two individuals.

A particular profile of preference orderings is represented by $\Pi_1^{t_1} \cup \Pi_2^{t_2}$. This is the formalization, in our language, of the profile $(\prec_1^{t_1}, \prec_2^{t_2})$ in the sense of Section 2.

In a similar manner, we can enumerate all the possible social orderings. However, there are 6^{36} number of social welfare functions. Hence, it would be simpler to define them in an abstract way (in the metalanguage) than to give a specific way to enumerate them[8]. Let Σ be a set consisting formulae of the form

$$P_s^t(u,v) \text{ or } \neg P_s^t(u,v), \tag{4.3}$$

where distinct $u,v \in \mathbf{A}$, $t \in \mathbf{6}^2$ and $s = 1, ..., 6^{36}$. First, for each $s = 1, ..., 6^{36}$, we assume that for all distinct $u,v,w \in \mathbf{A}$ and $t \in \mathbf{6}^2$;

Total: either $P_s^t(u,v) \in \Sigma$ or $P_s^t(v,u) \in \Sigma$;

Asymmetry: $P_s^t(u,v) \in \Sigma$ implies $\neg P_s^t(v,u) \in \Sigma$;

Transitivity: $P_s^t(u,v) \in \Sigma$ and $P_s^t(v,w) \in \Sigma$ imply $P_s^t(u,w) \in \Sigma$.

In addition, we assume the following:

Distinctiveness: for any distinct $s, s' = 1, ..., 6^{36}$, $P_s^t(u,v) \in \Sigma$ and $P_{s'}^t(u,v) \notin \Sigma$ for some $t \in \mathbf{6}^2$ and distinct $u,v \in \mathbf{A}^2$.

To save spaces, we introduce the set of *distinct ordered pairs* in $\mathbf{A}$, denoted by $\mathbf{A}^{2-}$.

It would be a good idea to see that these conditions determine Σ to capture the intended meaning of the "set of all social welfare functions". First, let Σ_s^t be the set:

$$\{P_s^t(u,v) \in \Sigma : (u,v) \in \mathbf{A}^{2-}\} \cup \{\neg P_s^t(u,v) \in \Sigma : (u,v) \in \mathbf{A}^{2-}\}.$$

This is the set of "social preferences" for given s and $t = (t_1, t_2)$. Also, the social welfare function, φ_s, named s is defined to be $\Sigma_s = \bigcup_{t \in \mathbf{6}^2} \Sigma_s^t$.

The following lemma states that the set Σ enumerates all possible social welfare functions; their names are $s = 1, ..., 6^{36}$. Thus, we have a complete enumeration of social welfare functions.

Lemma 4.1.(1): For any $s = 1, ..., 6^{36}$ and $t = (t_1, t_2) \in \mathbf{6}^2$, Σ_s^t can be expressed as: for some u, v, w with $\{u, v, w\} = \{\alpha, \beta, \gamma\}$,

$$\Sigma_s^t = \{P_s^t(u,v), \neg P_s^t(v,u), P_s^t(v,w), \neg P_s^t(w,v), P_s^t(u,w), \neg P_s^t(w,u)\}.$$

(2): If $s \neq s'$, then $\Sigma_s \neq \Sigma_{s'}$.

(3): Let $\varphi : SO^2 \to SO$ be an arbitrary social welfare function. Then, there is a unique s such that for any $t = (t_1, t_2) \in \mathbf{6}^2$, $\varphi(\prec_1^{t_1}, \prec_2^{t_2})(u,v)$ holds if and only if $P_s^t(u,v) \in \Sigma_s^t$.

Proof.(1): This follows from (4.3) and the three conditions.

(2): This follows from Distinctiveness.

[8] It is not difficult to enumerate all the social welfare functions in a concrete manner; one possiblity is to use the lexicographic ordering.

(3): This is obtained by (2) and by comparing the number of social welfare functions with $s = 1, ..., 6^{36}$. ■

We should mention the following lemma, which will be used in Section 7.

Lemma 4.2. The set $\Pi \cup \Sigma$ is consistent in classical logic.

This lemma can be proved by using the soundness theorem for classical logic, i.e., to find a truth assignment satisfying $\Pi \cup \Sigma$. See Kaneko [13], Sec.3.2. This implies that any subset of $\Pi \cup \Sigma$ is consistent. This fact will be referred as part of Lemma 4.2.

4.2. Three Conditions for a Social Welfare Function

In our formalized language, the three conditions for Arrow's theorem are formalized in the following way: Let s be any number from 1 to 6^{36}.

$PR(s)$: $\bigwedge_{t \in \mathbf{6}^2} \bigwedge_{(u,v) \in \mathbf{A}^{2-}} \left(R_1^{t_1}(u,v) \wedge R_2^{t_2}(u,v) \supset P_s^t(u,v)\right).$

$IIA(s)$: $\bigwedge_{t,t' \in \mathbf{6}^2} \bigwedge_{(u,v) \in \mathbf{A}^{2-}}$
$\left(R_1^{t_1}(u,v) \wedge R_2^{t_2}(v,u) \wedge P_s^t(u,v) \supset [R_1^{t'_1}(u,v) \wedge R_2^{t'_2}(v,u) \supset P_s^{t'}(u,v)]\right) \wedge$
$\left(R_1^{t_1}(u,v) \wedge R_2^{t_2}(v,u) \wedge P_s^t(v,u) \supset [R_1^{t'_1}(u,v) \wedge R_2^{t'_2}(v,u) \supset P_s^{t'}(v,u)]\right).$

$ND(s)$: $\neg \bigvee_{i=1,2} \bigwedge_{t \in \mathbf{6}^2} \bigwedge_{(u,v) \in \mathbf{A}^{2-}} \left(R_i^{t_i}(u,v) \supset P_s^t(u,v)\right).$

Recall the remark stated in (3.1) and (3.2), Arrow's impossibility theorem is stated as follows:

Theorem 4.3 (Arrow's Impossibility in the Formal System):

$$\vdash \Pi, \Sigma, \bigvee_{s=1}^{6^{36}} (PR(s) \wedge IIA(s) \wedge ND(s)) \to \quad . \tag{4.4}$$

All the profiles are expressed by Π and all the welfare functions are expressed by Σ. A contradiction is derived from the existence of a social welfare function satisfying these conditions and Π, Σ. A proof for an upper bound to be given in Section 8 will be also a proof of Theorem 4.3.

We obtain the following from (4.4) by an application of $(\to \neg)$:

$$\vdash \Pi, \Sigma \to \neg \bigvee_{s=1}^{6^{36}} (PR(s) \wedge IIA(s) \wedge ND(s)). \tag{4.5}$$

This is rather the standard statement of Arrow's theorem that there is no social welfare function satisfying the required three conditions. In this paper, we will evaluate the contemplexity of the sequent of (4.4) rather than (4.5).

5. Contemplexity for Arrow's Theorem

We give the contemplexity theorem for Arrow's Impossibility Theorem and compare it with the contemplexity required for the Cyclical Impossibility Theorem.

5.1. Upper and Lower Bounds

Now, we present the main theorem of the paper, which will be proved in Sections 7 and 8. We denote the sequent $\Pi, \Sigma, \bigvee_{s=1}^{6^{36}} (PR(s) \wedge IIA(s) \wedge ND(s)) \rightarrow$ by σ.

Theorem 5.1 (Upper and Lower Bounds): $6^{36} \leq \eta(\sigma) \leq 6^{37} + 420$.

This states that Arrow's theorem in the case with 2 individuals and 3 social alternatives necessarily requires a proof of width at least 6^{36} and at most $6^{37} + 420$. These numbers are much larger than the Avogadro number 6×10^{23}. If all these cases should be covered, it would be practically impossible to be treated. It is an implication that any proof of Arrow's impossibility theorem cannot be practically written down.

In fact, we can improve the evaluation of the lower bound to 3×6^{36}. In addition to the proof for the lower bound given in Section 7, however, we need another long proof for this multiplication of 3 with the lower bound. This improvement means that each of 6^{36} cases is essential for Arrow's theorem. However, we omit this improvement in this paper, since the effect of additional 3 on implications would be negligible.

Some reader doubts that something may be wrong with the above claim that Arrow's theorem cannot practically be proved, since Arrow [2] himself proved it (a more general version) and many other people checked it and/or reproved it. A resolution of this doubt is related to one point we mentioned in Section 1. In the case of proposition logic, all details and all cases are described in a proof, and "a free variable" representing "an arbitrary element" is excluded from the object-expressions. In nonformalized mathematics, a free variable is used so often even unconsciously: For example, "a similar manner" includes "a free variable". At the meta-level, we use free variables even in propositional logic.

In the nonformalized proofs of Arrow's theorem, such free variables are used a lot. In our proof of the upper bound given in Theorem 5.1, free variables will be used in metalanguage.

One strength of mathematical logic is a division of treatments between the object-level and meta-level. At the object-level, a free variable is a concept in predicate logic. Thus, it is an implication of Theorem 5.1 that Arrow's theorem involves so many steps if we adopt the propositional logic - the complete enumeration method - for its proof, and that practically the use of "a free variable" is inevitable. For a further development of inductive game theory and epistemic logic, we should take free variables more seriously and explicitly. One possibility is to incorporate predicate logic into them even with a finite structure. This will belong to a future project.

Now, let us talk about the proof of Theorem 5.1: It consists of two parts: (A): $6^{36} \leq \eta(\sigma)$; and (B) $\eta(\sigma) \leq 6^{37}+420$. They will be given in Sections 7 and 8, respectively.

Part A is genuinely a proof-theoretic argument; we evaluate a given cut-free proof of σ. The main part of the proof is to show that the proof has at least one application of ($\vee \rightarrow$) with its principal formula $\bigvee_{s=1}^{6^{36}} (PR(s) \wedge IIA(s) \wedge ND(s))$. This means that this inference has 6^{36} upper sequents, and thus, the proof tree has at least 6^{36} leaves. In other words, $\bigvee_{s=1}^{6^{36}} (PR(s) \wedge IIA(s) \wedge ND(s))$ is essential. Although this may sound trivial, it would take an accurate proof-theoretic evaluation of a given cut-proof. It will be remarked in Section 6.2 about the case where (cut)'s are allowed.

Part B is proved by giving a particular proof of Arrow's theorem. Our task is to evaluate it in terms of contemplexity. Therefore, our proof is more precise than the standard one in that we cannot abbreviate some cases as "similar". We should also avoid calling some cases "trivial", since those cases may be even more difficult than "nontrivial" cases. Of course, if some cases are truly parallel to some other cases by permutations of components, we would abbreviate them. By our proof, we will obtain some additional statement.

Theorem 5.2 (No Entanglement of the three Conditions): For each $s = 1, ..., 6^{36}$, either $\vdash \Sigma_s, PR(s) \rightarrow$, $\vdash \Sigma_s, IIA(s) \rightarrow$ or $\vdash \Sigma_s, ND(s) \rightarrow$.

That is, it is not the case that two or three conditions of $PR(s), IIA(s)$ and $ND(s)$ are entangled for the proof of inconsistency.

5.2. Comparison with the Cyclical Impossibility Theorem

In Section 3, we looked at a few examples for contemplexity, which show that these contemplexities are rather tiny. It may be a good idea to compare the Cyclical Impossibility Theorem with respect to contemplexity to Theorem 5.1. For this, transitivity and asymmetry are formulated as follows:

Transitivity (Tr) **for** D: $\bigwedge_{\{u,v,w\}=\mathbf{A}} (D(u,v) \wedge D(v,w) \supset D(u,w))$;

Asymmetry (As) **for** D: $\bigwedge_{(u,v)\in\mathbf{A}^2} (D(u,v) \supset \neg D(v,u))$.

Then we have the following theorem.

Theorem 5.3 (Cyclical Impossibility): Recall $\Phi = \{D(\alpha,\beta),\ D(\beta,\gamma),\ D(\gamma,\alpha)\}$.

(1): $\vdash \Phi, Tr, As \rightarrow$;

(2): $\eta(\Phi, Tr, As \rightarrow\ ) = 4$.

To obtain the exact statement of (2), we need more precise proof-theoretic developments. Here, we prove (1) and only (2$'$): $\eta(\Phi, Tr, As \rightarrow\ ) \leq 4$. The exact evaluation of (2) needs certain result given in Kaneko-Suzuki [17].

Theorem 5.3 shows that our formal approach separates Arrow's impossibility theorem from the cyclic impossibility theorem.

Proof of Theorem 5.3.(1) and (2′): Here, we can give a full detailed proof of the sequent in (1). We denote $D(\alpha,\beta), D(\beta,\gamma), ...$ by $D_{\alpha\beta}, D_{\beta\gamma}$ etc. First, we have

$$\dfrac{\dfrac{D_{\alpha\beta}\to D_{\alpha\beta}}{\Phi,D_{\alpha\gamma}\supset\neg D_{\gamma\alpha}\to D_{\alpha\beta}}(\mathrm{th})\quad \dfrac{D_{\beta\gamma}\to D_{\beta\gamma}}{\Phi,D_{\alpha\gamma}\supset\neg D_{\gamma\alpha}\to D_{\beta\gamma}}(\mathrm{th})}{\Phi,D_{\alpha\gamma}\supset\neg D_{\gamma\alpha}\to D_{\alpha\beta}\wedge D_{\beta\gamma}}(\to\wedge)$$

Then, combining this proof with the proof of $\Phi, D_{\alpha\gamma}, D_{\alpha\gamma}\supset\neg D_{\gamma\alpha}\to\ $, we have

$$\dfrac{\dfrac{\cdots}{\Phi,D_{\alpha\gamma}\supset\neg D_{\gamma\alpha}\to D_{\alpha\beta}\wedge D_{\beta\gamma}}(\to\wedge)\quad \dfrac{\dfrac{D_{\alpha\gamma}\to D_{\alpha\gamma}}{\Phi,D_{\alpha\gamma}\to D_{\alpha\gamma}}(\mathrm{th})\quad \dfrac{\dfrac{D_{\gamma\alpha}\to D_{\gamma\alpha}}{\neg D_{\gamma\alpha},D_{\gamma\alpha}\to}(\neg\to)}{\neg D_{\gamma\alpha},\Phi\to}(\mathrm{th})}{\Phi,D_{\alpha\gamma},D_{\alpha\gamma}\supset\neg D_{\gamma\alpha}\to}(\supset\to)}{\dfrac{\dfrac{\Phi,D_{\alpha\beta}\wedge D_{\beta\gamma}\supset D_{\alpha\gamma},D_{\alpha\gamma}\supset\neg D_{\gamma\alpha}\to}{\Phi,Tr,D_{\alpha\gamma}\supset\neg D_{\gamma\alpha}\to}(\wedge\to)}{\Phi,Tr,As\to}(\wedge\to)}(\supset\to)$$

In inference $(\supset\to)$ in the middle of the above proof, we used the fact that $D_{\gamma\alpha}\in\Phi$. This proof shows (1). Then, this proof has contemplexity 4. Hence, we have $\eta(\Phi, Tr, As \to\ ) \leq 4$. ■

6. Discussions and Various Remarks

6.1. Discussions

We have evaluated Arrow's impossibility theorem σ in terms of the contemplexity measure η. We gave a lower bound and an upper bound of the contemplexity $\eta(\sigma)$, which are 6^{36} and $6^{37}+420$, respectively. If we take these bounds faithfully, it would be practically impossible to have a proof of Arrow's theorem. As mentioned after Theorem 5.1, Arrow's theorem in the specific case will be proved in Section 8, and a more general version of it has been proved. This appears to be contradictory.

The above seeming contradiction may be resolved by finding that our theorem is formulated in a formalized propositional logic, while an actual proof uses "a free variable" expressing "something arbitrary". In mathematical logic, a free variable is a concept in predicate logic, and is typically regarded as necessary when some mathematical structure involves an infinite number of objects. Our result states that the use of a free variable or predicate logic is practically necessary even for finite problems.

A free variable will be a key for our future analysis of human thinking: It requires specific detailed instances and also some abstraction. It is an imWithout abstraction, even a finite problem would be too complicated to manage it practically. This is an implication of Theorem 5.1 that a free variable, *a fortiori*, predicate logic will be important for our future analysis of finite epistemic problems.

Once it is understood that contemplexity may give a stringent constraint on a player's thinking, it would be a problematic to confine our research to provability (validity in the semantical sense), since provability makes no distinction between large and small contemplexity. This fact requires us to rethink the present and traditional axiomatic attitude in the research of epistemic logic.

According this tradition, the belief set Γ for a player is also regarded as a set of axioms and must be well-prepared; for example, it satisfies some criteria: consistency, independence and some simplicity. However, independence and simplicity for Γ make a derivation of a theorem A from Γ to become long. Our theorem has an implication that a derivation is too long for the player. Also, even if he succeeds in deriving A from Γ, he may not recall the derivation. This suggests that for the consideration of a player's beliefs/knowledge, we should forget to consider requirement that the player can practices the derivation always unless it exceeds some bound. Rather, once he derives A from Γ, he can add A to Γ;

$$\text{from } \Gamma \text{ to } \Gamma \cup \{A\}.$$

Thus, the belief set Γ is regarded as changing with time.

Once we adopt the above view of changing the belief set, the additional A may be not only a theorem from the previous Γ but also something taught through communication to other people or formal education. From this point of view, the belief set Γ may be no longer independent or simple. It may consists of a lot of accumulated beliefs, in which case a derivation of a new statement may become shorter.

The treatment of an inconsistent belief set, discussed in Section 1.2, may be more meaningful. Originally, the player has a belief set, which is inconsistent objectively but appears to be no problem for him. He adds new facts, which are sometimes derived from his previous beliefs or sometimes else educated to him, to his belief set. His belief set is getting larger, and finally he notices inconsistency of his beliefs[9].

Thus, Theorem 5.1 has a lot of implicatons and suggestions for new research topics for game theory as well as epistemic logic (rather logic in general). However, those are new problems for these disciplines, and remain to be important open problems.

6.2. Remarks

Remark 1 (General but still Finite Version of Arrow's Theorem): When more individuals and more social alternatives, the lower bound part of Theorem 5.1 can be extended in a straightforward way. The key was the number of social welfare functions 6^{36}. In the case with 3 individuals and 3 social alternatives, the key number becomes

[9]The theory of belief-revision (cf., Gördenfors [6] and Schulte [22]) has a conceptual difficulty with the adoption of provability. If we take the above view, it would make more sense since belief revisions can be discussed in a dynamic context.

$6^{6^3} = 6^{216}$. In the general case with m individuals and k social alternatives, the key number becomes $(k!)^{(k!)^m}$.

So far, we do not have an upper bound part of the general version of Theorem 5.1. In an existing proof of Arrow's theorem (cf., Luce-Raiffa [9], Chap.14, Arrow *et al.* [3], Chap.1), a lot of meta-treatments such as a minimal decisive set are used and give difficulties to translate it into a formalization in propositional logic. It remains open to evaluate an upper bound of the general version.

Remark 2 (Embedding into an Epistemic Logic): As mentioned in Section 1, one of our motivations is to develop epistemic logic. We evaluated Arrow's impossibility theorem in classical logic, but this evaluation is directly translated into epistemic logic such as GL_{EF} of Kaneko-Suzuki [14]. That is, the proof of Arrow's theorem was done in the mind of player i. Our result means that Arrow's theorem is an example to show a gigantic contemplexity value from a relatively tiny problem: As far as his inferences are following the propositional logic, he cannot manage the proof. Thus, predicate logic will be important in the study of epistemic logic for a situation including no infinity.

Remark 3 (Total number of Sequents): Our contemplexity measure $\eta(P)$ is defined to be the width of the proof P. The reason for taking the width was explained in Section 3. Some reader may be interested in counting all the sequents in P. Let us modify $\eta(P)$ to the number of sequents in P, which is denoted by $\eta^T(P)$. In the same as (3.9), we define $\eta^T(\Gamma \to \Theta)$ to be the smallest number in $\{\eta^T(P) : P \text{ is a proof of } \Gamma \to \Theta\}$. Since $\eta(\Gamma \to \Theta) \leq \eta^T(\Gamma \to \Theta)$ in general, 6^{36} is still a lower bound of $\eta^T(\sigma)$. Looking at the proof given in Section 8, we can calculate one upper bound of $\eta^T(\sigma)$:

$$\eta^T(\sigma) \leq (6^{36} - 2) \times 19 + 2 \times 711 = 6^{36} \times 19 + 711. \tag{6.1}$$

Thus, we can use the total number of sequents in a proof for a measure of the size of a proof. But, for the reasons given in Section 3, we take the measure $\eta(P)$ as the definition of the size of P.

Remark 4 (Hilbert-Style Formulation): We have defined the contemplexity measure η in classical logic in the Gentzen-style sequent formulation. Analogously, we can use classical logic in the Hilbert-style. There are still many choices of a language and an axiomatic system. Among them, the system in the Hilbert-style given in Kaneko [13] is directly comparable with the system of the present paper. The translation between the Gentzen-style and Hilbert-style is possible, with the use of the cut-inference in the Gentzen-style. The lower bound part of Theorem 5.1 can be converted to the Hilbert-style classical logic. The upper bound part is not directly converted. Either to give a proof in the Hilbert-style once more or to translate each step of the proof in Section 8 to the Hilbert-style. Perhaps, the corresponding upper bound for the contemplexity becomes larger than $6^{37} + 420$.

Remark 5 (Length of a Proof): In the Hilbert-style, a proof is often formulated as a finite sequence of formulae (cf. Mendelson [20]). This case is similar to the case of counting all sequents in a proof tree mentioned in Remark 3. In the case, the length of a proof depends upon the axiomatic system of classical logic in the Hilbert-style. Nevertheless, we cannot be free from the lower and upper bounds such as 6^{36} and $6^{37}+420$ or the number given in (6.1).

Remark 6 (Language with Binary Conjunctions and Disjunctions): In the language $\mathcal{P}$ given adopted in Section 3, conjunction symbol $\wedge$ and disjunction symbol $\vee$ are applied to any finite multi-sets of formulae. Since we take conjunctions and disjunctions for many possible formulae, this definition has saved a lot in various expressions. However, this is not standard in the logic literature; it is more standard to allow only binary conjunctions and disjunctions. Here, we consider how the main result (Theorem 5.1) would change if we adopt binary conjunctions and disjunctions.

Now, we take the subset $\mathcal{P}_B$ of $\mathcal{P}$ defined by the restriction that every conjunctive or disjunction subformula in any formula is binary, i.e., $\wedge\{A_1, A_2\}$ or $\vee\{A_1, A_2\}$, where $\{A_1, A_2\}$ is a multi-set of formulae. Then, the Gentzen-system defined in Section 3 is restricted to the language $\mathcal{P}_B$. In this language, $\wedge\{A_1, ..., A_m\}$ is translated, by the repetition of the binary conjunctions, into:

$$\wedge\{\wedge\{\wedge ... \wedge \{\wedge\{A_1, A_2\}, A_3\}...\}, A_m\}, \tag{6.2}$$

and $\vee\{A_1, ..., A_m\}$ is similarly translated. With this transformation, the Gentzen-system within $\mathcal{P}_B$ is equivalent, with respect to provability, to that within $\mathcal{P}$. Nevertheless, they may differ with respect to the sizes of proofs.

Thus, the language $\mathcal{P}_B$ is quite inconvenient when we take a conjunction or disjunction of many formulae such as $PR(s) \wedge IIA(s) \wedge ND(s)$ and $\bigvee_{s=1}^{6^{36}} Ar(s)$. Also, we need the corresponding repetitions of $(\wedge \rightarrow)$ and $(\vee \rightarrow)$. Let us denote the formulae defined by (6.2) by $\bigwedge\{A_1, ..., A_m\}$, and correspondingly $\bigvee\{A_1, ..., A_m\}$. Then, for example, the inference in the language $\mathcal{P}$

$$\frac{\{A, \Gamma \rightarrow \Theta : A \in \Phi\}}{\vee\Phi, \Gamma \rightarrow \Theta}\ (\vee \rightarrow), \text{ where } \Phi = \{A_1, ..., A_m\}$$

can be changed to

$$\cfrac{\cfrac{\cfrac{\bullet \quad \bullet \quad \bullet}{\bigvee\{A_1, ..., A_{m-2}\}, \Gamma \rightarrow \Theta}\ (\vee \rightarrow) \qquad A_{m-1}, \Gamma \rightarrow \Theta}{\bigvee\{A_1, ..., A_{m-1}\}, \Gamma \rightarrow \Theta}\ (\vee \rightarrow) \qquad A_m, \Gamma \rightarrow \Theta}{\bigvee \Phi, \Gamma \rightarrow \Theta}\ (\vee \rightarrow)$$

The width does not change as the number of disjuncts m, but the depth of the inference becomes also m, though it was 2 in the language $\mathcal{P}$. Thus, this language is quite inconvenient.

Nevertheless, it would be important to check how Theorem 5.1 changes when we adopt the language $\mathcal{P}_B$. Our conjecture is that the claim remains the same, but the depth changes drastically. This is an important open problem.

7. Proof of the Lower Bound Part of Theorem 5.1

In Section 7.1, we summarize certain facts about any cut-free proof P of Arrow's theorem σ. Then, we prove in Section 7.2 that $6^{36} \leq \eta(\sigma)$: Any cut-free proof P of σ has at least one application of $(\vee \rightarrow)$ with its principal formula $\bigvee_{s=1}^{6^{36}} AR(s)$. This implies that this $(\vee \rightarrow)$ has 6^{36} uppersequents; *a fortiori*, proof P has at least 6^{36} leaves.

Throughout this section, let $P = (T, <, \psi)$ be any cut-free proof of $\sigma = \Pi, \Sigma, \bigvee_{s=1}^{6^{36}} AR(s) \rightarrow$. We will change ψ, but will not touch the tree structure $(T, <)$. Later, we will focus on certain subtrees of $(T, <)$, but we will consider them relative to the entire tree $(T, <)$. Recall that $<^I$ is the immediate predecessor part of $<$.

7.1. The Ascending Chain containing $\bigvee_{s=1}^{6^{36}} AR(s)$ in P

Consider the root node n_0 in the proof P. Now, we trace up nodes $n_0, n_1, ..., n_l$ in T so that

(1): $n_k <^I n_{k+1}$ for all $k = 0, ..., l-1$;

(2): each sequent $\psi(n_k)$ contains the formula $\bigvee_{s=1}^{6^{36}} AR(s)$ for $k = 0, ..., l$;

(3): each sequent $\psi(n_k)$ is the lower sequent of $(\neg \rightarrow)$, $(c\rightarrow)$, $(\rightarrow c)$ or (th) for $k = 0, ..., l-1$;

(4): at n_l, one of the following cases holds:

(4a): $\psi(n_l)$ is the lower sequent of $(\vee \rightarrow)$ with its principal formula $\bigvee_{s=1}^{6^{36}} AR(s)$;

(4b): $\psi(n_l)$ is the lower sequent of (th), and in this case, its upper sequent has no occurrences of $\bigvee_{s=1}^{6^{36}} AR(s)$.

If $\psi(n_l)$ is an initial sequent, then it would be $\psi(n_l) = \bigvee_{s=1}^{6^{36}} AR(s) \rightarrow \bigvee_{s=1}^{6^{36}} AR(s)$ by (2), but this is impossible by the subformula property (Theorem 3.2). Hence, it suffices to require (4a) and (4b). In fact, this sequence $n_0, ..., n_l$ is uniquely determined, since P has no branching in these nodes by (3). We call this sequence the *ascending chain containing* $\bigvee_{s=1}^{6^{36}} AR(s)$ from the root node n_0.

Recall that P is a cut-free proof and satisfies the subformula property (Theorem 3.2). The sequent $\psi(n_k)$ associated with each node n_k $(k = 0, ..., l)$ is expressed as

$$\psi(n_k) = \Gamma_k, \bigvee_{s=1}^{6^{36}} AR(s), ..., \bigvee_{s=1}^{6^{36}} AR(s) \rightarrow \Theta_k. \tag{7.1}$$

Also, since Π, Σ are literals (i.e., each is an atomic formulae or the negation formula of an atomic formula), so are Γ_k, Θ_k, and in particular, Θ_k are atomic formulae. We write this fact:

(5): for $k = 0, ..., l$, Γ_k is a subset of Π, Σ and $\neg\Theta_k := \{\neg A : A \text{ is in } \Theta\}$ is a subset of Π, Σ.

This will be used in the following.

7.2. Proof of $6^{36} \leq \eta(\sigma)$

Suppose, on the contrary, that $P = (T, <, \psi)$ has no applications of $(\vee \rightarrow)$ with its principal formula $\bigvee_{s=1}^{6^{36}} AR(s)$. Then, for the uppermost node n_l in the ascending chain $n_0, ..., n_l$, we have only the possibility (4b). Thus, for any node n_k in $n_0, n_1, ..., n_l$, $\bigvee_{s=1}^{6^{36}} AR(s)$ comes from the uppersequent n_{k+1} or it is introduced by (th). By (3), $\bigvee_{s=1}^{6^{36}} AR(s)$ is never a side formula of any operation inference rule in P.

Now, we can eliminate all occurrences of $\bigvee_{s=1}^{6^{36}} AR(s)$ in the sequents in $n_0, ..., n_l$. We define the new function ψ' so that if n is a node in $n_0, ..., n_l$,

$$\psi'(n_k) = \Gamma_k \rightarrow \Theta_k.$$

and otherwise $\psi'(n) = \psi(n)$. Now, we have $(X, <, \psi')$.

We show that $(X, <, \psi')$ is also a proof. For any node $n \in X$, define $X_n = \{n' \in X : n' = n \text{ or } n' > n\}$. Then, we show by induction over the tree structure $(X, <)$ from its leaves that the restriction $(X_n, <, \psi')$ is a proof for any $n \in X$.

When a node n is not in $n_0, ..., n_l$, the structure $(X_n, <, \psi')$ is not affected by the change of ψ to ψ' and is a proof.

Consider n_l. In P, $\psi'(n_l)$ must be the lower sequent of (th) and the upper sequent is given as $\psi'(n') = \psi(n')$ with $n_l <^I n'$. It is already seen that $(X_{n'}, <, \psi')$ is a proof. Since the thinning rule is allowed to have the same sequent in the upper and lower sequent, the following

$$\frac{\psi(n')}{\psi'(n_l)}$$

is also a legitimate thinning rule. Hence, $(X_{n_l}, <, \psi')$ is a proof.

Now, consider any node n_k in $n_0, ..., n_{l-1}$. The induction hypothesis is that the structure $(X_{n_{k-1}}, <, \psi')$ is a proof. We should consider all possible cases for inference rules. By (3), we should consider only four cases, i.e., $(\neg \rightarrow)$, $(\mathrm{c}\rightarrow)$, $(\rightarrow\mathrm{c})$ and (th) by (3): Here we consider only $(\mathrm{c}\rightarrow)$. The other cases are similar. It is expressed as

$$\frac{\Gamma_k, \xi, ..., \xi, \xi \rightarrow \Theta_k}{\Gamma_k, \xi, ..., \xi \rightarrow \Theta_k}\ (\mathrm{c} \rightarrow) \text{ or } \frac{\Gamma', A, A, \xi, ..., \xi \rightarrow \Theta_k}{\Gamma', A, \xi, ..., \xi \rightarrow \Theta_k}\ (\mathrm{c} \rightarrow).$$

where $\xi = \bigvee_{s=1}^{6^{36}} AR(s)$, and $\Gamma_{k-1} = \Gamma' \cup \{A, A\}$, $\Gamma_k = \Gamma' \cup \{A\}$ in the right case. Also, ξ does not occur in $\Gamma_{k-1}, \Gamma_k, \Theta_k$. In the left case, ξ is the contraction formula, and in the right case, it is A. Hence, these are changed into the following by eliminating all occurrences of $\xi = \bigvee_{s=1}^{6^{36}} AR(s)$;

$$\frac{\Gamma_k \to \Theta_k}{\Gamma_k \to \Theta_k} \text{ (th) or } \frac{\Gamma', A, A \to \Theta_k}{\Gamma', A \to \Theta_k} \text{ (c} \to \text{)}.$$

These are legitimate inferences. Hence, $(X_{n_k}, <, \psi')$ is a proof.

We have proved that $(X_{n_0}, <, \psi')$ is a proof of its endsequent is $\Pi, \Sigma \to$. However, this is not provable by Lemma 4.2.

Therefore, we have proved that $P = (X, <, \psi)$ has at least one application of $(\vee \to)$ with its principal formula $\bigvee_{s=1}^{6^{36}} AR(s)$. Thus, P has at least 6^{36} leaves. Since P is an arbitrary cut-free proof of σ, we have $6^{36} \leq \eta(\sigma)$.

8. An Upper Bound of $\eta(\sigma)$

One upper bound of $\eta(\sigma)$ can be found by giving and evaluating a particular proof of Arrow's theorem σ. Of course, the proof should be considered in the propositional Gentzen-style sequent calculus. However, since the proof itself is gigantic, we cannot write down a complete proof at once. We describe the proof, case by case, using "free variables" at the meta-level. In each case, we evaluate the size of the part of the proof, and finally we will sum up the contemplexities over those cases.

The last part of our proof of σ is as follows:

$$\frac{\left\{ \dfrac{\dfrac{\vdots}{\Pi, \Sigma_s, PR(s) \wedge IIA(s) \wedge ND(s) \to}}{\Pi, \Sigma, PR(s) \wedge IIA(s) \wedge ND(s) \to} (th) \right\}_{s=1}^{6^{36}}}{\Pi, \Sigma, \bigvee_{s=1}^{6^{36}} PR(s) \wedge IIA(s) \wedge ND(s) \to} (\vee \to)$$

Thus, we should evaluate an upper bound for $\eta(\Pi, \Sigma_s, PR(s) \wedge IIA(s) \wedge ND(s) \to \ )$ and then we will sum up these upper bounds for $s = 1$ to 6^{36}.

We divide the following argument into two cases: Dictator Case and Non-dictator Case. In the dictator case, we have a proof of

$$\text{Dictator}(s): \ \Pi, \Sigma_s, ND(s) \to$$

and in the non-dictator case, we have either

$$\text{Pareto}(s)\text{: } \Pi, \Sigma_s, PR(s) \to \quad \text{or} \quad \text{Ind.IA}(s)\text{: } \Pi, \Sigma_s, IIA(s) \to \quad .$$

In each case, we have the third lowest sequent $\Pi, \Sigma_s, PR(s) \wedge IIA(s) \wedge ND(s) \rightarrow$ from the endsequent in P by $(\wedge \rightarrow)$; for example, in the dictator case, we have

$$\dfrac{\dfrac{\cdot \quad \cdot \quad \cdot}{\Pi, \Sigma_s, ND(s) \rightarrow}}{\Pi, \Sigma_s, PR(s) \wedge IIA(s) \wedge ND(s) \rightarrow} \ (\wedge \rightarrow).$$

The dictator case is clear-cut, but the non-dictator case is entangled with subcases for Pareto(s) and IndIA(s). The contemplexities of the subproofs are 216, 3 and 6, respectively, for Dictator(s), Pareto(s) and IndIA(s).

8.1. Dictator Case

In the dictator case, a social ordering always coincides with the individual preference ordering of the dictator. Since we have only two individuals, there are also two dictator cases: We let s_1 and s_2 be the indices of the dictator social welfare functions, where 1 is the dictator for s_1 and 2 is the dictator for s_2. In the following, we consider only s_1. The other case is symmetric.

Now consider s_1. Then we have the following proof. Let $u, v \in \mathbf{A}$ be any distinct alternatives, and let $t \in \mathbf{6}^2$ be a profile. Then, there are two cases, (a): $P^t_{s_1}(u, v)$ is in Σ_{s_1} and (b): $\neg P^t_{s_1}(u, v)$ is in Σ_{s_1}. First, in case (a), we have

$$\dfrac{\dfrac{P^t_{s_1}(u, v) \rightarrow P^t_{s_1}(u, v)}{\Pi, \Sigma_{s_1}, R^t_1(u, v) \rightarrow P^t_{s_1}(u, v)} \ (th)}{\Pi, \Sigma_{s_1} \rightarrow R^t_1(u, v) \supset P^t_{s_1}(u, v)} \ (\rightarrow\supset)$$

In case (b), it holds also that $\neg R^t_1(u, v)$ is in Π. Then,

$$\dfrac{\dfrac{\dfrac{R^t_1(u, v) \rightarrow R^t_1(u, v)}{\neg R^t_1(u, v), R^t_1(u, v) \rightarrow} \ (\neg \rightarrow)}{\Pi, \Sigma_{s_1}, R^t_1(u, v) \rightarrow P^t_{s_1}(u, v)} \ (th)}{\Pi, \Sigma_{s_1} \rightarrow R^t_1(u, v) \supset P^t_{s_1}(u, v)} \ (\rightarrow\supset)$$

In either case, we have the same endsequent, for which the proof given above and denoted by $\pi_{(u,v),t}$ has only one uppermost sequent. We combine these proofs and have the following proof.

$$
\dfrac{\dfrac{\left\{\dfrac{\left\{\dfrac{\pi_{(u,v),t}}{\Pi, \Sigma_{s_1} \rightarrow R^t_1(u,v) \supset P^t_{s_1}(u,v)}\right\}_{(u,v)\in A^2-}}{\Pi, \Sigma_{s_1} \rightarrow \bigwedge_{(u,v)\in A^2}(R^t_1(u,v) \supset P^t_{s_1}(u,v))}\ (\rightarrow \wedge)\right\}_{t\in \mathbf{6}^2}}{\Pi, \Sigma_{s_1} \rightarrow \bigwedge_{t\in\mathbf{6}^2}\bigwedge_{(u,v)\in A^2-}(R^t_1(u,v) \supset P^t_{s_1}(u,v))}\ (\rightarrow \wedge)}{\dfrac{\Pi, \Sigma_{s_1} \rightarrow \bigvee_{i=1,2}\bigwedge_{t\in\mathbf{6}^2}\bigwedge_{(u,v)\in A^2-}(R^t_i(u,v) \supset P^t_{s_1}(u,v)}{\Pi, \Sigma_{s_1}, \neg\bigvee_{i=1,2}\bigwedge_{t\in\mathbf{6}^2}\bigwedge_{(u,v)\in A^2-}(R^t_i(u,v) \supset P^t_{s_1}(u,v)) \rightarrow}\ (\neg \rightarrow)}\ (\rightarrow \vee)
$$

This proof tree has 6^2 branches at the middle $(\rightarrow \wedge)$ and each branch has 6 further branches at the top $(\rightarrow \wedge)$. Hence, the contemplexity of the proof of $\Pi, \Sigma_s, ND(s) \rightarrow$ is $6 \times 6^2 = 216$.

In the case of s_2 (2-dictator), we have also contemplexity 216. Hence, for s_1 and s_2, we have 432 leaves.

8.2. Non-Dictator Case

Let s be an arbitrary number from 1 to 6^{36} with $s \neq s_1, s_2$. Consider the profile $R^t = (R^t_1, R^t_2)$ defined by:

$$
\begin{cases} R^t_1 : \alpha \quad \beta \quad \gamma \\ R^t_2 : \gamma \quad \beta \quad \alpha. \end{cases} \tag{8.1}
$$

This means that $R^t_1(\alpha, \beta)$ and $R^t_1(\beta, \gamma)$ are in Π and $R^t_2(\gamma, \beta)$ and $R^t_2(\beta, \alpha)$ are in Π. We will use the notation of (8.1) in the following. We will also fix $t = (t_1, t_2)$ in the following.

Then, there are 6 possibilities for social preferences in Σ^t_s:

Case 1A: $P^t_s(\alpha, \beta)$ and $P^t_s(\beta, \gamma)$; Case 2A: $P^t_s(\gamma, \beta)$ and $P^t_s(\beta, \alpha)$;
Case 1B: $P^t_s(\alpha, \gamma)$ and $P^t_s(\gamma, \beta)$; Case 2B: $P^t_s(\gamma, \alpha)$ and $P^t_s(\alpha, \beta)$;
Case 1C: $P^t_s(\beta, \alpha)$ and $P^t_s(\alpha, \gamma)$; Case 2C: $P^t_s(\beta, \gamma)$ and $P^t_s(\gamma, \alpha)$.

By Transitivity for Σ^t_s, each case has the third preference, e.g., $P^t_s(\alpha, \gamma) \in \Sigma^t_s$ in Case 1A. Every social ordering named as s from 1 to 6^{36} belongs to one of these six cases. Thus, the social orderings indexed by 1 to 6^{36} are classified into these six classes.

We consider Cases 1A, 1B, and 1C in the following. The other three cases can be treated by switching between individuals 1 and 2 in the following proofs, and then we have proofs for Cases 2A, 2B and 2C. Also, Cases 1B and 1C are similar, but Case 1A differs from Cases 1B and 1C.

Each case will be again divided into various subcases. In each subcase, we will construct a proof having the following form: either

$$\frac{\bullet \quad \bullet \quad \bullet}{\Pi, \Sigma_s, PR(s) \rightarrow} \quad \text{or} \quad \frac{\bullet \quad \bullet \quad \bullet}{\Pi, \Sigma_s, IIA(s) \rightarrow}.$$

Then, we calculate the contemplexity of it. We may write Π, Σ_s as Ξ_s to simplify the notation.

8.2.1. Cases 1A: $P_s^t(\alpha, \beta), P_s^t(\beta, \gamma)$ and $P_s^t(\alpha, \gamma)$ in Σ_s

Consider the other profile:

$$t'\text{-th profile} \begin{cases} 1 : \gamma & \beta & \alpha \\ 2 : \alpha & \beta & \gamma \end{cases}.$$

Then, there are three cases to be considered:

A0: $P_s^{t'}(\gamma, \beta)$ and $P_s^{t'}(\beta, \alpha)$ are in Σ_s;

A1: $P_s^{t'}(\alpha, \beta)$ is in Σ_s;

A2: $P_s^{t'}(\beta, \gamma)$ is in Σ_s.

Now, we consider the above three cases.

A0: $P_s^{t'}(\gamma, \beta), P_s^{t'}(\beta, \alpha) \in \Sigma_s$. In this case, individual 1' preferences and the social preferences coincide for profiles R^t and $R^{t'}$. But since $s \neq s_1, s_2$, for some profile $R^l = (R_1^l, R_2^l)$ and some pair of distinct alternatives $(x, y) \in \mathbf{A}^{2-}$, it holds that

$$R_1^l(x, y) \in \Pi \text{ but } P_s^l(y, x) \in \Sigma_s. \tag{8.2}$$

or

$$R_1^l(y, x) \in \Pi \text{ but } P_s^l(x, y) \in \Sigma_s. \tag{8.3}$$

Here, we have various cases for the choice of x, y. In the following, let (x, y) be $(\alpha, \beta), (\beta, \gamma)$ or (α, γ). In these three cases, we have the parallel proofs. Thus, we consider the case of $(x, y) = (\alpha, \beta)$. Also, assume (8.2).

Now, we have to think about two subcases: (a) $R_2^l(\alpha, \beta) \in \Pi$; and (b) $R_2^l(\beta, \alpha) \in \Pi$.

A0a: $R_2^l(\alpha, \beta) \in \Pi$. Then,

$$\dfrac{\dfrac{\dfrac{R_1^l(\alpha,\beta) \rightarrow R_1^l(\alpha,\beta)}{\Pi,\Sigma_s \rightarrow R_1^l(\alpha,\beta)}(th) \quad \dfrac{R_2^l(\alpha,\beta) \rightarrow R_2^l(\alpha,\beta)}{\Pi,\Sigma_s \rightarrow R_2^l(\alpha,\beta)}(th)}{\Pi,\Sigma_s \rightarrow R_1^l(\alpha,\beta) \wedge R_2^l(\alpha,\beta)}(\rightarrow\wedge) \quad \dfrac{\dfrac{P_s^l(\alpha,\beta) \rightarrow P_s^l(\alpha,\beta)}{\neg P_s^l(\alpha,\beta), P_s^l(\alpha,\beta) \rightarrow}(\neg\rightarrow)}{\Pi,\Sigma_s, P_s^l(\alpha,\beta) \rightarrow}(th)}{\dfrac{\Pi,\Sigma_s, R_1^l(\alpha,\beta) \wedge R_2^l(\alpha,\beta) \supset P_s^l(\alpha,\beta) \rightarrow}{\Pi,\Sigma_s, PR(s) \rightarrow}(\wedge\rightarrow)}(\supset\rightarrow)$$

This proof has the contemplexity 3.

A0b: $R_2^l(\beta,\alpha) \in \Pi$. First, we have the following two proofs:

$$\dfrac{\dfrac{R_1^t(\alpha,\beta)\to R_1^t(\alpha,\beta)}{\Pi,\Sigma_s\to R_1^t(\alpha,\beta)}(th)\quad \dfrac{R_2^l(\beta,\alpha)\to R_2^l(\beta,\alpha)}{\Pi,\Sigma_s\to R_2^t(\beta,\alpha)}(th)\quad \dfrac{P_2^t(\beta,\alpha)\to P_2^t(\beta,\alpha)}{\Pi,\Sigma_s\to P_2^t(\beta,\alpha)}(th)}{\Pi,\Sigma_s\to R_1^t(\alpha,\beta)\wedge R_2^t(\beta,\alpha)\wedge P_s^t(\alpha,\beta)}(\to\wedge) \tag{8.4}$$

and

$$\dfrac{\dfrac{\dfrac{R_1^l(\beta,\alpha)\to R_1^l(\beta,\alpha)}{\Pi,\Sigma_s\to R_1^l(\beta,\alpha)}\quad \dfrac{R_1^l(\alpha,\beta)\to R_1^l(\alpha,\beta)}{\Pi,\Sigma_s\to R_2^l(\alpha,\beta)}}{\Pi,\Sigma_s\to R_1^l(\beta,\alpha)\wedge R_2^l(\beta,\alpha)}(\to\wedge)\quad \dfrac{\dfrac{P_s^l(\alpha,\beta)\to P_s^l(\alpha,\beta)}{\neg P_s^l(\alpha,\beta),P_s^l(\alpha,\beta)\to}(\neg\to))}{\Pi,\Sigma_s,P_s^l(\alpha,\beta)\to}(th)}{\Pi,\Sigma_s,R_1^l(\beta,\alpha)\wedge R_2^l(\beta,\alpha)\supset P_s^l(\alpha,\beta)\to}(\supset\to) \tag{8.5}$$

Combining these two proofs, we have the following:

$$\dfrac{\dfrac{\dfrac{\bullet\ \bullet\ \bullet}{\Pi,\Sigma_s\to R_1^t(\alpha,\beta)\wedge R_2^t(\beta,\alpha)\wedge P_s^t(\alpha,\beta)}(\to\wedge)\quad \dfrac{\bullet\ \bullet\ \bullet}{\Pi,\Sigma_s,R_1^l(\beta,\alpha)\wedge R_2^l(\beta,\alpha)\supset P_s^l(\alpha,\beta)\to}(\supset\to)}{\Pi,\Sigma_s,R_1^t(\alpha,\beta)\wedge R_2^t(\beta,\alpha)\wedge P_s^t(\alpha,\beta)\supset(R_1^l(\beta,\alpha)\wedge R_2^l(\beta,\alpha)\supset P_s^l(\alpha,\beta))\to}(\supset\to)}{\Pi,\Sigma_s,IIA(s)\to} \tag{8.6}$$

The long formula in the second last sequent of (8.6) is a part of $IIA(s)$, and we obtain the last sequent. This case has contemplexity 6.

A1: Suppose $P_s^{t'}(\alpha,\beta) \in \Sigma_s$. Now, we consider following profile:

$$l\text{-}th\ \text{profile}\begin{cases} 1:\beta & \gamma \quad \alpha \\ 2:\gamma & \alpha \quad \beta \end{cases} \tag{8.7}$$

There are three cases to consider with respect to with P_s^l : A1a: $\neg P_s^l(\alpha,\beta) \in \Sigma_s$; A1b: $\neg P_s^l(\beta,\gamma) \in \Sigma_s$; and A1c:$P_s^l(\alpha,\beta), P_s^l(\beta,\gamma) \in \Sigma_s$.

A1a: Let $\neg P_s^l(\alpha,\beta) \in \Sigma_s$. Then, recalling $\Xi_s = \Pi,\Sigma_s$, we have the following proof without the names of inferences:

$$\dfrac{\dfrac{\dfrac{\pi_1^{t'}(\beta,\alpha)\quad \pi_2^{t'}(\alpha,\beta)\quad \pi_s^{t'}(\alpha,\beta)}{\Xi_s\to R_1^{t'}(\beta,\alpha)\wedge R_2^{t'}(\alpha,\beta)\wedge P_s^{t'}(\alpha,\beta)}\quad \dfrac{\dfrac{\dfrac{R_1^l(\beta,\alpha)\to R_1^l(\beta,\alpha)}{\Xi_s\to R_1^l(\beta,\alpha)}\quad \dfrac{R_2^l(\alpha,\beta)\to R_2^l(\alpha,\beta)}{\Xi_s\to R_2^l(\alpha,\beta)}}{\Xi_s\to R_1^l(\beta,\alpha)\wedge R_2^l(\alpha,\beta)}\quad \dfrac{\dfrac{P_s^l(\alpha,\beta)\to P_s^l(\alpha,\beta)}{\neg P_s^l(\alpha,\beta),P_s^l(\alpha,\beta)\to}}{\Xi_s,P_s^l(\alpha,\beta)\to}}{\Xi_s,[R_1^l(\beta,\alpha)\wedge R_2^l(\alpha,\beta)\supset P_s^l(\alpha,\beta)]\to}}{\Xi_s,R_1^{t'}(\beta,\alpha)\wedge R_2^{t'}(\alpha,\beta)\wedge P_s^{t'}(\alpha,\beta)\supset[R_1^l(\beta,\alpha)\wedge R_2^l(\alpha,\beta)\supset P_s^l(\alpha,\beta)]\to}}{\Xi_s,IIA(s)\to}$$

where $\pi_1^{t'}(\beta,\alpha), \pi_2^{t'}(\alpha,\beta), \pi_s^{t'}(\alpha,\beta)$ are similar to case A0b, and each has only one upper sequent. Hence the contemplexity here is 6.

A1b: Let $\neg P_s^l(\beta,\gamma) \in \Sigma_s$. Then, in this case, we compare the profile R^t with the profile R^l,

$$\dfrac{\dfrac{\dfrac{\pi_1^t(\beta,\gamma)\quad \pi_2^t(\gamma,\beta)\quad \pi_s^t(\beta,\gamma)}{\Xi_s\to R_1^t(\beta,\gamma)\wedge R_2^t(\gamma,\beta)\wedge P_s^t(\beta,\gamma)} \qquad \dfrac{\dfrac{\dfrac{R_1^l(\beta,\gamma)\to R_1^l(\beta,\gamma)}{\Xi_s\to R_1^l(\beta,\gamma)}\quad \dfrac{R_2^l(\gamma,\beta)\to R_2^l(\gamma,\beta)}{\Xi_s\to R_2^l(\gamma,\beta)}}{\Xi_s\to R_1^l(\beta,\gamma)\wedge R_2^l(\gamma,\beta)}\quad \dfrac{\dfrac{P_s^l(\beta,\gamma)\to P_s^l(\beta,\gamma)}{\neg P_s^l(\beta,\gamma),P_s^l(\beta,\gamma)\to}}{\Xi_s,P_s^l(\beta,\gamma)\to}}{\Xi_s,[R_1^l(\beta,\gamma)\wedge R_2^l(\gamma,\beta)\supset P_s^l(\beta,\gamma)]\to}}{\Xi_s,R_1^t(\beta,\gamma)\wedge R_2^t(\gamma,\beta)\wedge P_s^t(\beta,\gamma)\supset[R_1^l(\beta,\gamma)\wedge R_2^l(\gamma,\beta)\supset P_s^l(\beta,\gamma)]\to}}{\Xi_s,IIA(s)\to}$$

Here, the contemplexity is also 6.

A1c: $P_s^l(\alpha,\beta), P_s^l(\beta,\gamma) \in \Sigma_s$. In this case, we have $P_s^l(\alpha,\gamma) \in \Sigma_s$ and $\neg P_s^l(\gamma,\alpha) \in \Sigma_s$ by Transitivity and Asymmetry for Σ_s. Now, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{R_1^l(\gamma,\alpha)\to R_1^l(\gamma,\alpha)}{\Xi_s\to R_1^l(\gamma,\alpha)}(th)\quad \dfrac{R_2^l(\gamma,\alpha)\to R_2^l(\gamma,\alpha)}{\Xi_s\to R_2^l(\gamma,\alpha)}(th)}{\Xi_s\to R_1^l(\gamma,\alpha)\wedge R_2^l(\gamma,\alpha)}(\to\wedge)\qquad \dfrac{\dfrac{P_s^l(\gamma,\alpha)\to P_s^l(\gamma,\alpha)}{\neg P_s^l(\gamma,\alpha),P_s^l(\gamma,\alpha)\to}(\neg\to)}{\Xi_s,P_s^l(\gamma,\alpha)\to}(th)}{\Xi_s,R_1^l(\gamma,\alpha)\wedge R_2^l(\gamma,\alpha)\supset P_s^l(\gamma,\alpha)\to}(\supset\to)}{\Xi_s,PR(s)\to}(\wedge\to)$$

Hence, the contemplexity is 3.

A2: Suppose $P_s^{t'}(\beta,\gamma)$. In this case, we consider the following profile, rather than that of (8.7):

$$l\text{-th profile}\begin{cases} 1:\gamma & \alpha \quad \beta \\ 2:\beta & \gamma \quad \alpha \end{cases} \tag{8.8}$$

The remaining is similar to the cases of A1.

8.2.2. Case 1B: $P_s^t(\alpha,\gamma), P_s^t(\gamma,\beta)$ and $P_s^t(\alpha,\beta)$ in Σ_s

Consider another profile:

$$l\text{-}th\text{ profile}\begin{cases} 1:\beta & \alpha \quad \gamma \\ 2:\gamma & \beta \quad \alpha \end{cases}$$

Then, there are two cases to be considered: (1) $P_s^l(\alpha,\beta) \in \Sigma_s$; and (2) $\neg P_s^l(\alpha,\beta) \in \Sigma_s$.

B1: Suppose $P_s^l(\alpha,\beta) \in \Sigma_s$. Then,

$$\dfrac{\dfrac{\dfrac{\dfrac{R_1^l(\beta,\alpha)\to R_1^l(\beta,\alpha)}{\Xi_s\to R_1^l(\beta,\alpha)}(th)\quad \dfrac{R_2^l(\beta,\alpha)\to R_2^l(\beta,\alpha)}{\Xi_s\to R_2^l(\beta,\alpha)}(th)}{\Xi_s\to R_1^l(\beta,\alpha)\wedge R_2^l(\beta,\alpha)}(\to\wedge)\qquad \dfrac{\dfrac{P_s^l(\beta,\alpha)\to P_s^l(\beta,\alpha)}{\neg P_s^l(\beta,\alpha),P_s^l(\beta,\alpha)\to}(\neg\to)}{\Xi_s,P_s^l(\beta,\alpha)\to}(th)}{\Xi_s,R_1^l(\beta,\alpha)\wedge R_2^l(\beta,\alpha)\supset P_s^l(\beta,\alpha)\to}(\supset\to)}{\Xi_s,PR(s)\to}(\wedge\to)$$

We have contemplexity 3.

B2: Suppose $\neg P^l_s(\alpha,\beta) \in \Sigma_s$. Then, $P^l_s(\beta,\alpha) \in \Pi_s$ by Asymmetry for Π_s. We consider the two cases: (a) $P^l_s(\alpha,\gamma) \in \Pi_s$ and (b) $P^l_s(\gamma,\alpha) \in \Pi_s$.

B2a: $P^l_s(\alpha,\gamma) \in \Pi_s$: We have $P^l_s(\beta,\gamma) \in \Pi_s$ by Transitivity for Π_s and $\neg P^l_s(\gamma,\beta) \in \Pi_s$ by Asymmetry.

$$\dfrac{\dfrac{\dfrac{\pi^t_1(\beta,\gamma) \quad \pi^t_2(\gamma,\beta) \quad \pi^t_s(\gamma,\beta)}{\Xi_s \to R^t_1(\beta,\gamma) \wedge R^t_2(\gamma,\beta) \wedge P^t_s(\gamma,\beta)} \qquad \dfrac{\dfrac{\dfrac{R^l_1(\beta,\gamma)\to R^l_1(\beta,\gamma)}{\Xi_s\to R^l_1(\beta,\gamma)} \quad \dfrac{R^l_2(\gamma,\beta)\to R^l_2(\gamma,\beta)}{\Xi_s\to R^l_2(\gamma,\beta)}}{\Xi_s\to R^l_1(\beta,\gamma)\wedge R^l_2(\gamma,\beta)} \quad \dfrac{\dfrac{P^l_s(\gamma,\beta)\to P^l_s(\gamma,\beta)}{\neg P^l_s(\gamma,\beta), P^l_s(\gamma,\beta)\to}}{\Xi_s, P^l_s(\gamma,\beta)\to}}{\Xi_s, [R^l_1(\beta,\gamma)\wedge R^l_2(\gamma,\beta) \supset P^l_s(\gamma,\beta)]\to}}{\dfrac{\Xi_s, R^t_1(\beta,\gamma)\wedge R^t_2(\gamma,\beta)\wedge P^t_s(\gamma,\beta) \supset [R^l_1(\beta,\gamma)\wedge R^l_2(\gamma,\beta) \supset P^l_s(\gamma,\beta)]\to}{\Xi_s, IIA(s)\to}}$$

There are 6 leaves.

B2b: $P^l_s(\gamma,\alpha) \in \Pi_s$: Then, we have $\neg P^l_s(\alpha,\gamma) \in \Pi_s$ by Asymmetry. Then,

$$\dfrac{\dfrac{\dfrac{\pi^t_1(\alpha,\gamma) \quad \pi^t_2(\gamma,\alpha) \quad \pi^t_s(\alpha,\gamma)}{\Xi_s \to R^t_1(\alpha,\gamma) \wedge R^t_2(\gamma,\alpha) \wedge P^t_s(\alpha,\gamma)} \qquad \dfrac{\dfrac{\dfrac{R^l_1(\alpha,\gamma)\to R^l_1(\alpha,\gamma)}{\Xi_s\to R^l_1(\alpha,\gamma)} \quad \dfrac{R^l_2(\gamma,\alpha)\to R^l_2(\gamma,\alpha)}{\Xi_s\to R^l_2(\gamma,\alpha)}}{\Xi_s\to R^l_1(\alpha,\gamma)\wedge R^l_2(\gamma,\alpha)} \quad \dfrac{\dfrac{P^l_s(\alpha,\gamma)\to P^l_s(\alpha,\gamma)}{\neg P^l_s(\alpha,\gamma), P^l_s(\alpha,\gamma)\to}}{\Xi_s, P^l_s(\alpha,\gamma)\to}}{\Xi_s, [R^l_1(\alpha,\gamma)\wedge R^l_2(\gamma,\alpha) \supset P^l_s(\alpha,\gamma)]\to}}{\dfrac{\Xi_s, R^t_1(\alpha,\gamma)\wedge R^t_2(\gamma,\alpha)\wedge P^t_s(\alpha,\gamma) \supset [R^l_1(\alpha,\gamma)\wedge R^l_2(\gamma,\alpha) \supset P^l_s(\alpha,\gamma)]\to}{\Xi_s, IIA(s)\to}}$$

We have contemplexity 6.

8.2.3. Case 1.C: $P^t_s(\beta,\alpha), P^t_s(\alpha,\gamma)$ and $P^t_s(\beta,\gamma)$ in Σ_s

Consider another profile:

$$l\text{-}th \text{ profile} \begin{cases} 1: \alpha & \gamma & \beta \\ 2: \gamma & \beta & \alpha \end{cases}$$

Then, we should consider two cases: (1) $P^l_s(\beta,\gamma) \in \Sigma_s$; and (2) $P^l_s(\gamma,\beta) \in \Sigma_s$.

C1: $P^l_s(\beta,\gamma) \in \Sigma_s$. In this case, $\neg P^l_s(\gamma,\beta) \in \Sigma_s$ by Asymmetry for Σ_s. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{R^l_1(\gamma,\beta)\to R^l_1(\gamma,\beta)}{\Xi_s\to R^l_1(\gamma,\beta)}(th) \quad \dfrac{R^l_2(\gamma,\beta)\to R^l_2(\gamma,\beta)}{\Xi_s\to R^l_2(\gamma,\beta)}(th)}{\Xi_s\to R^l_1(\gamma,\beta)\wedge R^l_2(\gamma,\beta)}(\to\wedge) \qquad \dfrac{\dfrac{P^l_s(\gamma,\beta)\to P^l_s(\gamma,\beta)}{\neg P^l_s(\gamma,\beta), P^l_s(\gamma,\beta)\to}(\neg\to)}{\Xi_s, P^l_s(\gamma,\beta)\to}(th)}{\Xi_s, R^l_1(\gamma,\beta)\wedge R^l_2(\gamma,\beta)\supset P^l_s(\gamma,\beta)\to}(\supset\to)}{\Xi_s, PR(s)\to}(\wedge\to)$$

We have contemplexity 3.

C2: $P^l_s(\gamma,\beta) \in \Sigma_s$. In this case, we have two cases: (a): $P^l_s(\beta,\alpha) \in \Sigma_s$ and (b) $P^l_s(\beta,\alpha) \notin \Sigma_s$.

C2a: $P_s^l(\beta,\alpha)\in\Sigma_s$. In this case, $P_s^l(\gamma,\alpha)\in\Sigma_s$ by Transitivity for Σ_s. Then

$$\dfrac{\dfrac{\dfrac{\pi_1^t(\alpha,\gamma)\quad \pi_2^t(\gamma,\alpha)\quad \pi_s^t(\alpha,\gamma)}{\Xi_s\to R_1^t(\alpha,\gamma)\wedge R_2^t(\gamma,\alpha)\wedge P_s^t(\alpha,\gamma)}\qquad \dfrac{\dfrac{\dfrac{R_1^l(\alpha,\gamma)\to R_1^l(\alpha,\gamma)}{\Xi_s\to R_1^l(\alpha,\gamma)}\quad \dfrac{R_2^l(\gamma,\alpha)\to R_2^l(\gamma,\alpha)}{\Xi_s\to R_2^l(\gamma,\alpha)}}{\Xi_s\to R_1^l(\alpha,\gamma)\wedge R_2^l(\gamma,\alpha)}\quad \dfrac{\dfrac{P_s^l(\alpha,\gamma)\to P_s^l(\alpha,\gamma)}{\neg P_s^l(\alpha,\gamma), P_s^l(\alpha,\gamma)\to}}{\Xi_s, P_s^l(\alpha,\gamma)\to}}{\Xi_s,[R_1^l(\alpha,\gamma)\wedge R_2^l(\gamma,\alpha)\supset P_s^l(\alpha,\gamma)]\to}}{\Xi_s, R_1^{t1}(\alpha,\gamma)\wedge R_2^{t2}(\gamma,\alpha)\wedge P_s^t(\alpha,\gamma)\supset [R_1^{l1}(\alpha,\gamma)\wedge R_2^{l2}(\gamma,\alpha)\supset P_s^l(\alpha,\gamma)]\to}}{\Xi_s, IIA(s)\to}$$

We have contemplexity 6.

C2b: $P_s^l(\beta,\alpha)\notin\Sigma_s$. In this case, $P_s^l(\alpha,\beta)\in\Sigma_s$ by Asymmetry. Then,

$$\dfrac{\dfrac{\dfrac{\pi_1^t(\alpha,\beta)\quad \pi_2^t(\beta,\alpha)\quad \pi_s^t(\beta,\alpha)}{\Xi_s\to R_1^t(\alpha,\beta)\wedge R_2^t(\beta,\alpha)\wedge P_s^t(\beta,\alpha)}\qquad \dfrac{\dfrac{\dfrac{P_s^l(\beta,\alpha)\to P_s^l(\beta,\alpha)}{\neg P_s^l(\beta,\alpha), P_s^l(\beta,\alpha)\to}}{\Xi_s, P_s^l(\beta,\alpha)\to}\quad \dfrac{\dfrac{R_1^l(\alpha,\beta)\to R_1^l(\alpha,\beta)}{\Xi_s\to R_1^l(\alpha,\beta)}\quad \dfrac{R_2^l(\beta,\alpha)\to R_2^l(\beta,\alpha)}{\Xi_s\to R_2^l(\beta,\alpha)}}{\Xi_s\to R_1^l(\alpha,\beta)\wedge R_2^l(\beta,\alpha)}}{\Xi_s,[R_1^l(\alpha,\beta)\wedge R_2^l(\beta,\alpha)\supset P_s^l(\beta,\alpha)]\to}}{\Xi_s, R_1^t(\alpha,\beta)\wedge R_2^t(\beta,\alpha)\wedge P_s^t(\beta,\alpha)\supset [R_1^l(\alpha,\beta)\wedge R_2^l(\beta,\alpha)\supset P_s^l(\beta,\alpha)]\to}}{\Xi_s, IIA(s)\to}$$

We have contemplexity 6.

8.2.4. Calculation of An Upper Bound for $\rho(\sigma)$

The total number of social welfare functions is 6^{36} as already pointed out. Among them, there are two dictator cases s_1 and s_2, in which case a proof tree was given in Section 8.1.2. In the other cases, proofs are given in Section 8.1.3, where we have contemplexity 3 for $PR(s)$ and 6 for $IIA(s)$. Hence, we have

$$\eta(\sigma(s)) \quad\leq\quad 216 \text{ for } s = s_1, s_2;$$

$$\eta(\sigma(s)) \quad\leq\quad 6 \text{ for } s \neq s_1, s_2.$$

Since we have 6^{36} cases, we combining the above:

$$\eta(\sigma)\leq(6^{36}-2)\times 6+2\times 6^3=6^{37}+420.$$

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