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DYNAMIC ANALYSIS OF A UNIFIED MULTIVARIATE COUNTING PROCESS AND ITS ASYMPTOTIC BEHAVIOR

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Abstract

The class of counting processes constitutes a significant part of applied probability. The classic counting processes of importance include a Poisson process, a non-homogeneous Poisson process (NHPP) and a renewal process. More sophisticated counting processes have been developed in order to accommodate a wider range of applications. All of these counting processes seem to be quite different on surface, forcing one to understand each of them separately. The purpose of this paper is to develop a unified multivariate counting process which would contain all of the above examples as special cases. The dynamic behavior of the unified multivariate counting process is captured through analysis of the underlying Laplace transform generating functions, yielding asymptotic results. As an application, a manufacturing system with certain maintenance policies is considered, where the unified multivariate counting process enables one to determine numerically the optimal maintenance policy minimizing the total cost.

Keywords: Unified multivariate counting process; dynamic analysis; asymptotic behavior; NHPP; semi-Markov process; Laplace transform generating function

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0. Introduction

A stochastic process $\{N(t)\} : t \geq 0$ is called a counting process when $N(t)$ is non-negative, right continuous and monotone non-decreasing with $N(0) = 0$. The classic counting processes of importance include a Poisson process, a non-homogeneous Poisson process (NHPP) and a renewal process. More sophisticated counting processes have been developed in order to accommodate a wider range of applications. A Markov renewal process, for example, extends an ordinary renewal process in that the interarrival time between two successive arrivals has a probability distribution depending on the state transition of the underlying Markov chain, see e.g. Pyke [19, 20]. In Masuda and Sumita [15], the number of entries of a semi-Markov process into a subset of the state space is analyzed while Lucantoni, Meier-Hellstern and Neuts [13] develops a Markovian arrival process where a Markov chain defined on $\mathcal{J} = G \cup B$ with $G \cap B = \phi, G \neq \phi$ and $B \neq \phi$ is replaced to a state $i \in G$ as soon as it enters a state $j \in B$ with probability $\tilde{p}_{ji}$ and the counting process describes the number of such replacements occurred in $[0, t]$. In an age-dependent counting process generated from a renewal process studied by Sumita and Shanthikumar [23], items arrive according to an NHPP which is interrupted and reset at random epochs governed by a renewal process. All of these counting processes seem to be quite different on surface, forcing one to understand each of them separately. The purpose of this paper is to develop a unified multivariate counting process which would contain all of the above examples as special cases. The dynamic behavior of the unified multivariate counting process is captured through analysis of the underlying Laplace transform generating functions, yielding asymptotic results. The unified multivariate counting process finds many applications in various areas including communication networks and system reliability models.

We consider a system where items arrive according to an NHPP. This arrival stream is interrupted from time to time where the interruptions are governed by a finite semi-Markov process $J(t)$ on $\mathcal{J} = \{0, 1, 2, \cdots, J\}$. Whenever a state transition of the semi-Markov process occurs from i to j, the intensity function of the NHPP is switched from $\lambda_i(x)$ to $\lambda_j(x)$ with an initial value reset to $\lambda_j(0)$. In other words, the arrivals of items are generated by the NHPP with $\lambda_i(x)$ when the semi-Markov process is in state i with x denoting the time since the last entry into state i. Of particular interest in analysis of such systems are the multivariate counting processes $\underline{M}(t) = [M_i(t)]_{i\in\mathcal{J}}$ and $\underline{\underline{N}}(t) = [N_{ij}(t)]_{i,j\in\mathcal{J}}$ where $M_i(t)$ counts the cumulative number of items that have arrived in $[0, t]$ while the semi-Markov process is in state i and $N_{ij}(t)$ represents the cumulative number of the state transitions of the semi-Markov process from i to j in $[0, t]$. The multivariate counting process unifies many existing counting processes in that they can be derived as special cases of the multivariate counting process, as we will see.

Applications of such systems can be found, for example, in modern communication networks. One may consider a high speed communication link for transmitting video signals between two locations. Video sequences are transmitted as streams of binary data that vary over time in traffic intensity according to the level of movement, the frequency of scene changes, and the level of transmission quality. Consequently, efficient transmission of video traffic can be achieved through variable bit rate coding. In this coding scheme, data packets are not generated at a constant rate from the original sequence, but rather at varying rates. By doing so, one achieves less fluctuation in transmission quality level and, at the same time, transmission capacity can be freed up whenever possible. As in Maglaris et al. [14], such a mechanism may be implemented by using multimode encoders where each mode reflects a certain level of data compression, and the change between modes is governed by the underlying video sequence according to buffer occupancy levels. A system of this sort can be described in the above framework with $M(t) = \sum_{i\in\mathcal{J}} M_i(t)$ representing the number of packet arrivals at the origination site and $N_{ij}(t)$ describing the number of the encoder changes in $[0, t]$. The state of the underlying semi-Markov process at time t then corresponds to the current mode of the encoder. Other types of applications include system reliability models where the semi-Markov process describes the status of the system under consideration while the interupptions correspond to system failures and replacements. A cost function associated with such a system may then be constructed from the multivariate counting processes $\underline{M}(t)$ and $\underline{\underline{N}}(t)$ presented and analysis in this paper.

The structure of this paper is as follows. In Section 1, key transform results of various existing counting processes are summarized. A detailed description of the unified multivariate counting process is provided in Section 2 and its dynamic behavior is analyzed in Section 3 by examining the probabilistic flow of the underlying stochastic processes and deriving transform results invoving Laplace transform generating functions. Section 4 is devoted to derivation of the existing counting processes of Section 1 as special cases of the unified counting process. Asymptotic analysis is provided in Section 5 and some numerical examples are given in Section 6.

1. Various Counting Processes of Interest

In this section, we summarize key transform results of various counting processes of interest, which will be shown to be special cases of the unified counting process proposed in this paper. We begin the discussion with one of the most classical arrival processes, the Poisson process.

1.1. Poisson Process

A Poisson process of intensity λ is characterized by a sequence of independently and identically distributed (i.i.d.) exponential random variables $(X_j)_{j=1}^{\infty}$ with common probability density function (p.d.f.) $f_X(x) = \lambda e^{-\lambda x}$. Let $S_n = \sum_{j=1}^{n} X_j$. Then, the associated Poisson process $N(t) : t \geq 0$ is defined as a counting process satisfying

$$N(t) = n \quad \Longleftrightarrow \quad S_n \leq t < S_{n+1} \ . \tag{1.1.1}$$

If a system has an exponential lifetime of mean λ^{-1} and is renewed instantaneously upon failure, X_j represents the lifetime of the j-th renewal cycle. The Poisson process $N(t) : t \geq 0$ then counts the number of failures that have occurred by time t.

Let $p_n(t) = \mathrm{P}[N(t) = n \mid N(0) = 0]$ and define the probability generating function (p.g.f.) $\pi(v, t)$ by

$$\pi(v,t) = \mathrm{E}[v^{N(t)}] = \sum_{n=0}^{\infty} p_n(t) v^n \ . \tag{1.1.2}$$

It can be seen, see e.g. Karlin and Taylor [7], that

$$\frac{d}{dt} p_n(t) = -\lambda p_n(t) + \lambda p_{n-1}(t) \tag{1.1.3}$$

where $p_n(t) = 0$ for $n < 0$. Multiplying v^n on both sides of (1.1.3) and summing from 0 to ∞, one then finds that

$$\frac{\partial}{\partial t} \pi(v,t) = -\lambda(1-v)\pi(v,t) \ . \tag{1.1.4}$$

Since $p_n(0) = \delta_{\{n=0\}}$ where $\delta_{\{P\}} = 1$ if statement P is true and $\delta_{\{P\}} = 0$ otherwise, one has $\pi(v, 0) = 1$. Equation (1.1.4) can then be solved as

$$\pi(v,t) = e^{-\lambda t(1-v)}; \quad p_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \ . \tag{1.1.5}$$

1.2. Non-homogeneous Poisson Process (NHPP)

An NHPP $M(t) : t \geq 0$ differs from a Poisson process in that the failure intensity of the system is given as a function of time t. Accordingly, Equation (1.1.3) should be rewritten as

$$\frac{d}{dt}p_m(t) = -\lambda(t)p_m(t) + \lambda(t)p_{m-1}(t) \ . \tag{1.2.1}$$

By taking the generating function of Equation (1.2.1), one finds that

$$\frac{\partial}{\partial t}\pi(u,t) = -\lambda(t)(1-u)\pi(u,t) \ . \tag{1.2.2}$$

With $L(t) = \int_0^t \lambda(y)dy$, this equation can be solved as

$$\pi(u,t) = e^{-L(t)(1-u)}; \quad p_m(t) = e^{-L(t)}\frac{L(t)^m}{m!} \ . \tag{1.2.3}$$

The reader is referred to Ross [21] for further discussions of NHPPs.

1.3. Markov Modulated Poisson Process(MMPP)

Let $J(t) : t \geq 0$ be a Markov chain in continuous time on $\mathcal{J} = \{0, \cdots, J\}$ governed by a transition rate matrix $\underline{\underline{\nu}} = [\nu_{ij}]$. Let $\underline{\lambda}^\top = [\lambda_0, \cdots, \lambda_J]$ and define the associated diagonal matrix $\underline{\underline{\lambda}}_D = [\delta_{\{i=j\}}\lambda_i]$. An MMPP $M(t) : t \geq 0$ characterized by $(\underline{\underline{\nu}}, \underline{\underline{\lambda}}_D)$ is a pure jump process where jumps of $M(t)$ occur according to a Poisson process with intensity λ_i whenever the Markov chain $J(t)$ is in state i.

Let $\nu_i = \sum_{j \in \mathcal{J}} \nu_{ij}$ and define $\underline{\underline{\nu}}_D = [\delta_{\{i=j\}}\nu_i]$. The infinitesimal generator $\underline{\underline{Q}}$ associated with the Markov chain $J(t)$ is then given by

$$\underline{\underline{Q}} = -\underline{\underline{\nu}}_D + \underline{\underline{\nu}} \ . \tag{1.3.1}$$

For $i, j \in \mathcal{J}$, let

$$\underline{\underline{p}}(k,t) = [p_{ij}(k,t)] \ ; \tag{1.3.2}$$

$$p_{ij}(k,t) = \mathrm{P}[M(t) = k, J(t) = j \mid J(0) = i, M(0) = 0] \ ,$$

and define the associated matrix generating function $\underline{\underline{\pi}}(u,t)$ by

$$\underline{\underline{\pi}}(u,t) = \sum_{k=0}^{\infty} \underline{\underline{p}}(k,t)u^k \ . \tag{1.3.3}$$

It can be seen that

$$\frac{\partial}{\partial t} p_{ij}(k,t) = -(\lambda_j + \nu_j) p_{ij}(k,t) + \sum_{r\in\mathcal{J}} p_{ir}(k,t)\nu_{rj} + \lambda_j p_{ij}(k-1,t) \ .$$

In matrix notation, this can be rewritten as

$$\frac{\partial}{\partial t}\underline{\underline{p}}(k,t) = -\underline{\underline{p}}(k,t)\left\{\underline{\underline{\lambda}}_D + \underline{\underline{\nu}}_D - \underline{\underline{\nu}}\right\} + \underline{\underline{p}}(k-1,t)\underline{\underline{\lambda}}_D \ . \tag{1.3.4}$$

By taking the generating function of (1.3.4) together with (1.3.1), one sees that

$$\frac{\partial}{\partial t}\underline{\underline{\pi}}(u,t) = \underline{\underline{\pi}}(u,t)\left\{\underline{\underline{Q}} - (1-u)\underline{\underline{\lambda}}_D\right\} \ . \tag{1.3.5}$$

Since $M(0) = 0$, one has $\underline{\underline{\pi}}(u,0) = \underline{\underline{I}}$, where $\underline{\underline{I}} = [\delta_{\{i=j\}}]$ is the identity matrix, so that the above differential equation can be solved as

$$\underline{\underline{\pi}}(u,t) = e^{\left\{\underline{\underline{Q}}-(1-u)\underline{\underline{\lambda}}_D\right\}t} = \sum_{k=0}^{\infty}\frac{t^k}{k!}\left\{\underline{\underline{Q}} - (1-u)\underline{\underline{\lambda}}_D\right\}^k \ , \tag{1.3.6}$$

where $\underline{\underline{A}}^0 \stackrel{\text{def}}{=} \underline{\underline{I}}$ for any square matrix $\underline{\underline{A}}$. It should be noted that $\underline{\underline{\pi}}(1,t) = e^{\underline{\underline{Q}}t}$, which is the transition probability matrix of $J(t)$ as it should be. By taking the Laplace transform of both sides of (1.3.6), $\hat{\underline{\underline{\pi}}}(u,s) = \int_0^\infty e^{-st}\underline{\underline{\pi}}(u,t)$ is given by

$$\hat{\underline{\underline{\pi}}}(u,s) = \left\{s\underline{\underline{I}} - \underline{\underline{Q}} + (1-u)\underline{\underline{\lambda}}_D\right\}^{-1} \ . \tag{1.3.7}$$

In general, the interarrival times generated by an MMPP are not independent nor identically distributed. In multimedia computer and communication networks, data packets are mingled together with voice and image packets generated from analogue sources. Since arrival patterns of such packets differ from each other, MMPPs have provided useful means to model arrival processes of packets in multimedia computer and communication networks, see e.g. Heffes and Lucantoni [6] and Sriram and Whitt [22]. Burman and Smith [1], and Knessl, Matkowsky, Schuss and Tier [12] studied a single server queuing system with an MMPP arrival process and general i.i.d. service times. Neuts, Sumita and Takahashi [18] established characterization theorems for an MMPP to be a renewal process in terms of lumpability of the underlying Markov chain $J(t)$. The reader is referred to Neuts [17] for further discussions of MMPP.

An MMPP can be extended by replacing the underlying Markov chain in continuous time by a semi-Markov process. This process is denoted by SMMPP. To the best knowledge of the author, SMMPP has not been studied in the literature. As we will see, both MMPP and SMMPP will be proven to be special cases of the unified multivariate counting process proposed in this paper.

1.4. Renewal Process

Renewal processes can be considered as a generalization of Poisson processes in that a sequence of i.i.d. exponential random variables are replaced by that of any i.i.d. nonnegative random variables with common distribution function $A(x)$. The resulting counting process $N(t) : t \geq 0$ is still characterized by (1.1.1). Let $p_n(t) = \mathrm{P}[N(t) = n \mid N(0) = 0]$ as before. One then sees that,

$$p_n(t) = A^{(n)}(t) - A^{(n+1)}(t) \tag{1.4.1}$$

where $A^{(n)}(t)$ denotes the n-fold convolution of $A(x)$ with itself, i.e. $A^{(n+1)}(t) = \int_0^t A^{(n)}(t-x)dA(x)$ and $A^{(0)}(t) = U(t)$ which is the step function defined as $U(t) = 1$ for $t \geq 0$ and $U(t) = 0$ else.

Let $\pi_n(s) = \int_0^\infty e^{-st} p_n(t)dt$ and $\alpha(s) = \int_0^\infty e^{-st} dA(t)$. By taking the Laplace transform of both sides of (1.4.1), it follows that

$$\pi_n(s) = \frac{1-\alpha(s)}{s}\alpha(s)^n \ .$$

By taking the generating function of the above equation with $\pi(v,s) = \sum_{n=0}^\infty \pi_n(s)v^n$, one has

$$\pi(v,s) = \frac{1-\alpha(s)}{s}\frac{1}{1-v\alpha(s)} \ . \tag{1.4.2}$$

The reader is referred to Cox [5], or Karlin and Taylor [7] for further discussions of renewal processes.

1.5. Markov Renewal Process (MRP)

An MRP is an extension of an ordinary renewal process in that, in the interval $[0,t)$, the former describes the recurrence statistics for intermingling classes of epochs of an underlying semi-Markov process, whereas the latter counts the number of recurrences for a single recurrent class of epochs. More specifically, let $J(t) : t \geq 0$ be a semi-Markov process on $\mathcal{J} = \{0, \cdots, J\}$ governed by a matrix p.d.f. $\underline{\underline{a}}(x)$ where $\underline{\underline{a}}(x) \geq \underline{\underline{0}}$ and $\int_0^\infty \underline{\underline{a}}(x)dx = \underline{\underline{a}}_0$ is a stochastic matrix which is ergodic. Let $\varepsilon(\ell)$ be a recurrent class consisting of the entries of the semi-Markov process to state ℓ for $\ell \in \mathcal{J}$, and define $\tilde{N}_{\ell r}(t)$ to be a counting process describing the number of recurrences for $\varepsilon(r)$ given that there was an epoch of $\varepsilon(\ell)$ at time $t = 0$. Then $\underline{\tilde{N}}_\ell(t) = [\tilde{N}_{\ell 0}(t), \cdots, \tilde{N}_{\ell J}(t)]$ is called an MRP.

The study of MRPs can be traced back to early 1960s represented by the two original papers by Pyke [19, 20], followed by Keilson [8, 9], Keilson and Wishart [10, 11], Çınlar [2, 3], McLean and Neuts [16]. Since then, the area attracted many researchers and a survey paper by Çınlar [4] in 1975 already included more than 70 leading references. The study has been largely focused on the matrix renewal function $\underline{\underline{H}}(t) = [H_{\ell r}(t)]$ with $H_{\ell r}(t) = \mathrm{E}[\tilde{N}_{\ell r}(t)]$, the associated matrix renewal density, and the limit theorems. For example, one has the following result concerning the Laplace transform of $\underline{\underline{H}}(t)$ by Keilson [9],

$$\mathcal{L}\left\{\underline{\underline{H}}(t)\right\} = \frac{1}{s}\underline{\underline{\alpha}}(s)\left[\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\right]^{-1} , \tag{1.5.1}$$

where $\underline{\underline{\alpha}}(s)$ is the Laplace transform of $\underline{\underline{a}}(t)$. The unified multivariate counting process of this paper contains an MRP as a special case and provides more information based on dynamic analysis of the underlying probabilistic flows.

1.6. Number of Entries of a Semi-Markov Process into a Subset of the State Space (NESMPS)

Another type of counting processes associated with a semi-Markov process on $\mathcal{J} = \{0, \cdots, J\}$ governed by a matrix p.d.f. $\underline{\underline{a}}(x)$ is studied in Masuda and Sumita [15], where the state space $\mathcal{J}$ is decomposed into a set of good states $G(\neq \phi)$ and a set of bad states $B(\neq \phi)$ satisfying $\mathcal{J} = G \cup B$ and $G \cap B = \phi$. The counting process $N_{GB}(t)$ is then defined to describe the number of entries of $J(t)$ into B by time t.

While $N_{GB}(t)$ is a special case of MRPs, the detailed analysis is provided in [15], yielding much more information. More specifically, let $X(t)$ be the age process associated with $J(t)$, i.e.

$$X(t) = t - \sup\{\tau : J(t) \mid_{\tau-}^{\tau+} \neq 0,\ 0 < \tau \leq t\} \tag{1.6.1}$$

where $f(x) \mid_{x-}^{x+} = f(x+) - f(x-)$, and define

$$\underline{\underline{F}}_n(x,t) = [F_{n:ij}(x,t)] \tag{1.6.2}$$

where

$$\begin{aligned} F_{n:ij}(x,t) \quad &= \quad \mathrm{P}[X(t) \leq x, N_{GB}(t) = n, J(t) = j \mid \\ & \qquad X(0) = N_{GB}(0) = 0, J(0) = i]\ . \end{aligned} \tag{1.6.3}$$

One then has

$$\underline{\underline{f}}_n(x,t) = \frac{\partial}{\partial x}\underline{\underline{F}}_n(x,t)\ . \tag{1.6.4}$$

The associated matrix Laplace transform generating function can then be defined as

$$\underline{\underline{\varphi}}(v,w,s) = \sum_{n=0}^{\infty} v^n \int_0^{\infty}\int_0^{\infty} e^{-wt-st}\underline{\underline{f}}_n(x,t)dxdt\ . \tag{1.6.5}$$

It has been shown in Masuda and Sumita [15] that

$$\underline{\underline{\varphi}}(v,w,s) = \frac{1}{w+s}\cdot\underline{\underline{\gamma}}_0(s)\left\{\underline{\underline{I}} - v\underline{\underline{\beta}}(s)\right\}^{-1}\left\{\underline{\underline{I}} - \underline{\underline{\alpha}}_D(w+s)\right\}\ . \tag{1.6.6}$$

Here, with $\underline{\underline{\alpha}}(s) = \int_0^{\infty} e^{-st}\underline{\underline{a}}(t)dt$ and $\underline{\underline{\alpha}}_{CD}(s) = [\alpha_{ij}(s)]_{\ i\in C, j\in D}$ for $C, D \subset \mathcal{J}$, the following notation is employed:

$$\underline{\underline{\alpha}}(s) = \begin{bmatrix} \underline{\underline{\alpha}}_{GG}(s) & \underline{\underline{\alpha}}_{GB}(s) \\ \underline{\underline{\alpha}}_{BG}(s) & \underline{\underline{\alpha}}_{BB}(s) \end{bmatrix}\ ; \tag{1.6.7}$$

$$\underline{\underline{\alpha}}_D(s) = [\delta_{\{i=j\}}\alpha_i(s)] \text{ with } \alpha_i(s) = \sum_{j\in\mathcal{J}} \alpha_{ij}(s)\ ;$$

$$\underline{\underline{\chi}}_G(s) = \left\{\underline{\underline{I}}_{GG} - \underline{\underline{\alpha}}_{GG}(s)\right\}^{-1}\ ;\quad \underline{\underline{\chi}}_B(s) = \left\{\underline{\underline{I}}_{BB} - \underline{\underline{\alpha}}_{BB}(s)\right\}^{-1}\ ; \tag{1.6.8}$$

$$\underline{\underline{\beta}}(s) = \begin{bmatrix} \underline{\underline{0}}_{BB} & \underline{\underline{0}}_{BG} \\ \underline{\underline{\alpha}}_{GB}\underline{\underline{\chi}}_B(s) & \underline{\underline{\alpha}}_{GB}\underline{\underline{\chi}}_B(s)\underline{\underline{\alpha}}_{BG}\underline{\underline{\chi}}_G(s) \end{bmatrix}\ ; \tag{1.6.9}$$

and

$$\underline{\underline{\gamma}}_0(s) = \begin{bmatrix} \underline{\underline{\alpha}}_{BB}\underline{\underline{\chi}}_B(s) & \underline{\underline{\chi}}_B(s)\underline{\underline{\alpha}}_{BG}\underline{\underline{\chi}}_G(s) \\ \underline{\underline{0}}_{GB} & \underline{\underline{\alpha}}_{GG}\underline{\underline{\chi}}_G(s) \end{bmatrix} + \underline{\underline{I}}\ . \tag{1.6.10}$$

As we will see, the unified multivariate counting process proposed in this paper enables one to deal with multi-dimensional generalization of NESMPSs as special cases.

1.7. Markovian Arrival Process (MAP)

As for Poisson processes, a renewal process requires interarrival times to form a sequence of i.i.d. nonnegative random variables. As we have seen, a class of MMPPs enables one to avoid this requirement by introducing different Poisson arrival rates depending on the state of the underlying Markov chain. An alternative way to avoid this i.i.d. requirement is to adapt a class of MAPs, originally introduced by Lucantoni, Meier-Hellstern and Neuts [13]. We discuss here a slightly generalized version of MAPs in that a set of absorbing states is not necessarily a singleton set.

Let $J^*(t) : t \geq 0$ be an absorbing Markov chain on $\mathcal{J}^* = G \cup B$ with $G \neq \phi, B \neq \phi$ and $G \cap B = \phi$, where all states in B are absorbing. Without loss of generality, we assume that $G = \{0, \cdots, m\}$, and $B = \{m+1, \cdots, m+K\}$. For notational convenience, the following transition rate matrices are introduced.

$$\underline{\underline{\nu}}^*_{GG} = [\nu_{ij}]_{\ i,\ j \in G} \ ; \quad \underline{\underline{\nu}}^*_{GB} = [\nu_{ir}]_{\ i \in G,\ r \in B} \ . \tag{1.7.1}$$

The entire transition rate matrix $\underline{\underline{\nu}}^*$ governing $J^*(t)$ is then given by

$$\underline{\underline{\nu}}^* = \begin{bmatrix} \underline{\underline{\nu}}^*_{GG} & \underline{\underline{\nu}}^*_{GB} \\ \underline{\underline{0}} & \underline{\underline{0}} \end{bmatrix} \ . \tag{1.7.2}$$

A replacement Markov chain $J(t) : t \geq 0$ on G is now generated from $J^*(t) : t \geq 0$. Starting from a state in G, the process $J(t)$ coincides with $J^*(t)$ within G. As soon as $J^*(t)$ reaches state $r \in B$, it is instantaneously replaced at state $j \in G$ with probability $\tilde{p}_{rj}$ and the process continues. Let

$$\underline{\underline{C}} = \underline{\underline{\nu}}^*_{GG} \ , \quad \underline{\underline{D}} = \underline{\underline{\nu}}^*_{GB} \underline{\underline{\tilde{p}}}_{BG} \tag{1.7.3}$$

where $\underline{\underline{\tilde{p}}}_{BG} = [\tilde{p}_{rj}]_{\ r \in B,\ j \in G}$. Then the transition rate matrix $\underline{\underline{\nu}}$ and the infinitesimal generator $\underline{\underline{Q}}$ of $J(t)$ are given as

$$\underline{\underline{\nu}} = \underline{\underline{C}} + \underline{\underline{D}} \ ; \quad \underline{\underline{Q}} = -\underline{\underline{\nu}}_D + \underline{\underline{\nu}} \ , \tag{1.7.4}$$

where

$$\underline{\underline{\nu}}_D = \underline{\underline{C}}_D + \underline{\underline{D}}_D \ , \tag{1.7.5}$$

with

$$\underline{\underline{C}}_D = \left[\delta_{\{i=j\}} c_i\right] ; \; c_i = \sum_{j \in G} c_{ij} , \tag{1.7.6}$$

$$\underline{\underline{D}}_D = \left[\delta_{\{i=j\}} d_i\right] ; \; d_i = \sum_{j \in G} d_{ij} .$$

Let $\underline{\underline{p}}(t)$ be the transition probability matrix of $\underline{\underline{J}}(t)$ with its Laplace transform defined by $\underline{\underline{\pi}}(s) = \int_0^\infty e^{-st} \underline{\underline{p}}(t) dt$. From the Kolmogorov forward equation $\frac{d}{dt}\underline{\underline{p}}(t) = \underline{\underline{p}}(t)\underline{\underline{Q}}$ with $\underline{\underline{p}}(0) = \underline{\underline{I}}$, one has

$$\underline{\underline{\pi}}(s) = \left\{ s\underline{\underline{I}} - \underline{\underline{Q}} \right\}^{-1} . \tag{1.7.7}$$

Let $N_{MAP}(t) : t \geq 0$ be the counting process keeping the record of the number of replacements in $[0, t)$ and define

$$f_{ij}(k,t) = \mathrm{P}\left[N_{MAP}(t) = k, J(t) = j \mid J(0) = i\right] \; , \; i, j \in G . \tag{1.7.8}$$

By analyzing the probabilistic flow at state j at time $t + \Delta$, it can be seen that

$$\begin{aligned} f_{ij}(k, t+\Delta) &= f_{ij}(k,t) \left\{ 1 - \sum_{\ell \in G} (c_{j\ell} + d_{j\ell})\Delta \right\} + \sum_{\ell \in G} f_{i\ell}(k,t) c_{\ell j} \Delta \\ &\quad + \sum_{\ell \in G} f_{i\ell}(k-1,t) d_{\ell j} \Delta + o(\Delta) . \end{aligned}$$

It then follows that

$$\begin{aligned} \frac{\partial}{\partial t} f_{ij}(k,t) &= -f_{ij}(k,t) \sum_{\ell \in G} (c_{j\ell} + d_{j\ell}) + \sum_{\ell \in G} f_{i\ell}(k,t) c_{\ell j} \\ &\quad + \sum_{\ell \in G} f_{i\ell}(k-1,t) d_{\ell j} . \end{aligned} \tag{1.7.9}$$

In matrix notation, the above equation can be rewritten as

$$\frac{\partial}{\partial t} \underline{\underline{f}}(k,t) = -\underline{\underline{f}}(k,t) \underline{\underline{\nu}}_D + \underline{\underline{f}}(k,t) \underline{\underline{C}} + \underline{\underline{f}}(k-1,t) \underline{\underline{D}} . \tag{1.7.10}$$

We now introduce the following matrix Laplace transform generating function:

$$\underline{\underline{\varphi}}(v,s) = \left[\varphi_{ij}(v,s)\right] ; \quad \varphi_{ij}(v,s) = \int_0^\infty e^{-st} \sum_{k=0}^\infty f_{ij}(k,t) v^k dt . \tag{1.7.11}$$

Taking the Laplace transform with respect to t and the generating function with respect to k of both sides of (1.7.10), one has

$$\underline{\underline{\varphi}}(v,s) \left\{ s\underline{\underline{I}} + \underline{\underline{\nu}}_D - \underline{\underline{C}} - v\underline{\underline{D}} \right\} = \underline{\underline{I}} ,$$

which can be solved as

$$\underline{\underline{\varphi}}(v,s) = \left\{ s\underline{\underline{I}} - \underline{\underline{Q}} + (1-v)\underline{\underline{D}} \right\}^{-1} . \tag{1.7.12}$$

We note from (1.7.7) and (1.7.12) that $\underline{\underline{\varphi}}(1,s) = \underline{\underline{\pi}}(s)$ as it should be.

It may be worth noting that an MMPP is a special case of an MAP, which can be seen in the following manner. Let an MAP be defined on $G \cup B$ where $G = \{0_G, \cdots, J_G\}$ and $B = \{0_B, \cdots, J_B\}$. Transitions within G is governed by $\underline{\underline{\nu}}$. An entry into $j_B \in B$ is possible only from $j_G \in G$. When this occurs, the Markov process is immediately replaced at the entering state j_G. The counting process for the number of such replacements then has the Laplace transform generating function $\underline{\underline{\varphi}}(v,s)$ given in (1.7.12) where $\underline{\underline{D}}$ is replaced by $\underline{\underline{\lambda}}_D$, which coincides with $\hat{\underline{\underline{\pi}}}(u,s)$ of (1.3.7) for MMPPs, proving the claim.

1.8. Age-dependent Counting Process Generated from a Renewal Process (ACPGRP)

An age-dependent counting process generated from a renewal process has been introduced and studied by Sumita and Shanthikumar [23], where items arrive according to an NHPP which is interrupted and reset at random epochs governed by a renewal process. More specifically, let $N(t) : t \geq 0$ be a renewal process associated with a sequence of i.i.d. nonnegative random variables $(X_j)_{j=1}^{\infty}$ with common p.d.f. $a(x)$. The age process $X(t)$ is then defined by

$$X(t) = t - \sup\{\tau : N(t) \,|_{\tau-}^{\tau+} = 1, 0 < \tau \leq t\} . \tag{1.8.1}$$

In other words, $X(t)$ is the elapsed time since the last renewal. We next consider an NHPP $Z(x)$ governed by an intensity function $\lambda(x)$. If we define

$$L(x) = \int_0^x \lambda(y)dy , \tag{1.8.2}$$

one has

$$g(x,k) = \mathrm{P}[Z(x) = k] = \exp(-L(x))\frac{L(x)^k}{k!}, \quad k = 0,1,2,\cdots . \tag{1.8.3}$$

Of interest, then, is a counting process $\{M(t) : t \geq 0\}$ characterized by

$$\begin{aligned} &\mathrm{P}\left[M(t+\Delta) - M(t) = 1 \mid M(t) = m, N(t) = 1, X(t) = x\right] \\ &= \lambda(x)\Delta + o(\Delta),\ (m,n,x) \in S,\ \Delta > 0 . \end{aligned} \tag{1.8.4}$$

Here $S = \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{R}_+$ where $\mathbb{Z}_+$ is the set of nonnegative integers and $\mathbb{R}_+$ is the set of nonnegative real numbers.

Since the counting process $M(t)$ is governed by the intensity function depending on the age process $X(t)$ of the renewal process $N(t)$, it is necessary to analyze the trivariate process $[M(t), N(t), X(t)]$. Let the multivariate transform of $[M(t), N(t), X(t)]$ be defined by

$$\varphi(u, v, w, s) = \int_0^\infty e^{-st}\mathrm{E}\left[u^{M(t)}v^{N(t)}e^{-wX(t)}\right] dt \ . \tag{1.8.5}$$

It has been shown in Sumita and Shanthikumar [23] that

$$\varphi(u, v, w, s) = \frac{\beta^*(u, w+s)}{1 - v\beta(u, s)} \ , \tag{1.8.6}$$

where we define, for $m \geq 0$ with $\bar{A}(t) = \int_0^\infty a(x)dx$,

$$b_m(t) = a(t)g(t, m) \ ; \quad b_m^*(t) = \bar{A}(t)g(t, m) \ , \tag{1.8.7}$$

$$\beta_m(s) = \int_0^\infty e^{-st}b_m(t)dt \ ; \quad \beta_m^*(s) = \int_0^\infty e^{-st}b_m^*(t)dt \ , \tag{1.8.8}$$

and

$$\beta(u, s) = \sum_{m=0}^\infty \beta_m(s)u^m \ ; \quad \beta^*(u, s) = \sum_{m=0}^\infty \beta_m^*(s)u^m \ . \tag{1.8.9}$$

The Laplace transform generating function of $M(t)$ defined by

$$\pi(u, s) = \int_0^\infty e^{-st}\mathrm{E}\left[u^{M(t)}\right] dt \tag{1.8.10}$$

is then given by $\pi(u, s) = \varphi(u, 1, 0, s)$, so that one has from (1.8.6)

$$\pi(u, s) = \frac{\beta^*(u, s)}{1 - \beta(u, s)} \ . \tag{1.8.11}$$

This class of counting processes denoted by ACPGRP may be extended where the underlying renewal process is replaced by an MMPP or an SMMPP. We define the former as a class of age-dependent counting processes governed by an MMPP, denoted by Markov modulated age-dependent non-homogeneous Poisson process (MMANHPP), and the latter as a class of age-dependent counting processes governed by an SMMPP, denoted by semi-Markov modulated age-dependent non-homogeneous Poisson process (SMMANHPP). The two extended classes are new and become special cases of the unified counting process proposed in this paper as we will see.

All of the counting processes discussed in this section are summarized in Figure 1.

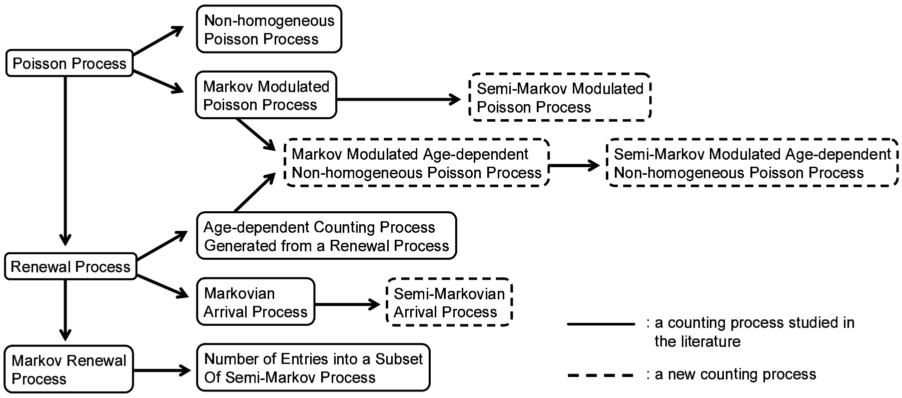

FIGURE 1: Various Counting Processes

2. A Unified Multivariate Counting Process $[\underline{M}(t),\ \underline{\underline{N}}(t)]$

In this section, we propose a stochastic process representing a unified multivariate counting process, which would contain all of the counting processes discussed in Section 1 as special cases. More specifically, we consider a system where items arrive according to an NHPP. This arrival stream is governed by a finite semi-Markov process on $\mathcal{J} = \{0, \cdots, J\}$ in that the intensity function of the NHPP depends on the current state of the semi-Markov process. That is, when the semi-Markov process is in state i with the current dwell time of x, items arrive according to a Poisson process with intensity $\lambda_i(x)$. If the semi-Markov process switches its state from i to j, the intensity function $\lambda_i(x)$ is interrupted, the intensity function at state j is reset to $\lambda_j(0)$, and the arrival process resumes. Of particular interest would be the multivariate counting processes $\underline{M}(t) = [M_0(t), \cdots, M_J(t)]^\top$ and $\underline{\underline{N}}(t) = [N_{ij}(t)]$ with $M_i(t)$ and $N_{ij}(t)$ counting the number of items that have arrived in state i by time t and the number of transitions of the semi-Markov process from state i to state j by time t respectively. The two counting processes $\underline{M}(t)$ and $\underline{\underline{N}}(t)$ enable one to introduce a variety of interesting performance indicators as we will see.

Formally, let $J(t) : t \geq 0$ be an semi-Markov process on $\mathcal{J} = \{0, \cdots, J\}$ governed by a matrix cumulative distribution function (c.d.f.)

$$\underline{\underline{A}}(x) = [A_{ij}(x)]\,, \tag{2.1}$$

which is assumed to be absolutely continuous with the matrix probability density function (p.d.f.)

$$\underline{\underline{a}}(x) = [a_{ij}(x)] = \frac{d}{dx}\underline{\underline{A}}(x). \tag{2.2}$$

It should be noted that, if we define $A_i(x)$ and $\bar{A}_i(x)$ by

$$A_i(x) = \sum_{j \in \mathcal{J}} A_{ij}(x); \quad \bar{A}_i(x) = 1 - A_i(x) \ , \tag{2.3}$$

then $A_i(x)$ is an ordinary c.d.f. and $\bar{A}_i(x)$ is the corresponding survival function. The hazard rate functions associated with the semi-Markov process are then defined as

$$\eta_{ij}(x) = \frac{a_{ij}(x)}{\bar{A}_i(x)}, \quad i, j \in \mathcal{J}. \tag{2.4}$$

For notational convenience, the transition epochs of the semi-Markov process are denoted by $\tau_n, n \geq 0$, with $\tau_0 = 0$. The age process $X(t)$ associated with the semi-Markov process is then defined as

$$X(t) = t - \max\{\tau_n : 0 \leq \tau_n \leq t\}. \tag{2.5}$$

At time t with $J(t) = i$ and $X(t) = x$, the intensity function of the NHPP is given by $\lambda_i(x)$. For the cumulative arrival intensity function $L_i(x)$ in state i, one has

$$L_i(x) = \int_0^x \lambda_i(y) dy \ . \tag{2.6}$$

The probability of observing k arrivals in state i within the current age of x can then be obtained as

$$g_i(x, k) = e^{-L_i(x)} \frac{L_i(x)^k}{k!}, \quad k = 0, 1, 2, \cdots, \quad i \in \mathcal{J}. \tag{2.7}$$

Of interest are the multivariate counting processes

$$\underline{M}(t) = [M_0(t), \cdots, M_J(t)]^\top , \tag{2.8}$$

where $M_i(t)$ represents the total number of items that have arrived by time t while the semi-Markov process stayed in state i, and

$$\underline{\underline{N}}(t) = [N_{ij}(t)] , \tag{2.9}$$

with $N_{ij}(t)$ denoting the number of transitions of the semi-Markov process from state i to state j by time t. It is obvious that $N_i(t) \stackrel{\text{def}}{=} \sum_{\ell \in \mathcal{J}} N_{\ell i}(t)$ denotes the number of entries into state i by time t. The initial state is not included in $N_{*i}(t)$ for any $i \in \mathcal{J}$. In other words, if $J(0) = i$, $N_{*i}(t)$ remains 0 until the first return of the semi-Markov process to state i. In the next section, we will analyze the dynamic behavior of $\left[\underline{M}(t), \underline{\underline{N}}(t)\right]$, yielding the Laplace transform generating function. In the subsequent section, all of the counting processes discussed in Section 1 would be expressed in terms of $\underline{M}(t)$ and $\underline{\underline{N}}(t)$, thereby providing a unified approach for studying various counting processes. The associated asymptotic behavior as $t \to \infty$ would also be discussed.

3. Dynamic Analysis of $[\underline{M}(t),\ \underline{\underline{N}}(t)]$

The purpose of this section is to examine the dynamic behavior of the multivariate stochastic process $\left[\underline{M}(t), \underline{\underline{N}}(t)\right]$ introduced in Section 2 by observing its probabilistic flow in the state space. Figure 2 below depicts a typical sample path of the multivariate process.

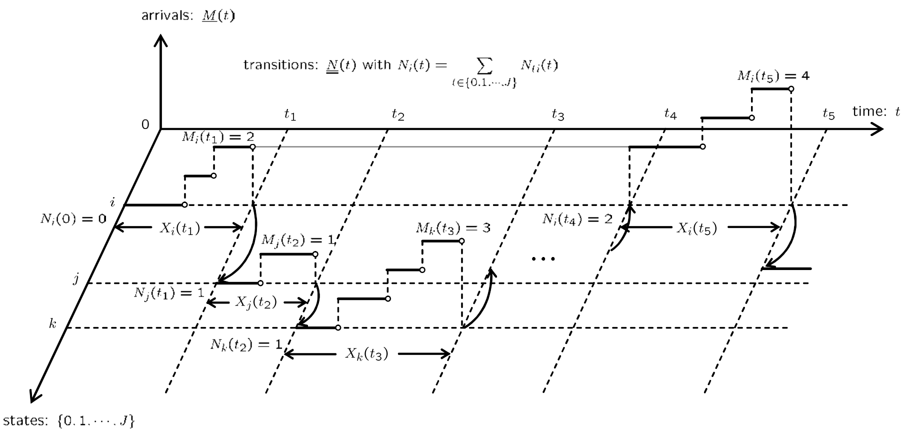

FIGURE 2: Typical Sample Path of $[\underline{M}(t),\ \underline{\underline{N}}(t)]$

Since $\left[\underline{M}(t), \underline{\underline{N}}(t)\right]$ is not Markov, we employ the method of supplementary variables. More specifically, the multivariate stochastic process $\left[\underline{M}(t), \underline{\underline{N}}(t), X(t), J(t)\right]$ is considered. This multivariate stochastic process is Markov and has the state space $S = \mathbb{Z}_+^{J+1} \times \mathbb{Z}_+^{(J+1)\times(J+1)} \times \mathbb{R}_+ \times \mathcal{J}$, where $\mathbb{Z}_+^{J+1}$ and $\mathbb{Z}_+^{(J+1)\times(J+1)}$ are the set of $(J+1)$ and $(J+1) \times (J+1)$ dimensional non-negative integer vectors, $\mathbb{R}_+$ is the set of non-negative real numbers and $\mathcal{J} = \{0, \cdots, J\}$. Let $F_{ij}(\underline{m}, \underline{\underline{n}}, x, t)$ be the joint probability distribution of $\left[\underline{M}(t), \underline{\underline{N}}(t), X(t), J(t)\right]$ defined by

$$F_{ij}(\underline{m}, \underline{\underline{n}}, x, t) = \\ \mathrm{P}\left[\underline{M}(t) = \underline{m}, \underline{\underline{N}}(t) = \underline{\underline{n}}, X(t) \le x, J(t) = j \mid \underline{M}(0) = \underline{0}, \underline{\underline{N}}(0) = \underline{\underline{0}}, J(0) = i\right] . \tag{3.1}$$

In order to assure the differenciability of $F_{ij}(\underline{m}, \underline{\underline{n}}, x, t)$ with respect to x, we assume that $X(0)$ has an absolutely continuous initial distribution function $D(x)$ with p.d.f. $d(x) = \frac{d}{dx}D(x)$. (If $X(0) = 0$ with probability 1, we consider a sequence of initial distribution functions $\{D_k(x)\}_{k=1}^{\infty}$ satisfying $D_k(x) \to U(x)$ as $k \to \infty$ where $U(x) = 1$ for $x \ge 0$ and $U(x) = 0$ otherwise. The desired results can be obtained through this limiting process.) One can then define

$$f_{ij}(\underline{m}, \underline{\underline{n}}, x, t) = \frac{\partial}{\partial x} F_{ij}(\underline{m}, \underline{\underline{n}}, x, t) . \tag{3.2}$$

By interpreting the probabilistic flow of the multivariate process $\left[\underline{M}(t), \underline{\underline{N}}(t), X(t), J(t)\right]$ in its state space, one can establish the following equations:

$$f_{ij}(\underline{m}, \underline{\underline{n}}, x, t) = \delta_{\{i=j\}}\delta_{\{\underline{m}=m_i\underline{1}_i\}}\delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}} d(x-t)\frac{\bar{A}_i(x)}{\bar{A}_i(x-t)} g_i(t, m_i) \\ + \left(1 - \delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}}\right) \sum_{k=0}^{m_j} f_{ij}(\underline{m} - k\underline{1}_j, \underline{\underline{n}}, 0+, t-x)\bar{A}_j(x) g_j(x, k) \; ; \tag{3.3}$$

$$f_{ij}(\underline{m}, \underline{\underline{n}}, 0+, t) = \left(1 - \delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}}\right) \sum_{\ell \in \mathcal{J}} \int_0^{\infty} f_{i\ell}(\underline{m}, \underline{\underline{n}} - \underline{\underline{1}}_{\ell j}, x, t)\eta_{\ell j}(x) dx \; ; \tag{3.4}$$

$$f_{ij}(\underline{m}, \underline{\underline{n}}, x, 0) = \delta_{\{i=j\}}\delta_{\{\underline{m}=\underline{0}\}}\delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}} d(x) , \tag{3.5}$$

where $\underline{1}_i$ is the column vector whose i-th element is equal to 1 with all other elements being 0 , $\underline{\underline{1}}_{ij} = \underline{1}_i\underline{1}_j^{\top}$ and $f_{ij}(\underline{m}, \underline{\underline{n}}, 0+, t) = 0$ for $\underline{\underline{N}} \le \underline{\underline{0}}$.

The first term of the right hand side of Equation (3.3) represents the case that $J(t)$ has not left the initial state $J(0) = i$ by time t $\left[\delta_{\{i=j\}} = 1 \text{ and } \delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}} = 1\right]$ and there have been m_i items arrived during time t $\left[\delta_{\{\underline{m}=m_i\underline{1}_i\}} = 1\right]$, provided that $J(0) = i$ and $X(0) = x - t$. The second term corresponds to the case that $J(t)$ made at least one transition from $J(0) = i$ by time t $\left[\delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}} = 0\right]$, the multivariate process $\left[\underline{M}(t), \underline{\underline{N}}(t), X(t), J(t)\right]$ just entered the state $\left[\underline{m} - k\underline{1}_j, \underline{\underline{n}}, 0+, j\right]$ at time $t - x$, $J(t)$

remained in the state j until time t with $X(t) = x$, and there have been k items arrived during the current age x, provided that $J(0) = i$ and $X(0) = x - t$. For the multivariate process at $[\underline{m}, \underline{\underline{n}}, 0+, j]$ at time t, it must be at $\left[\underline{m}, \underline{\underline{n}} - \underline{\underline{1}}_{\ell j}, x, \ell\right]$ followed by a transition from ℓ to j at time t which increases $N_{\ell j}(t)$ by one, explaining (3.4). Equations (3.5) merely describe the initial condition that $[\underline{M}(0), \underline{\underline{N}}(0), X(0), J(0)] = [\underline{0}, \underline{\underline{0}}, x, i]$.

In what follows, the dynamic behavior of the multivariate process $[\underline{M}(t), \underline{\underline{N}}(t), X(t), J(t)]$ would be captured by establishing the associated Laplace transform generating functions based on (3.3), (3.4) and (3.5). For notational convenience, the following matrix functions are employed.

(3.6) $\underline{\underline{b}}_k(t) = [b_{k:ij}(t)]$; $b_{k:ij}(t) = a_{ij}(t) g_i(t,k)$,

(3.7) $$\underline{\underline{b}}^*_{k:D}(t) = \left[\delta_{\{i=j\}} b^*_{k:i}(t)\right] = \begin{bmatrix} b^*_{k:0}(t) & & \\ & \ddots & \\ & & b^*_{k:J}(t) \end{bmatrix} \; ; \quad b^*_{k:i}(t) = \bar{A}_i(t) g_i(t,k) \; ,$$

(3.8) $$\underline{\underline{r}}^*_k(t) = \left[r^*_{k:ij}(t)\right] \; ; \quad r^*_{k:ij}(t) = g_i(t,k) \int_0^\infty d(x-t) \frac{a_{ij}(x)}{\bar{A}_i(x-t)} dx \; ,$$

(3.9) $$\underline{\underline{\beta}}_k(s) = [\beta_{k:ij}(s)] \; ; \quad \beta_{k:ij}(s) = \int_0^\infty e^{-st} b_{k:ij}(t) dt \; ,$$

(3.10) $$\underline{\underline{\beta}}(\underline{u}, s) = [\beta_{ij}(u_i, s)] \; ; \quad \beta_{ij}(u_i, s) = \sum_{k=0}^\infty \beta_{k:ij}(s) u_i^k \; ,$$

(3.11) $$\underline{\underline{\beta}}^*_{k:D}(s) = \begin{bmatrix} \beta^*_{k:0}(s) & & \\ & \ddots & \\ & & \beta^*_{k:J}(s) \end{bmatrix} \; ; \quad \beta^*_{k:i}(s) = \int_0^\infty e^{-st} b^*_{k:i}(t) dt \; ,$$

(3.12) $$\underline{\underline{\beta}}^*_D(\underline{u}, s) = \begin{bmatrix} \beta^*_0(u_0, s) & & \\ & \ddots & \\ & & \beta^*_J(u_J, s) \end{bmatrix} \; ; \quad \beta^*_i(u_i, s) = \sum_{k=0}^\infty \beta^*_{k:i}(s) u_i^k \; ,$$

(3.13) $$\underline{\underline{\rho}}^*_k(s) = \left[\rho^*_{k:ij}(s)\right] \; ; \quad \rho^*_{k:ij}(s) = \int_0^\infty e^{-st} r^*_{k:ij}(t) dt \; ,$$

(3.14) $$\underline{\underline{\rho}}^*(\underline{u}, s) = \left[\rho^*_{ij}(u_i, s)\right] \; ; \quad \rho^*_{ij}(u_i, s) = \sum_{k=0}^\infty \rho^*_{k:ij}(s) u_i^k \; ,$$

(3.15) $$\underline{\underline{\xi}}(\underline{m},\underline{\underline{n}},0+,s) = \left[\xi_{ij}(\underline{m},\underline{\underline{n}},0+,s)\right] \ ;$$
$$\xi_{ij}(\underline{m},\underline{\underline{n}},0+,s) = \int_0^\infty e^{-st} f_{ij}(\underline{m},\underline{\underline{n}},0+,t)dt \ ,$$

(3.16) $$\hat{\underline{\underline{\xi}}}(\underline{u},\underline{\underline{v}},0+,s) = \left[\hat{\xi}_{ij}(\underline{u},\underline{\underline{v}},0+,s)\right] \ ;$$
$$\hat{\xi}_{ij}(\underline{u},\underline{\underline{v}},0+,s) = \sum_{\underline{m}\in\mathbb{Z}_+^{J+1}} \ \sum_{\underline{\underline{n}}\in\mathbb{Z}_+^{(J+1)\times(J+1)}\setminus\{\underline{\underline{0}}\}} \xi_{ij}(\underline{m},\underline{\underline{n}},0+,s)\underline{u}^{\underline{m}}\underline{\underline{v}}^{\underline{\underline{n}}} \ ,$$

(3.17) $$\underline{\underline{\varphi}}(\underline{m},\underline{\underline{n}},w,s) = \left[\varphi_{ij}(\underline{m},\underline{\underline{n}},w,s)\right] \ ;$$
$$\varphi_{ij}(\underline{m},\underline{\underline{n}},w,s) = \int_0^\infty \int_0^\infty e^{-wx}e^{-st} f_{ij}(\underline{m},\underline{\underline{n}},x,t)dtdx \ ,$$

(3.18) $$\hat{\underline{\underline{\varphi}}}(\underline{u},\underline{\underline{v}},w,s) = \left[\hat{\varphi}_{ij}(\underline{u},\underline{\underline{v}},w,s)\right] \ ;$$
$$\hat{\varphi}_{ij}(\underline{u},\underline{\underline{v}},w,s) = \sum_{\underline{m}\in\mathbb{Z}_+^{J+1}} \ \sum_{\underline{\underline{n}}\in\mathbb{Z}_+^{(J+1)\times(J+1)}} \varphi_{ij}(\underline{m},\underline{\underline{n}},w,s)\underline{u}^{\underline{m}}\underline{\underline{v}}^{\underline{\underline{n}}} \ ,$$

where $\underline{u}^{\underline{m}}\underline{\underline{v}}^{\underline{\underline{n}}} = \prod_{i\in\mathcal{J}} u_i^{m_i} \prod_{(i,j)\in\mathcal{J}\times\mathcal{J}\setminus\{(0,0)\}} v_{ij}^{n_{ij}}$. We are now in a position to prove the main theorem of this section.

Theorem 1. *Let $X(0) = 0$. Then:*

(3.19) $$\hat{\underline{\underline{\xi}}}(\underline{u},\underline{\underline{v}},0+,s) = \tilde{\underline{\underline{\beta}}}(\underline{u},\underline{\underline{v}},s)\left\{\underline{\underline{I}} - \tilde{\underline{\underline{\beta}}}(\underline{u},\underline{\underline{v}},s)\right\}^{-1} \ ;$$

(3.20) $$\hat{\underline{\underline{\varphi}}}(\underline{u},\underline{\underline{v}},w,s) = \left\{\underline{\underline{I}} - \tilde{\underline{\underline{\beta}}}(\underline{u},\underline{\underline{v}},s)\right\}^{-1} \underline{\underline{\beta}}_D^*(\underline{u},w+s) \ ,$$

where $\tilde{\underline{\underline{\beta}}}(\underline{u},\underline{\underline{v}},s) = [v_{ij}\cdot\beta_{ij}(u_i,s)]$.

Proof. First, we assume that $X(0)$ has a p.d.f. $d(x)$. Substituting (3.3) and (3.5) into (3.4), one sees that

$$\begin{aligned}
&f_{ij}(\underline{m},\underline{\underline{n}},0+,t)\\
=\ &\left(1-\delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}}\right)\left\{\sum_{\ell\in\mathcal{J}}\int_0^\infty \delta_{\{i=\ell\}}\delta_{\{\underline{m}=m_i\underline{1}_i\}}\delta_{\{\underline{\underline{n}}=\underline{\underline{1}}_{\ell j}\}}d(x-t)\frac{\bar{A}_i(x)}{\bar{A}_i(x-t)}g_i(t,m_i)\eta_{\ell j}(x)dx\right.\\
&\left.+\sum_{\ell\in\mathcal{J}}\left(1-\delta_{\{\underline{\underline{n}}=\underline{\underline{1}}_{\ell j}\}}\right)\int_0^\infty\sum_{k=0}^{m_\ell} f_{i\ell}(\underline{m}-k\underline{1}_\ell,\underline{\underline{n}}-\underline{\underline{1}}_{\ell j},0+,t-x)\bar{A}_\ell(x)g_\ell(x,k)\eta_{\ell j}(x)dx\right\}\\
=\ &\left(1-\delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}}\right)\left\{\delta_{\{\underline{m}=m_i\underline{1}_i\}}\delta_{\{\underline{\underline{n}}=\underline{\underline{1}}_{ij}\}}g_i(t,m_i)\int_0^\infty d(x-t)\frac{a_{ij}(x)}{\bar{A}_i(x-t)}dx\right.\\
&\left.+\sum_{\ell\in\mathcal{J}}\left(1-\delta_{\{\underline{\underline{n}}=\underline{\underline{1}}_{\ell j}\}}\right)\sum_{k=0}^{m_\ell}\int_0^\infty f_{i\ell}(\underline{m}-k\underline{1}_\ell,\underline{\underline{n}}-\underline{\underline{1}}_{\ell j},0+,t-x)a_{\ell j}(x)g_\ell(x,k)dx\right\} \ .
\end{aligned}$$

Consequently, one sees from (3.6) and (3.8) that

$$
\begin{aligned}
(3.21)\qquad & f_{ij}(\underline{m},\underline{\underline{n}},0+,t)\\
= & \left(1-\delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}}\right)\Bigg\{\delta_{\{\underline{m}=m_i\underline{1}_i\}}\delta_{\{\underline{\underline{n}}=\underline{\underline{1}}_{ij}\}}r^*_{m_i:ij}(t)\\
& +\sum_{\ell\in\mathcal{J}}\left(1-\delta_{\{\underline{\underline{n}}=\underline{\underline{1}}_{\ell j}\}}\right)\sum_{k=0}^{m_\ell}\int_0^\infty f_{i\ell}(\underline{m}-k\underline{1}_\ell,\underline{\underline{n}}-\underline{\underline{1}}_{\ell j},0+,t-x)b_{k:\ell j}(x)dx\Bigg\}
\end{aligned}
$$

where $a_{ij}(x)=\bar{A}_i(x)\eta_{ij}(x)$ is employed from (2.4). By taking the Laplace transform of both sides of (3.21) with respect to t, it follows that

$$
\begin{aligned}
(3.22)\qquad & \xi_{ij}(\underline{m},\underline{\underline{n}},0+,s)\\
= & \left(1-\delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}}\right)\Bigg\{\delta_{\{\underline{m}=m_i\underline{1}_i\}}\delta_{\{\underline{\underline{n}}=\underline{\underline{1}}_{ij}\}}\rho^*_{m_i:ij}(s)\\
& +\sum_{\ell\in\mathcal{J}}\left(1-\delta_{\{\underline{\underline{n}}=\underline{\underline{1}}_{\ell j}\}}\right)\sum_{k=0}^{m_\ell}\xi_{i\ell}(\underline{m}-k\underline{1}_\ell,\underline{\underline{n}}-\underline{\underline{1}}_{\ell j},0+,s)\beta_{k:\ell j}(s)\Bigg\}\ .
\end{aligned}
$$

By taking the multivariate generating function of (3.22) with respect to $\underline{m}$ and $\underline{\underline{n}}$, it can be seen that

$$
\begin{aligned}
\hat{\xi}_{ij}(\underline{u},\underline{\underline{v}},0+,s) &= \sum_{\underline{m}\in\mathbb{Z}_+^{J+1}}\ \sum_{\underline{\underline{n}}\in\mathbb{Z}_+^{(J+1)\times(J+1)}\setminus\{\underline{\underline{0}}\}}\xi_{ij}(\underline{m},\underline{\underline{n}},0+,s)\underline{u}^{\underline{m}}\underline{\underline{v}}^{\underline{\underline{n}}}\\
&= \sum_{\underline{m}\in\mathbb{Z}_+^{J+1}}\ \sum_{\underline{\underline{n}}\in\mathbb{Z}_+^{(J+1)\times(J+1)}\setminus\{\underline{\underline{0}}\}}\Bigg\{\delta_{\{\underline{m}=m_i\underline{1}_i\}}\delta_{\{\underline{\underline{n}}=\underline{\underline{1}}_{ij}\}}\rho^*_{m_i:ij}(s)\underline{u}^{\underline{m}}\underline{\underline{v}}^{\underline{\underline{n}}}\\
&\quad +\sum_{\ell\in\mathcal{J}}\left(1-\delta_{\{\underline{\underline{n}}=\underline{\underline{1}}_{\ell j}\}}\right)\sum_{k=0}^{m_\ell}\xi_{i\ell}(\underline{m}-k\underline{1}_\ell,\underline{\underline{n}}-\underline{\underline{1}}_{\ell j},0+,s)\beta_{k:\ell j}(s)\underline{u}^{\underline{m}}\underline{\underline{v}}^{\underline{\underline{n}}}\Bigg\}\ .
\end{aligned}
$$

It then follows from (3.10), (3.14), (3.16) and the discrete convolution property that

$$
\begin{aligned}
& \hat{\xi}_{ij}(\underline{u},\underline{\underline{v}},0+,s)\\
= & \sum_{m_i=0}^{\infty}u_i^{m_i}v_{ij}\rho^*_{m_i:ij}(s)\\
& +\sum_{\ell\in\mathcal{J}}\left(v_{\ell j}\sum_{\underline{m}\in\mathbb{Z}_+^{J+1}}\sum_{k=0}^{m_\ell}\sum_{\underline{\underline{n}}\in\mathbb{Z}_+^{(J+1)\times(J+1)}\setminus\{\underline{\underline{0}}\}}\xi_{i\ell}(\underline{m}-k\underline{1}_\ell,\underline{\underline{n}},0+,s)\beta_{k:\ell j}(s)\underline{u}^{\underline{m}}\underline{\underline{v}}^{\underline{\underline{n}}}\right)\\
= & v_{ij}\rho^*_{ij}(u_i,s)+\sum_{\ell\in\mathcal{J}}v_{\ell j}\hat{\xi}_{i\ell}(\underline{u},\underline{\underline{v}},0+,s)\beta_{\ell j}(u_\ell,s)\ .
\end{aligned}
$$

The last expression can be rewritten in matrix form and one has

$$
\underline{\underline{\hat{\xi}}}(\underline{u},\underline{\underline{v}},0+,s)=\underline{\underline{\tilde{\rho}}}^*(\underline{u},\underline{\underline{v}},s)+\underline{\underline{\hat{\xi}}}(\underline{u},\underline{\underline{v}},0+,s)\underline{\underline{\tilde{\beta}}}(\underline{u},\underline{\underline{v}},s)\ ,
$$

which can be solved for $\hat{\underline{\underline{\xi}}}(\underline{u}, \underline{\underline{v}}, 0+, s)$ as

$$\hat{\underline{\underline{\xi}}}(\underline{u}, \underline{\underline{v}}, 0+, s) = \tilde{\underline{\underline{\rho}}}^*(\underline{u}, \underline{\underline{v}}, s)\left\{\underline{\underline{I}} - \tilde{\underline{\underline{\beta}}}(\underline{u}, \underline{\underline{v}}, s)\right\}^{-1} , \tag{3.23}$$

where $\tilde{\underline{\underline{\rho}}}^*(\underline{u}, \underline{\underline{v}}, s) = \left[v_{ij} \cdot \rho^*_{ij}(u_i, s)\right]$.

Next, we introduce the following double Laplace transform

$$\varepsilon_{k:i}(w, s) = \int_0^\infty \int_0^\infty e^{-wx} e^{-st} d(x-t) \frac{\bar{A}_i(x)}{\bar{A}_i(x-t)} g_i(t, k) dt dx , \tag{3.24}$$

and the associated diagonal matrix

$$\underline{\underline{\varepsilon}}_D(\underline{u}, w, s) = \begin{bmatrix} \varepsilon^*_0(u_0, w, s) & & \\ & \ddots & \\ & & \varepsilon^*_J(u_J, w, s) \end{bmatrix} ; \tag{3.25}$$

$$\varepsilon_i(u_i, w, s) = \sum_{k=0}^\infty \varepsilon_{k:i}(w, s) u_i^k .$$

By taking the double Laplace transform of (3.3), one sees from (3.7) and (3.24) that

$$\begin{aligned} \varphi_{ij}(\underline{m}, \underline{\underline{n}}, w, s) &= \delta_{\{i=j\}} \delta_{\{\underline{m}=m_i \underline{1}_i\}} \delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}} \varepsilon_{m_i:i}(w, s) \\ &+ \left(1 - \delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}}\right) \sum_{k=0}^{m_j} \xi_{ij}(\underline{m} - k\underline{1}_j, \underline{\underline{n}}, 0+, s) \beta^*_{k:j}(w+s) . \end{aligned} \tag{3.26}$$

By taking the double generating function, this then leads to

$$\begin{aligned} &\hat{\varphi}_{ij}(\underline{u}, \underline{\underline{v}}, w, s) \\ = &\sum_{\underline{m} \in \mathbb{Z}_+^{J+1}} \sum_{\underline{\underline{n}} \in \mathbb{Z}_+^{(J+1)\times(J+1)}} \varphi_{ij}(\underline{m}, \underline{\underline{n}}, w, s) \underline{u}^{\underline{m}} \underline{\underline{v}}^{\underline{\underline{n}}} \\ = &\delta_{\{i=j\}} \sum_{m_i=0}^\infty u_i^{m_i} \varepsilon_{m_i:i}(w, s) \\ &+ \sum_{\underline{m} \in \mathbb{Z}_+^{J+1}} \sum_{\underline{\underline{n}} \in \mathbb{Z}_+^{(J+1)\times(J+1)} \setminus \{\underline{\underline{0}}\}} \left(\sum_{k=0}^{m_j} \xi_{ij}(\underline{m} - k\underline{1}_j, \underline{\underline{n}}, 0+, s) \beta^*_{k:j}(w+s) \underline{u}^{\underline{m}} \underline{\underline{v}}^{\underline{\underline{n}}} \right) \\ = &\delta_{\{i=j\}} \varepsilon_i(u_i, w, s) + \hat{\xi}_{ij}(\underline{u}, \underline{\underline{v}}, 0+, s) \beta^*_j(u_j, w+s) . \end{aligned}$$

By rewriting the last expression in matrix form, it can be seen that

$$\hat{\underline{\underline{\varphi}}}(\underline{u}, \underline{\underline{v}}, w, s) = \underline{\underline{\varepsilon}}_D(\underline{u}, w, s) + \hat{\underline{\underline{\xi}}}(\underline{u}, \underline{\underline{v}}, 0+, s) \underline{\underline{\beta}}^*_D(\underline{u}, w+s) . \tag{3.27}$$

We now consider the limiting process $D(x) \to U(x)$. For the p.d.f., this limiting process becomes $d(x) \to \delta(x)$ where $\delta(x)$ is the Delta function defined as a unit function for convolution, i.e. $f(x) = \int f(x-\tau)\delta(\tau)d\tau$ for any function f. Accordingly, it can be seen from (3.8) that $r^*_{k:ij}(t) \to b_{k:ij}(t)$. This in turn implies from (3.24) that $\varepsilon_{k:i}(w,s) \to \beta^*_{k:i}(w+s)$. Consequently, it follows in matrix form that $\underline{\underline{\tilde{\rho}}}^*(\underline{u},\underline{v},s) \to \underline{\underline{\tilde{\beta}}}(\underline{u},\underline{v},s)$ and $\underline{\underline{\varepsilon}}_D(\underline{u},w,s) \to \underline{\underline{\beta}}^*_D(\underline{u},w+s)$. From (3.23), it can be readily seen that $\underline{\underline{\hat{\xi}}}(\underline{u},\underline{v},0+,s) \to \underline{\underline{\tilde{\beta}}}(\underline{u},\underline{v},s)\left\{\underline{\underline{I}} - \underline{\underline{\tilde{\beta}}}(\underline{u},\underline{v},s)\right\}^{-1}$, proving (3.19). One also sees from (3.24) that

$$\begin{aligned}
\underline{\underline{\hat{\varphi}}}(\underline{u},\underline{v},w,s) &\to \underline{\underline{\beta}}^*_D(\underline{u},w+s) + \underline{\underline{\hat{\xi}}}(\underline{u},\underline{v},0+,s)\underline{\underline{\beta}}^*_D(\underline{u},w+s) \\
&= \left[\underline{\underline{I}} + \underline{\underline{\tilde{\beta}}}(\underline{u},\underline{v},s)\left\{\underline{\underline{I}} - \underline{\underline{\tilde{\beta}}}(\underline{u},\underline{v},s)\right\}^{-1}\right]\underline{\underline{\beta}}^*_D(\underline{u},w+s) \\
&= \left[\underline{\underline{I}} - \underline{\underline{\tilde{\beta}}}(\underline{u},\underline{v},s) + \underline{\underline{\tilde{\beta}}}(\underline{u},\underline{v},s)\right]\left\{\underline{\underline{I}} - \underline{\underline{\tilde{\beta}}}(\underline{u},\underline{v},s)\right\}^{-1}\underline{\underline{\beta}}^*_D(\underline{u},w+s) \\
&= \left\{\underline{\underline{I}} - \underline{\underline{\tilde{\beta}}}(\underline{u},\underline{v},s)\right\}^{-1}\underline{\underline{\beta}}^*_D(\underline{u},w+s)\ ,
\end{aligned}$$

which proves (3.20), completing the proof.

In the next section, it will be shown that all the transform results obtained in Section 1 can be derived from Theorem 1.

4. Derivation of the Special Cases from the Unified Counting Process

We are now in a position to demonstrate the fact that the proposed multivariate counting process introduced in Section 2 and analysed in Section 3 indeed unifies the existing counting processes discussed in Section 1. We will do so by deriving the transform results of Section 1 from Theorem 1.

4.1. Derivation of Poisson Process

Let $N(t)$ be a Poisson process with intensity λ as discussed in Section 1.1. From (1.1.5), one sees that $\hat{\pi}(v,s) = \int_0^\infty e^{-st}\mathrm{E}[v^{N(t)}]dt = \int_0^\infty e^{-st}\pi(v,t)dt$ is given by

$$\hat{\pi}(v,s) = \frac{1}{s+\lambda(1-v)}\ . \tag{4.1.1}$$

For the unified counting process, we consider a single state Markov chain in continuous time where only the number of self transitions in $(0,t]$ is counted. More specifically, let $\mathcal{J} = \{0\}$, $\underline{\underline{N}}(t) = [N_{00}(t)]$, $u_0 = 1$, $v_{00} = v$, $w = 0$, $\lambda_0(t) = 0$ and $a_{00}(x) = \lambda e^{-\lambda x}$. We note from (2.7) that $\lambda_0(t) = 0$ implies $g_0(x,k) = \delta_{\{k=0\}}$ so that

$b_{k:00}(t) = \delta_{\{k=0\}} a_{00}(t)$ from (3.6). This then implies $\beta_{00}(1,s) = \lambda/(s+\lambda)$. Similarly, since $\bar{A}_0(x) = e^{-\lambda x}$, one has $\beta_0^*(1,s) = 1/(s+\lambda)$. It then follows from Theorem 1 that

$$\hat{\varphi}_0(1, v, 0+, s) = \frac{1}{1 - v\beta_{00}(1,s)} \beta_0^*(1,s) = \frac{1}{s + \lambda(1-v)} \ . \tag{4.1.2}$$

Hence, from (4.1.1) and (4.1.2), one concludes that $\hat{\pi}(v,s) = \hat{\varphi}_0(1, v, 0+, s)$.

4.2. Derivation of NHPP

Let $M(t)$ be an NHPP of Section 1.2 characterized by a time dependent intensity function $\lambda(t)$. It can be seen from (1.2.3) that $\hat{\pi}(u,s) = \int_0^\infty e^{-st} \mathrm{E}[u^{M(t)}]dt = \int_0^\infty e^{-st}\pi(u,t)dt$ is given by

$$\hat{\pi}(u,s) = \int_0^\infty e^{-st} e^{-L(t)(1-u)} dt \ . \tag{4.2.1}$$

In order to see that $M(t)$ can be viewed as a special case of the proposed multivariate counting process, we first consider a single state semi-Markov process where the dwell time in the state is deterministic given by T. The marginal counting process $\underline{M}(t) = [M_0(t)]$ then converges in distribution to $M(t)$ as $T \to \infty$ as we show next.

Let $\mathcal{J} = \{0\}$, $u_0 = u$, $v_{00} = 1$, $w = 0$, and $\lambda_0(x) = \lambda(x)$. We define the delta function $\delta(t)$ as the unit function for convolution operations, i.e., for any integrable function f on $(0, \infty)$, one has $f(t) = \int_0^\infty \delta(t-x) f(x) dx$. It then follows that $a_{00}(x) = \delta(x-t)$ and therefore $b_{k:00}(t) = \delta(t-T) g_0(t,k)$ from (3.6). This in turn implies that

$$\begin{aligned} \beta_{00}(u,s) &= \int_0^\infty e^{-st}\delta(t-T)e^{-L_0(t)(1-u)}dt \\ &= e^{-\{sT + L_0(T)(1-u)\}} \ . \end{aligned} \tag{4.2.2}$$

Let $U(t)$ be the step function defined by $U(t) = 0$ if $t < 0$ and $U(t) = 1$ otherwise. Since $\bar{A}_{00}(t) = 1 - U(t-T) = \delta_{\{0 \le t < T\}}$, one sees that

$$\begin{aligned} \beta_0^*(u,s) &= \int_0^\infty e^{-st}\delta_{\{0\le t<T\}} e^{-L_0(t)(1-u)}dt \\ &= \int_0^T e^{-st} e^{-L_0(t)(1-u)} dt \ . \end{aligned} \tag{4.2.3}$$

Theorem 1 together with (4.2.2) and (4.2.3) then leads to

$$\hat{\varphi}_{00}(u, 1, 0+, s) = \frac{1}{1 - \beta_{00}(u,s)} \beta_0^*(u,s) = \frac{\int_0^T e^{-st} e^{-L_0(t)(1-u)}dt}{1 - e^{-\{sT + L_0(T)(1-u)\}}} \ . \tag{4.2.4}$$

Now it can be readily seen that $\hat{\varphi}_{00}(u, 1, 0+, s)$ in (4.2.4) converges to $\hat{\pi}(u,s)$ in (4.2.1) as $T \to \infty$, proving the claim.

4.3. Derivation of MMPP

Let $M(t)$ be an MMPP of Section 1.3 characterized by $(\underline{\underline{\nu}}, \underline{\underline{\lambda}}_D)$. We show that the Laplace transform generating function $\hat{\pi}(u,s) = \int_0^\infty e^{-st}\mathrm{E}[u^{M(t)}]dt$ given in (1.3.7) can be derived as a special case of Theorem 1.

For the unified multivariate counting process, let $\mathcal{J} = \{0, \cdots, J\}$, $M(t) = \sum_{i\in\mathcal{J}} M_i(t)$, $\underline{u} = u\underline{1}$, $\underline{\underline{v}} = \underline{\underline{1}}$, $w = 0$, $\lambda_i(t) = \lambda_i$, and $\underline{\underline{a}}(x) = [e^{-\nu_i x}\nu_{ij}] = e^{-\underline{\underline{\nu}}_D x}\underline{\underline{\nu}}$. From (2.3) one sees that $\underline{\underline{\bar{A}}}_D(x) = \underline{\underline{I}} - \underline{\underline{A}}_D = \left[\delta_{\{i=j\}}\left\{1 - \sum_{j\in\mathcal{J}} A_{ij}(x)\right\}\right] = \left[\delta_{\{i=j\}}e^{-\nu_i x}\right] = e^{-\underline{\underline{\nu}}_D x}$. Therefore, one sees from (3.6), (3.9) and (3.10) that

$$\text{(4.3.1)} \qquad \begin{aligned} \underline{\underline{\beta}}(u\underline{1}, s) &= \sum_{k=0}^{\infty}\int_0^\infty e^{-st}\underline{\underline{g}}_D(t,k)\underline{\underline{a}}(t)dt\ u^k\underline{1} \\ &= \left\{s\underline{\underline{I}} + \underline{\underline{\nu}}_D + (1-u)\underline{\underline{\lambda}}_D\right\}^{-1}\underline{\underline{\nu}} \end{aligned}$$

and similarly one has, from (3.7), (3.11) and (3.12),

$$\text{(4.3.2)} \qquad \begin{aligned} \underline{\underline{\beta}}^*_D(u\underline{1}, s) &= \sum_{k=0}^{\infty}\int_0^\infty e^{-st}\underline{\underline{\bar{A}}}_D(t)\underline{\underline{g}}_D(t,k)dt\ u^k\underline{1} \\ &= \left\{s\underline{\underline{I}} + \underline{\underline{\nu}}_D + (1-u)\underline{\underline{\lambda}}_D\right\}^{-1}. \end{aligned}$$

It then follows from Theorem 1, (4.3.1) and (4.3.2) that

$$\begin{aligned} \underline{\underline{\hat{\varphi}}}(u\underline{1}, \underline{\underline{1}}, 0+, s) &= \left\{\underline{\underline{I}} - \underline{\underline{\tilde{\beta}}}(u\underline{1}, \underline{\underline{1}}, s)\right\}^{-1}\underline{\underline{\beta}}^*_D(u\underline{1}, s) \\ &= \left\{\underline{\underline{I}} - \underline{\underline{\beta}}(u\underline{1}, s)\right\}^{-1}\underline{\underline{\beta}}^*_D(u\underline{1}, s) \\ &= \left\{s\underline{\underline{I}} + \underline{\underline{\nu}}_D + (1-u)\underline{\underline{\lambda}}_D - \underline{\underline{\nu}}\right\}^{-1} \\ &= \left\{s\underline{\underline{I}} - \underline{\underline{Q}} + (1-u)\underline{\underline{\lambda}}_D\right\}^{-1}, \end{aligned}$$

which coincides with (1.3.7) as expected.

4.4. Derivation of Renewal Process

In order to demonstrate that a renewal process is a special case of the unified multivariate counting process, we follow the line of the arguments for the case of Poisson processes. Namely, let $\mathcal{J} = \{0\}$, $N(t) = N_0(t)$, $u_0 = 1$, $v_{00} = v$, $w = 0$,

$a_0(x) = a(x)$ and $\bar{A}_0(x) = 1 - \int_0^x a(y)dy$. From Theorem 1, one has

$$\begin{aligned}\varphi_0(1, v, 0+, s) &= \frac{1}{1 - v\beta(1, s)}\beta^*(1, s) \\ &= \frac{1}{1 - v\int_0^\infty e^{-st}a(t)dt} \times \int_0^\infty e^{-st}\bar{A}(t)dt \\ &= \frac{1}{1 - v\alpha(s)} \cdot \frac{1 - \alpha(s)}{s} \ ,\end{aligned}$$

which agrees with (1.4.2).

4.5. Derivation of MRP

Let $\underline{\tilde{N}}_\ell(t) = [\tilde{N}_{\ell 0}(t), \cdots, \tilde{N}_{\ell J}(t)]$ be an MRP discussed in Section 1.5. We recall that $\tilde{N}_{\ell r}(t)$ describes the number of entries of the semi-Markov process into state r in $(0, t]$ given that $J(0) = \ell$. For the unified multivariate counting process, $N_{ij}(t)$ counts the number of transitions from state i to state j in $(0, t]$. Hence, one has $\tilde{N}_{\ell r}(t) = \sum_{i \in \mathcal{J}} N_{ir}(t)$ provided that $J(0) = \ell$. Accordingly, we set $\underline{\underline{v}} = [\tilde{v}_0\underline{1}, \cdots, \tilde{v}_r\underline{1}, \cdots, \tilde{v}_J\underline{1}]$, i.e. $v_{ir} = \tilde{v}_r$ for all $i \in \mathcal{J}$. With $w = 0+$, $\underline{u} = \underline{1}$, $\lambda_\ell(t) = 0$ for all $\ell \in \mathcal{J}$, one has $\beta_{\ell r}(1, s) = \alpha_{\ell r}(s)$ and $\beta^*_\ell = \{1 - \alpha_\ell(s)\}/s$ from (3.6), (3.7) and (3.9) through (3.12), where $\alpha_\ell(s) = \sum_{r \in \mathcal{J}} \alpha_{\ell r}(s)$. Let $\underline{\underline{\tilde{v}}}_D = [\delta_{\{\ell = r\}}\tilde{v}_r]$. It then follows from Theorem 1 that

$$\begin{aligned}\underline{\underline{\hat{\varphi}}}(\underline{1}, \underline{\underline{v}}, 0+, s) &= \left\{\underline{\underline{I}} - [\tilde{v}_r\alpha_{\ell r}(s)]\right\}^{-1} \times \left[\delta_{\{\ell = r\}}\frac{1 - \alpha_\ell(s)}{s}\right] \\ &= \left\{\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\underline{\underline{\tilde{v}}}_D\right\}^{-1} \times \left[\delta_{\{\ell = r\}}\frac{1 - \alpha_\ell(s)}{s}\right] .\end{aligned} \tag{4.5.1}$$

It should be noted that, with $\underline{\tilde{v}} = [\tilde{v}_0, \cdots, \tilde{v}_J]^\top$ and $\underline{\tilde{v}}^{\underline{\tilde{N}}(t)} = \prod_{r \in \mathcal{J}} \tilde{v}_r^{\tilde{N}_{\ell r}(t)}$, the $\ell - r$ element of $\underline{\underline{\hat{\varphi}}}(\underline{1}, \underline{\underline{v}}, 0+, s)$ in (4.5.1) can be written as

$$\hat{\varphi}_{\ell r}(\underline{1}, \underline{\underline{v}}, 0+, s) = \mathcal{L}\left\{\mathrm{E}\left[\underline{\tilde{v}}^{\underline{\tilde{N}}(t)}, J(t) = r \mid J(0) = \ell\right]\right\} . \tag{4.5.2}$$

We now focus on $\tilde{N}_{\ell r}(t)$ for $\ell = 0, 1, \cdots, J$. In doing so, let $\underline{\underline{\tilde{v}}}_{D:j} \stackrel{\text{def}}{=} [\underline{1}_0, \cdots, \tilde{v}_j\underline{1}_j, \cdots, \underline{1}_J]$ and define

$$\underline{\psi}(\tilde{v}_r, s) \stackrel{\text{def}}{=} \begin{bmatrix}\mathcal{L}\left\{\mathrm{E}\left[\tilde{v}_r^{\tilde{N}_r(t)} \mid J(0) = 0\right]\right\} \\ \vdots \\ \mathcal{L}\left\{\mathrm{E}\left[\tilde{v}_r^{\tilde{N}_r(t)} \mid J(0) = J\right]\right\}\end{bmatrix} . \tag{4.5.3}$$

It then follows from (4.5.1) through (4.5.3) that

$$\begin{aligned}\underline{\psi}(\tilde{v}_r, s) &= \underline{\underline{\hat{\varphi}}}(\underline{1}, \underline{\underline{\tilde{v}}}_{D:r}, 0+, s)\underline{1} \\ &= \left\{\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\underline{\underline{\tilde{v}}}_{D:r}\right\}^{-1} \times \left[\delta_{\{\ell=r\}}\frac{1-\alpha_\ell(s)}{s}\right] \times \underline{1} \ ,\end{aligned}$$

i.e., one has

$$\underline{\psi}(\tilde{v}_r, s) = \frac{1}{s}\left\{\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\underline{\underline{\tilde{v}}}_{D:r}\right\}^{-1} \times \left\{\underline{1} - \underline{\underline{\alpha}}(s)\underline{1}\right\} \ . \tag{4.5.4}$$

We recall that $H_{\ell r}(t) = \mathrm{E}[\tilde{N}_{\ell r}(t)]$, which can be obtained by differencing $\underline{\psi}(\tilde{v}_r, s)$ with respect to $\tilde{v}_r$ at $\tilde{v}_r = 1$. More formally, one has

$$\left.\frac{\partial}{\partial \tilde{v}_r}\underline{\psi}(\tilde{v}_r, s)\right|_{\tilde{v}_r=1} = \begin{bmatrix}\mathcal{L}\left\{\mathrm{E}\left[\tilde{N}_r(t) \mid J(0)=0\right]\right\} \\ \vdots \\ \mathcal{L}\left\{\mathrm{E}\left[\tilde{N}_r(t) \mid J(0)=J\right]\right\}\end{bmatrix} \ , \tag{4.5.5}$$

which is the r-th column of $\mathcal{L}\{\underline{\underline{H}}(t)\}$ given in (1.5.1). By noting that

$$\frac{d}{dx}\left\{\underline{\underline{I}} - \underline{\underline{f}}(x)\right\}^{-1} = \left\{\underline{\underline{I}} - \underline{\underline{f}}(x)\right\}^{-1}\left\{\frac{d}{dx}\underline{\underline{f}}(x)\right\}\left\{\underline{\underline{I}} - \underline{\underline{f}}(x)\right\}^{-1} \ ,$$

one sees from (4.5.4) that

$$\begin{aligned}&\left.\frac{\partial}{\partial \tilde{v}_j}\underline{\psi}(\tilde{v}_j, s)\right|_{\tilde{v}_j=1} \\ &= \frac{1}{s}\left\{\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\right\}^{-1}\begin{bmatrix} & \alpha_{0j}(s) & \\ \underline{\underline{0}} & \vdots & \underline{\underline{0}} \\ & \alpha_{Jj}(s) & \end{bmatrix}\left\{\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\right\}^{-1}\left\{\underline{1} - \underline{\underline{\alpha}}(s)\underline{1}\right\} \ .\end{aligned} \tag{4.5.6}$$

Since $\left\{\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\right\}^{-1} = \sum_{k=0}^{\infty}\underline{\underline{\alpha}}(s)^k$ and $\underline{\underline{\alpha}}^0(s) = \underline{\underline{I}}$, it can be seen that

$$\left\{\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\right\}^{-1}\left\{\underline{1} - \underline{\underline{\alpha}}(s)\underline{1}\right\} = \sum_{k=0}^{\infty}\underline{\underline{\alpha}}(s)^k\left\{\underline{1} - \underline{\underline{\alpha}}(s)\underline{1}\right\} = \underline{1} \ . \tag{4.5.7}$$

Substituting (4.5.7) into (4.5.6), one then concludes that

$$\left.\frac{\partial}{\partial \tilde{v}_j}\underline{\psi}(\tilde{v}_j, s)\right|_{\tilde{v}_j=1} = \frac{1}{s}\left\{\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\right\}^{-1}\begin{bmatrix}\alpha_{0j}(s) \\ \vdots \\ \alpha_{Jj}(s)\end{bmatrix} \ .$$

This in turn implies that

$$\left[\left.\frac{\partial}{\partial \tilde{v}_0}\underline{\psi}(\tilde{v}_0, s)\right|_{\tilde{v}_0=1}, \cdots, \left.\frac{\partial}{\partial \tilde{v}_J}\underline{\psi}(\tilde{v}_J, s)\right|_{\tilde{v}_J=1}\right] = \frac{1}{s}\left\{\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\right\}^{-1}\underline{\underline{\alpha}}(s) \ , \tag{4.5.8}$$

which agrees with $\mathcal{L}\{\underline{\underline{H}}(t)\}$ of (1.5.1), completing the derivation.

4.6. Derivation of NESMPS

As in Subsection 1.6, let the state space $\mathcal{J}$ of $J(t)$ be decomposed into a set of good states $G(\neq \phi)$ and a set of bad states $B(\neq \phi)$ satisfying $J = G \cup B$ and $G \cap B = \phi$. The counting process $N_{GB}(t)$ of Subsection 1.6 describes the number of entries of $J(t)$ into B by time t. The Laplace transform generating function of the joint probability of $N_{GB}(t)$, the age process $X(t)$ and $J(t)$ is given in (1.6.6).

In the context of the unified multivariate counting process $\left[\underline{M}(t), \underline{\underline{N}}(t)\right]$ discussed in Section 2, one expects to have

$$N_{GB}(t) = \sum_{i \in G} \sum_{j \in B} N_{ij}(t) \ . \tag{4.6.1}$$

In order to prove (4.6.1) formally, we set

$$\underline{u} = \underline{1} \ ; \ \text{and} \quad \underline{\underline{v}} = \begin{bmatrix} \underline{\underline{1}}_{BB} & \underline{\underline{1}}_{BG} \\ v\underline{\underline{1}}_{GB} & \underline{\underline{1}}_{GG} \end{bmatrix} \ . \tag{4.6.2}$$

From (2.7), (3.6), (3.9) and (3.10), one has $\beta_{ij}(1, s) = \alpha_{ij}(s)$ so that

$$\underline{\underline{\tilde{\beta}}}(\underline{1}, \underline{\underline{v}}, s) = \begin{bmatrix} \underline{\underline{\alpha}}_{BB}(s) & \underline{\underline{\alpha}}_{BG}(s) \\ v\underline{\underline{\alpha}}_{GB}(s) & \underline{\underline{\alpha}}_{GG}(s) \end{bmatrix} \ , \tag{4.6.3}$$

where $\underline{\underline{\tilde{\beta}}}(\underline{u}, \underline{\underline{v}}, s)$ is as given in Theorem 1. Similarly, it can be seen from (2.7), (3.7), (3.11) and (3.12) that $\beta_i^*(1, w+s) = \{1 - \alpha_i(w+s)\}/(w+s)$ and hence

$$\underline{\underline{\beta}}^*_D(\underline{1}, w+s) = \frac{1}{w+s} \left\{ \underline{\underline{I}} - \underline{\underline{\alpha}}_D(w+s) \right\} \ . \tag{4.6.4}$$

Substituting (4.6.3) and (4.6.4) into (3.20), it then follows that

$$\underline{\underline{\hat{\varphi}}}(\underline{1}, \underline{\underline{v}}, w, s) = \frac{1}{w+s} \left\{ \underline{\underline{I}} - \underline{\underline{\tilde{\beta}}}(\underline{1}, \underline{\underline{v}}, s) \right\}^{-1} \left\{ \underline{\underline{I}} - \underline{\underline{\alpha}}_D(w+s) \right\} \ . \tag{4.6.5}$$

By comparing (1.6.6) with (4.6.5), Equation (4.6.1) holds true if and only if

$$\underline{\underline{\gamma}}_0(s) \left\{ \underline{\underline{I}} - v\underline{\underline{\beta}}(s) \right\}^{-1} = \left\{ \underline{\underline{I}} - \underline{\underline{\tilde{\beta}}}(\underline{1}, \underline{\underline{v}}, s) \right\}^{-1} \ . \tag{4.6.6}$$

From (4.6.3), one sees that

$$\underline{\underline{I}} - \underline{\underline{\tilde{\beta}}}(\underline{1}, \underline{\underline{v}}, s) = \begin{bmatrix} \underline{\underline{\chi}}_B^{-1}(s) & -\underline{\underline{\alpha}}_{BG}(s) \\ -v\underline{\underline{\alpha}}_{GB}(s) & \underline{\underline{\chi}}_G^{-1}(s) \end{bmatrix} \ , \tag{4.6.7}$$

where $\underline{\underline{\chi}}_G(s)$ and $\underline{\underline{\chi}}_B(s)$ are as given in (1.6.8). We define the inverse matrix of (4.6.7) by

$$\left\{\underline{\underline{I}} - \underline{\underline{\tilde{\beta}}}(\underline{1}, \underline{v}, s)\right\}^{-1} = \begin{bmatrix} \underline{\underline{C}}_{BB}(v,s) & \underline{\underline{C}}_{BG}(v,s) \\ \underline{\underline{C}}_{GB}(v,s) & \underline{\underline{C}}_{GG}(v,s) \end{bmatrix} . \tag{4.6.8}$$

Since the multiplication of the two matrices in (4.6.7) and (4.6.8) yields the identity matrix, it follows that

$$\begin{cases} \underline{\underline{\chi}}_B^{-1}(s)\underline{\underline{C}}_{BB}(v,s) - \underline{\underline{\alpha}}_{BG}(s)\underline{\underline{C}}_{GB}(v,s) = \underline{\underline{I}}_{BB} \\ \underline{\underline{\chi}}_B^{-1}(s)\underline{\underline{C}}_{BG}(v,s) - \underline{\underline{\alpha}}_{BG}(s)\underline{\underline{C}}_{GG}(v,s) = \underline{\underline{I}}_{BG} \\ -v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{C}}_{BB}(v,s) + \underline{\underline{\chi}}_G^{-1}(s)\underline{\underline{C}}_{GB}(v,s) = \underline{\underline{I}}_{GB} \\ -v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{C}}_{BG}(v,s) + \underline{\underline{\chi}}_G^{-1}(s)\underline{\underline{C}}_{GG}(v,s) = \underline{\underline{I}}_{GG} \end{cases} .$$

Solving the above equations for $\underline{\underline{C}}_{**}(v,s)$, one has

$$\begin{aligned} \underline{\underline{C}}_{BB}(v,s) &= \underline{\underline{\chi}}_B(s) + v\underline{\underline{\chi}}_B(s)\underline{\underline{\alpha}}_{BG}(s)\underline{\underline{\chi}}_G(s) \\ &\times \left\{\underline{\underline{I}}_{GG} - v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_B(s)\underline{\underline{\alpha}}_{BG}(s)\underline{\underline{\chi}}_G(s)\right\}^{-1} \underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_B(s) , \end{aligned} \tag{4.6.9}$$

$$\begin{aligned} \underline{\underline{C}}_{BG}(v,s) &= \underline{\underline{\chi}}_B(s)\underline{\underline{\alpha}}_{BG}(s)\underline{\underline{\chi}}_G(s) \\ &\times \left\{\underline{\underline{I}}_{GG} - v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_B(s)\underline{\underline{\alpha}}_{BG}(s)\underline{\underline{\chi}}_G(s)\right\}^{-1} , \end{aligned} \tag{4.6.10}$$

$$\begin{aligned} \underline{\underline{C}}_{GB}(v,s) &= v\underline{\underline{\chi}}_G(s) \\ &\times \left\{\underline{\underline{I}}_{GG} - v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_B(s)\underline{\underline{\alpha}}_{BG}(s)\underline{\underline{\chi}}_G(s)\right\}^{-1} \underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_B(s) , \end{aligned} \tag{4.6.11}$$

and

$$\underline{\underline{C}}_{GG}(v,s) = \underline{\underline{\chi}}_G(s)\left\{\underline{\underline{I}}_{GG} - v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_B(s)\underline{\underline{\alpha}}_{BG}(s)\underline{\underline{\chi}}_G(s)\right\}^{-1} . \tag{4.6.12}$$

We next turn our attention to the left hand side of Equation (4.6.6). From (1.6.9), one sees that

$$\begin{aligned} &\underline{\underline{I}} - v\underline{\underline{\beta}}(s) \\ &= \begin{bmatrix} \underline{\underline{I}}_{BB} & \underline{\underline{0}}_{BG} \\ -v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_B(s) & \underline{\underline{I}}_{GG} - v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_B(s)\underline{\underline{\alpha}}_{BG}(s)\underline{\underline{\chi}}_G(s) \end{bmatrix} . \end{aligned} \tag{4.6.13}$$

As before, we define the inverse matrix of (4.6.13) as

$$\left\{\underline{\underline{I}} - v\underline{\underline{\beta}}(s)\right\}^{-1} = \begin{bmatrix} \underline{\underline{D}}_{BB}(v,s) & \underline{\underline{D}}_{BG}(v,s) \\ \underline{\underline{D}}_{GB}(v,s) & \underline{\underline{D}}_{GG}(v,s) \end{bmatrix} . \tag{4.6.14}$$

Multiplying the two matrices in (4.6.13) and (4.6.14) then yields

$$\begin{cases} \underline{\underline{D}}_{BB}(v,s) = \underline{\underline{I}}_{BB} \\ \underline{\underline{D}}_{BG}(v,s) = \underline{\underline{0}}_{BG} \\ -v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_{B}(s)\underline{\underline{D}}_{BB}(v,s) \\ \qquad + \left\{\underline{\underline{I}}_{GG} - v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_{B}(s)\underline{\underline{\alpha}}_{BG}(s)\underline{\underline{\chi}}_{G}(s)\right\}\underline{\underline{D}}_{GB}(v,s) = \underline{\underline{0}}_{GB} \\ -v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_{B}(s)\underline{\underline{D}}_{BG}(v,s) \\ \qquad + \left\{\underline{\underline{I}}_{GG} - v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_{B}(s)\underline{\underline{\alpha}}_{BG}(s)\underline{\underline{\chi}}_{G}(s)\right\}\underline{\underline{D}}_{GG}(v,s) = \underline{\underline{I}}_{GG} \end{cases},$$

which in turn can be solved for $\underline{\underline{D}}_{**}(v,s)$ as

$$\text{(4.6.15)} \quad \underline{\underline{D}}_{BB}(v,s) = \underline{\underline{I}}_{BB}\ ,$$

$$\text{(4.6.16)} \quad \underline{\underline{D}}_{BG}(v,s) = \underline{\underline{0}}_{BG}\ ,$$

$$\text{(4.6.17)} \quad \underline{\underline{D}}_{GB}(v,s) = v\left\{\underline{\underline{I}}_{GG} - v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_{B}(s)\underline{\underline{\alpha}}_{BG}(s)\underline{\underline{\chi}}_{G}(s)\right\}^{-1}\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_{B}(s)\ ,$$

and

$$\text{(4.6.18)} \qquad \underline{\underline{D}}_{GG}(v,s) = \left\{\underline{\underline{I}}_{GG} - v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_{B}(s)\underline{\underline{\alpha}}_{BG}(s)\underline{\underline{\chi}}_{G}(s)\right\}^{-1}\ .$$

Let the left hand side matrix of (4.6.6) be described as

$$\text{(4.6.19)} \qquad \underline{\underline{\gamma}}_{0}(s)\left\{\underline{\underline{I}} - v\underline{\underline{\beta}}(s)\right\}^{-1} = \begin{bmatrix} \underline{\underline{Z}}_{BB}(v,s) & \underline{\underline{Z}}_{BG}(v,s) \\ \underline{\underline{Z}}_{GB}(v,s) & \underline{\underline{Z}}_{GG}(v,s) \end{bmatrix}\ .$$

From (1.6.10) and (4.6.14) through (4.6.18), one sees that

$$\begin{aligned} \underline{\underline{Z}}_{BB}(v,s) &= \underline{\underline{I}}_{BB} + \underline{\underline{\alpha}}_{BB}(s)\underline{\underline{\chi}}_{B}(s) + v\underline{\underline{\chi}}_{B}(s)\underline{\underline{\alpha}}_{BG}(s)\underline{\underline{\chi}}_{G}(s) \\ &\quad \times \left\{\underline{\underline{I}}_{GG} - v\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_{B}(s)\underline{\underline{\alpha}}_{BG}(s)\underline{\underline{\chi}}_{G}(s)\right\}^{-1}\underline{\underline{\alpha}}_{GB}(s)\underline{\underline{\chi}}_{B}(s)\ . \end{aligned}$$

From (1.6.8), one easily sees that $\underline{\underline{\chi}}_{B}(s) = \underline{\underline{I}}_{BB} + \underline{\underline{\alpha}}_{BB}(s)\underline{\underline{\chi}}_{B}(s)$, and hence $\underline{\underline{Z}}_{BB}(v,s) = \underline{\underline{C}}_{BB}(v,s)$ from (4.6.9). The fact that $\underline{\underline{Z}}_{BG}(v,s) = \underline{\underline{C}}_{BG}(v,s)$ is straightforward from (4.6.10). With $\underline{\underline{\chi}}_{G}(s) = \underline{\underline{I}}_{GG} + \underline{\underline{\alpha}}_{GG}(s)\underline{\underline{\chi}}_{G}(s)$, one has $\underline{\underline{Z}}_{GB}(v,s) = \underline{\underline{C}}_{GB}(v,s)$ from (4.6.11) and $\underline{\underline{Z}}_{GG}(v,s) = \underline{\underline{C}}_{GG}(v,s)$ from (4.6.12), completing the proof for (4.6.6).

4.7. Derivation of MAP

We recall that an MAP is constructed from an absorbing Markov chain $J^*(t)$ in continuous time on $\mathcal{J} = G \cup B$, with B being a set of absorbing states, governed by

$\underline{\underline{\nu}}^*$ defined in (1.7.2). A replacement Markov chain $J(t)$ is then generated from $J^*(t)$, where $J(t)$ coincides with $J^*(t)$ within G starting from a state in G. As soon as $J^*(t)$ reaches state $r \in B$, it is instantaneously replaced at state $j \in G$ with probability $\tilde{p}_{rj}$ and the process continues.

In order to deduce an MAP from the unified multivariate counting process $[\underline{M}(t), \underline{\underline{N}}(t)]$ as a special case, we start from (3.20) in Theorem 1, where the underlying semi-Markov process is now the replacement Marcov chain $J(t)$ discussed above. This replacement Marcov chain is defined on G governed by $\underline{\underline{\nu}} = \underline{\underline{C}} + \underline{\underline{D}}$ with $\underline{\underline{C}} = [c_{ij}] = [\nu_{ij}]_{i,j\in G}$ and $\underline{\underline{D}} = [d_{ij}] = \left[\sum_{r\in B} \nu_{ir}\tilde{p}_{rj}\right]_{i,j\in G}$ as in (1.7.3). We note that $\beta_{ij}(1,s) = \alpha_{ij}(s)$ from (2.7), (3.6), (3.9) and (3.10), and $\beta_i^*(1,s) = \{1-\alpha_i(s)\}/s$ from (2.7), (3.7), (3.11) and (3.12). Hence, it follows that

$$\text{(4.7.1)} \qquad \hat{\underline{\underline{\varphi}}}(\underline{1}, \underline{\underline{v}}, 0+, s) \;=\; \left\{\underline{\underline{I}} - [v_{ij}\cdot\alpha_{ij}(s)]\right\}^{-1} \left[\delta_{\{i=j\}}\cdot\frac{1-\alpha_i(s)}{s}\right] .$$

Let $\underline{\underline{\nu}}_D = \underline{\underline{C}}_D + \underline{\underline{D}}_D$ as in (1.7.5) and (1.7.6). Since $J(t)$ is a Markov chain, the dwell time in state i is independent of the next state and is exponentially distributed with parameter $\nu_i = c_i + d_i$. Consequently, one has

$$\text{(4.7.2)} \qquad \alpha_i(s) = \frac{\nu_i}{s+\nu_i}; \quad \alpha_{ij}(s) = \frac{c_{ij}+d_{ij}}{\nu_i}\alpha_i(s) = \frac{c_{ij}+d_{ij}}{s+\nu_i} .$$

Substituting (4.7.2) into (4.7.1) and noting $\left[\delta_{\{i=j\}}\cdot\frac{1-\alpha_i(s)}{s}\right]^{-1} = s\underline{\underline{I}} + \underline{\underline{\nu}}_D$, it follows that

$$\text{(4.7.3)} \qquad \hat{\underline{\underline{\varphi}}}(\underline{1}, \underline{\underline{v}}, 0+, s) = \left\{s\underline{\underline{I}} - \underline{\underline{Q}} + \underline{\underline{C}} + \underline{\underline{D}} - [v_{ij}(c_{ij}+d_{ij})]\right\}^{-1} ,$$

where $\underline{\underline{Q}}$ in (1.7.4) is employed.

As it stands, the Laplace transform generating function of (4.7.3) describes the matrix counting process $\underline{\underline{N}}(t) = [N_{ij}(t)]$ where v_{ij} corresponds to $N_{ij}(t)$. For $N_{MAP}(t)$ of Subsection 1.7, it is only necessary to count the number of replacements in $(0,t]$. Given that state $j \in G$ is visited from the current state $i \in G$, this move is direct within G with probability $c_{ij}/(c_{ij}+d_{ij})$, and such a move involves replacement with probability $d_{ij}/(c_{ij}+d_{ij})$. Accordingly, we set

$$\text{(4.7.4)} \qquad v_{ij} = \frac{c_{ij}}{c_{ij}+d_{ij}} + \frac{d_{ij}}{c_{ij}+d_{ij}}v .$$

Substitution of (4.7.4) into 4.7.3) then leads to

$$\underline{\underline{\hat{\varphi}}}(\underline{1}, \underline{v}, 0+, s) = \left\{ s\underline{\underline{I}} - \underline{\underline{Q}} + (1-v)\underline{\underline{D}} \right\}^{-1} ,$$

which coincides with $\underline{\underline{\varphi}}(v, s)$ of (1.7.12), as expected.

4.8. Derivation of ACPGRP

The age-dependent counting process of Sumita and Shanthikumar [23] has been extended in this paper where the underlying renewal process is replaced by a semi-Markov process with state dependent non-homogeneous hazard functions. Accordingly, the original model can be treated as a special case of the unified multivariate counting process proposed in this paper by setting $\mathcal{J} = \{0\}$, $M(t) = M_0(t)$, $N(t) = N_{00}(t)$, $X(t) = X_0(t)$, $u_0 = u$, $v_{00} = v$. With this specification, from Theorem 1, one sees that

$$\hat{\varphi}_{00}(u, v, w, s) = \frac{\beta^*(u, w+s)}{1 - v\beta(u, s)} .$$

It then follows that

$$\pi(u, s) = \hat{\varphi}_{00}(u, 1, 0, s) = \frac{\beta^*(u, s)}{1 - \beta(u, s)} ,$$

which coincides with Equation (1.8.11).

5. Asymptotic Analysis

Let $\mathcal{A}, \mathcal{M}$ and $\mathcal{N}$ be arbitrary subsets of the state space $\mathcal{J}$ of the underlying semi-Markov process, and define

$$M_{\mathcal{A}}(t) = \sum_{i \in \mathcal{A}} M_i(t) \; ; \quad N_{\mathcal{M}\mathcal{N}}(t) = \sum_{i \in \mathcal{M}} \sum_{j \in \mathcal{N}} N_{ij}(t) \; , \tag{5.1}$$

where $M_{\mathcal{A}}(t)$ describes the total number of items arrived in $[0, t]$ according to the non-homogeneous Poisson processes within $\mathcal{A}$ and $N_{\mathcal{M}\mathcal{N}}(t)$ denotes the number of transitions from any state in $\mathcal{M}$ to any state in $\mathcal{N}$ occurred in $[0, t]$. Appropriate choice of $\mathcal{A}, \mathcal{M}$ and $\mathcal{N}$ would then enable one to analyze processes of interest in a variety of applications. In the variable bit rate coding scheme for video transmission explained in Section 1, for example, one may be interested in the packet arrival stream for a specified mode of the encoder represented by $M_{\mathcal{A}}(t)$. In reliability models with a structure as outlined in Section 1.8, the underlying semi-Markov process may describe the state of a production machine. Minimal repairs would take place whenever the system state is in $\mathcal{A}$ at the cost of c, while component replacements would be forced at the cost of d if the machine state entries $\mathcal{N} \subset \mathcal{J}$. The total maintenance cost $S(t)$ is then given by $S(t) = cM_{\mathcal{A}}(t) + dN_{\mathcal{J}\mathcal{N}}(t)$. A simplified version of this cost structure has been analyzed by Sumita and Shanthikumar [23] where the underlying semi-Markov process is merely a renewal process with $\mathcal{J} = \mathcal{A} = \mathcal{N} = \{1\}$. The purpose of this section is to study a more general cost structure specified by

$$S(t) = cM_{\mathcal{A}}(t) + dN_{\mathcal{M}\mathcal{N}}(t) \ , \tag{5.2}$$

with focus on the Laplace transform generating function and the related moment asymptotic behaviors of $M_{\mathcal{A}}(t), N_{\mathcal{M}\mathcal{N}}(t)$ and $S(t)$ based on Theorem 1.

For notational simplicity, we introduce the following vectors and matrices. Let $\mathcal{A}, \mathcal{M}$ and $\mathcal{N} \subset \mathcal{J}$ with their compliments defined respectively by $\mathcal{A}^C = \mathcal{J} \setminus \mathcal{A}$, $\mathcal{M}^C = \mathcal{J} \setminus \mathcal{M}$ and $\mathcal{N}^C = \mathcal{J} \setminus \mathcal{N}$. The cardinality of a set A is denoted by $|A|$.

$$\underline{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{J+1} \ ; \quad \underline{\underline{1}} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{(J+1)\times(J+1)} \ , \tag{5.3}$$

$$\underline{u}(\mathcal{A}) = \begin{bmatrix} u_i \end{bmatrix} \in \mathbb{R}^{J+1} \text{ with } u_i = \begin{cases} u & \text{if } i \in \mathcal{A} \\ 1 & \text{else} \end{cases} \ ; \tag{5.4}$$

$$\underline{\underline{v}}(\mathcal{M}, \mathcal{N}) = \begin{bmatrix} v_{ij} \end{bmatrix} \in \mathbb{R}^{(J+1)\times(J+1)} \text{ with } v_{ij} = \begin{cases} v & \text{if } i \in \mathcal{M}, \ j \in \mathcal{N} \\ 1 & \text{else} \end{cases} \ .$$

Submatrices of $\underline{\underline{A}} \in \mathbb{R}^{(J+1)\times(J+1)}$ are denoted by

$$\underline{\underline{A}}_{\mathcal{A}\bullet} = \begin{bmatrix} A_{ij} \end{bmatrix}_{i\in\mathcal{A}, j\in\mathcal{J}} \in \mathbb{R}^{|\mathcal{A}|\times(J+1)} \ ; \quad \underline{\underline{A}}_{\bullet\mathcal{A}} = \begin{bmatrix} A_{ij} \end{bmatrix}_{i\in\mathcal{J}, j\in\mathcal{A}} \in \mathbb{R}^{(J+1)\times|\mathcal{A}|} \ ; \tag{5.5}$$

$$\underline{\underline{A}}_{\mathcal{M}\mathcal{N}} = \begin{bmatrix} A_{ij} \end{bmatrix}_{i\in\mathcal{M}, j\in\mathcal{N}} \in \mathbb{R}^{|\mathcal{M}|\times|\mathcal{N}|} \ ,$$

so that one has

$$\underline{\underline{A}} = \begin{bmatrix} \underline{\underline{A}}_{\mathcal{M}\mathcal{N}} & \underline{\underline{A}}_{\mathcal{M}\mathcal{N}^C} \\ \underline{\underline{A}}_{\mathcal{M}^C\mathcal{N}} & \underline{\underline{A}}_{\mathcal{M}^C\mathcal{N}^C} \end{bmatrix} \ , \tag{5.6}$$

with understanding that the states are arranged appropriately.

Let $\underline{\underline{A}}_k = \int_0^\infty x^k \underline{\underline{a}}(x)dx,\ k = 0, 1, 2, \cdots$. Throughout the paper, we assume that $||\underline{\underline{A}}_k|| < \infty$ for $0 \le k \le 2$. In particular, one has $\underline{\underline{A}}_0 = \underline{\underline{A}}(\infty)$ which is stochastic. Let $\underline{e}^\top$ be the normalized left eigenvector of $\underline{\underline{A}}_0$ associated with eigenvalue 1 so that $\underline{e}^\top \underline{\underline{A}}_0 = \underline{e}^\top$ and $\underline{e}^\top \underline{1} = 1$. The Taylor expansion of the Laplace transform $\underline{\underline{\alpha}}(s)$ is then given by

$$\underline{\underline{\alpha}}(s) = \underline{\underline{A}}_0 - s\ \underline{\underline{A}}_1 + \frac{s^2}{2}\underline{\underline{A}}_2 + \underline{\underline{o}}(s^2)\ . \tag{5.7}$$

We recall from Theorem 1 that

$$\hat{\underline{\underline{\varphi}}}(\underline{u}, \underline{v}, w, s) = \left\{\underline{\underline{I}} - \tilde{\underline{\underline{\beta}}}(\underline{u}, \underline{v}, s)\right\}^{-1} \underline{\underline{\beta}}^*_D(\underline{u}, w + s)\ , \tag{5.8}$$

where

$$\tilde{\underline{\underline{\beta}}}(\underline{u}, \underline{v}, s) = [v_{ij} \cdot \beta_{ij}(u_i, s)]\ . \tag{5.9}$$

From (2.6), (2.7), (3.6), (3.9) and (3.10), one sees that

$$\beta_{ij}(u, s) = \int_0^\infty e^{-st} e^{-L_i(t)(1-u)} a_{ij}(t)dt\ . \tag{5.10}$$

Similarly, it follows from (2.6), (2.7), (3.7), (3.11) and (3.12) that

$$\beta^*_i(u, s) = \int_0^\infty e^{-st} e^{-L_i(t)(1-u)} \bar{A}_i(t)dt\ . \tag{5.11}$$

The r-th order partial derivatives of $\beta_{ij}(u, s)$ and $\beta^*_i(u, s)$ with respect to u at $u = 1$ are then given by

$$\zeta_{r:ij}(s) \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial u}\right)^r \beta_{ij}(u, s)\Big|_{u=1} = \int_0^\infty e^{-st} L^r_i(t) a_{ij}(t)dt\ , \quad r = 1, 2\ ; \tag{5.12}$$

$$\zeta^*_{r:i}(s) \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial u}\right)^r \beta^*_i(u, s)\Big|_{u=1} = \int_0^\infty e^{-st} L^r_i(t) \bar{A}_i(t)dt\ , \quad r = 1, 2\ . \tag{5.13}$$

In matrix form, Equations (5.12) and (5.13) can be written as

$$\underline{\underline{\zeta}}_r(s) = \int_0^\infty e^{-st} \underline{\underline{L}}^r_D(t) \underline{\underline{a}}(t)dt\ , \quad r = 1, 2\ ; \tag{5.14}$$

$$\underline{\underline{\zeta}}^*_{r:D}(s) = \int_0^\infty e^{-st} \underline{\underline{L}}^r_D(t) \underline{\underline{\bar{A}}}_D(t)dt\ , \quad r = 1, 2\ . \tag{5.15}$$

Let $\underline{\underline{\Phi}}_{r:k} = \int_0^\infty t^k \underline{\underline{L}}_D^r(t)\underline{\underline{a}}(t)dt$ and $\underline{\underline{\Phi}}^*_{r:D:k} = \int_0^\infty t^k \underline{\underline{L}}_D^r(t)\bar{\underline{\underline{A}}}_D(t)dt$, $r = 1,2$. The Taylor expansion of $\underline{\underline{\zeta}}_r(s)$ and $\underline{\underline{\zeta}}^*_{r:D}(s)$ are then given by

$$\underline{\underline{\zeta}}_r(s) = \underline{\underline{\Phi}}_{r:0} - s\,\underline{\underline{\Phi}}_{r:1} + \frac{s^2}{2}\underline{\underline{\Phi}}_{r:2} + \underline{\underline{o}}(s^2)\ , \quad r = 1,2\ ; \tag{5.16}$$

$$\underline{\underline{\zeta}}^*_{r:D}(s) = \underline{\underline{\Phi}}^*_{r:D:0} - s\,\underline{\underline{\Phi}}^*_{r:D:1} + \frac{s^2}{2}\underline{\underline{\Phi}}^*_{r:D:2} + \underline{\underline{o}}(s^2)\ , \quad r = 1,2\ . \tag{5.17}$$

In order to prove the main results of this section, the following theorem of Keilson [9] plays a key role.

Theorem 2. *(Keilson [9])*
As $s \to 0+$, one has

$$\left\{\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\right\}^{-1} = \frac{1}{s}\ \underline{\underline{H}}_1 + \underline{\underline{H}}_0 + \underline{\underline{o}}(1)\ , \tag{5.18}$$

where

$$\underline{\underline{H}}_1 = \frac{1}{m}\ \underline{1}\ \underline{e}^\top, \quad m = \underline{e}^\top \underline{\underline{A}}_1 \underline{1}\ ,$$
$$\underline{\underline{H}}_0 = \underline{\underline{H}}_1\left(-\underline{\underline{A}}_1 + \frac{1}{2}\ \underline{\underline{A}}_2\underline{\underline{H}}_1\right) + \left(\underline{\underline{Z}}_0 - \underline{\underline{H}}_1\underline{\underline{A}}_1\underline{\underline{Z}}_0\right)\left(\underline{\underline{A}}_0 - \underline{\underline{A}}_1\underline{\underline{H}}_1\right) + \underline{\underline{I}}\ ,$$
$$\underline{\underline{Z}}_0 = \left(\underline{\underline{I}} - \underline{\underline{A}}_0 + \underline{1}\cdot\underline{e}^\top\right)^{-1}\ .$$

Accordingly, one has

$$\underline{\underline{I}} + \sum_{k=1}^\infty \underline{\underline{a}}^{(k)}(t) = t\underline{\underline{H}}_1 + \underline{\underline{H}}_0 + \underline{\underline{o}}(1) \tag{5.19}$$

where $\underline{\underline{a}}^{(k+1)}(t) = \int_0^t \underline{\underline{a}}^{(k)}(t-x)\underline{\underline{a}}(x)dx$, $k \geq 1$, with $\underline{\underline{a}}^{(1)} = \underline{\underline{a}}(t)$.

We are now in a position to prove the first key theorem of this section.

Theorem 3. *Let $\underline{p}^\top(0)$ be an initial probability vector of the underlying semi-Markov process. As $t \to \infty$, one has*

$$\mathrm{E}\left[M_{\mathcal{A}}(t)\right] = \underline{p}^\top(0)\left\{t\ \underline{\underline{P}}_1 + \underline{\underline{P}}_0\right\}\underline{1} + o(1)\ , \tag{5.20}$$

$$\mathrm{E}\left[N_{\mathcal{M}\mathcal{N}}(t)\right] = \underline{p}^\top(0)\left\{t\ \underline{\underline{Q}}_1 + \underline{\underline{Q}}_0\right\}\underline{1} + o(1)\ , \tag{5.21}$$

where

$$\underline{\underline{P}}_1 = \underline{\underline{H}}_{1:\bullet\mathcal{A}}\underline{\underline{\Phi}}_{1:0:\mathcal{A}\bullet}\ , \quad \underline{\underline{P}}_0 = \underline{\underline{H}}_{0:\bullet\mathcal{A}}\underline{\underline{\Phi}}_{1:0:\mathcal{A}\bullet} - \underline{\underline{H}}_{1:\bullet\mathcal{A}}\underline{\underline{\Phi}}_{1:1:\mathcal{A}\bullet} + \underline{\underline{H}}_{1:\bullet\mathcal{A}}\underline{\underline{\Phi}}^*_{1:D:0:\mathcal{A}\bullet}\ ,$$
$$\underline{\underline{Q}}_1 = \underline{\underline{H}}_{1:\bullet\mathcal{M}}\left[\underline{\underline{A}}_{0:\mathcal{M}\mathcal{N}},\ \underline{\underline{0}}_{\mathcal{M}\mathcal{N}^C}\right]\ ,$$
$$\underline{\underline{Q}}_0 = \underline{\underline{H}}_{0:\bullet\mathcal{M}}\left[\underline{\underline{A}}_{0:\mathcal{M}\mathcal{N}},\ \underline{\underline{0}}_{\mathcal{M}\mathcal{N}^C}\right] - \underline{\underline{H}}_{1:\bullet\mathcal{M}}\left[\underline{\underline{A}}_{1:\mathcal{M}\mathcal{N}},\ \underline{\underline{0}}_{\mathcal{M}\mathcal{N}^C}\right]\ .$$

Proof. We first note from (5.8) together with (5.4) that

$$\mathcal{L}\left\{\mathrm{E}\left[u^{M_{\mathcal{A}}(t)}\right]\right\} = \underline{p}^{\top}(0)\ \underline{\underline{\hat{\varphi}}}\left(\underline{u}(\mathcal{A}), \underline{1}, 0, s\right)\ \underline{1}\ ; \tag{5.22}$$

$$\mathcal{L}\left\{\mathrm{E}\left[v^{N_{\mathcal{M}\mathcal{N}}(t)}\right]\right\} = \underline{p}^{\top}(0)\ \underline{\underline{\hat{\varphi}}}\left(\underline{1}, \underline{v}(\mathcal{M},\mathcal{N}), 0, s\right)\ \underline{1}\ , \tag{5.23}$$

By taking the partial derivatives of (5.22) and (5.23) with respect to u at $u = 1$ and v at $v = 1$ respectively, one has

$$\mathcal{L}\Big\{\mathrm{E}\big[M_{\mathcal{A}}(t)\big]\Big\} = \underline{p}^{\top}(0)\left\{\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\right\}^{-1}\left\{\frac{1}{s}\begin{bmatrix}\underline{\underline{\zeta}}_{1:\mathcal{A}\bullet}(s)\\ \underline{\underline{0}}_{\mathcal{A}^C\bullet}\end{bmatrix} + \begin{bmatrix}\underline{\underline{\zeta}}^{*}_{1,D:\mathcal{A}\bullet}(s)\\ \underline{\underline{0}}_{\mathcal{A}^C\bullet}\end{bmatrix}\right\}\underline{1}\ ; \tag{5.24}$$

$$\mathcal{L}\Big\{\mathrm{E}\big[N_{\mathcal{M}\mathcal{N}}(t)\big]\Big\} = \frac{1}{s}\ \underline{p}^{\top}(0)\left\{\underline{\underline{I}} - \underline{\underline{\alpha}}(s)\right\}^{-1}\begin{bmatrix}\underline{\underline{\alpha}}_{\mathcal{M}\mathcal{N}}(s) & \underline{\underline{0}}_{\mathcal{M}\mathcal{N}^C}\\ \underline{\underline{0}}_{\mathcal{M}^C\mathcal{N}} & \underline{\underline{0}}_{\mathcal{M}^C\mathcal{N}^C}\end{bmatrix}\underline{1}\ . \tag{5.25}$$

Theorem 2 of Keilson [9] combined with (5.7), (5.14) and (5.15) then yields the Laplace transform expansions of (5.24) and (5.25), and the theorem follows by taking the inversion of the Laplace transform expansions.

The next theorem can be shown in a similar manner by differentiating (5.22) and (5.23) twice with respect to u at $u = 1$ and v at $v = 1$ respectively, and the proof is omitted here.

Theorem 4. *As $t \to \infty$, one has*

$$\begin{aligned}&\mathrm{E}\Big[M_{\mathcal{A}}(t)\big(M_{\mathcal{A}}(t) - 1\big)\Big]\\ &\qquad = \underline{p}^{\top}(0)\left\{t^2\underline{\underline{P}}_1^2 + t\left(2\ \underline{\underline{P}}_1\underline{\underline{P}}_0 + 2\ \underline{\underline{\hat{P}}}_0\underline{\underline{P}}_1 + \underline{\underline{P}}_2\right)\right\}\underline{1} + o(t)\ ;\end{aligned} \tag{5.26}$$

$$\begin{aligned}&\mathrm{E}\Big[N_{\mathcal{M}\mathcal{N}}(t)\big(N_{\mathcal{M}\mathcal{N}}(t) - 1\big)\Big]\\ &\qquad = \underline{p}^{\top}(0)\left\{t^2\ \underline{\underline{Q}}_1^2 + 2t\left(\underline{\underline{Q}}_1\underline{\underline{Q}}_0 + \underline{\underline{Q}}_0\underline{\underline{Q}}_1\right)\right\}\underline{1} + o(t)\ ,\end{aligned} \tag{5.27}$$

where $\underline{\underline{\hat{P}}}_0 = \underline{\underline{H}}_{0:\bullet\mathcal{A}}\underline{\underline{\Phi}}_{1:0:\mathcal{A}\bullet} - \underline{\underline{H}}_{1:\bullet\mathcal{A}}\underline{\underline{\Phi}}_{1:1:\mathcal{A}\bullet}$, $\underline{\underline{P}}_2 = \underline{\underline{H}}_{1:\bullet\mathcal{A}}\underline{\underline{\Phi}}_{2:0:\mathcal{A}\bullet}$ *and other matrices are as defined in Theorem 3.*

Theorems 3 and 4 then lead to the following theorem providing the asymptotic expansions of $\mathrm{Var}\big[M_{\mathcal{A}}(t)\big]$ and $\mathrm{Var}\big[N_{\mathcal{M}\mathcal{N}}(t)\big]$.

Theorem 5. *As $t \to \infty$, one has*

$$\mathrm{Var}\Big[M_{\mathcal{A}}(t)\Big] = t\ \underline{p}^{\top}(0)\ \underline{\underline{U}}_0\ \underline{1} + o(t)\ , \tag{5.28}$$

$$\mathrm{Var}\Big[N_{\mathcal{M}\mathcal{N}}(t)\Big] = t\ \underline{p}^{\top}(0)\ \underline{\underline{V}}_0\ \underline{1} + o(t)\ , \tag{5.29}$$

where

$$\underline{\underline{U}}_0 = 2\,\underline{\underline{P}}_1\underline{\underline{P}}_0 + \underline{\underline{P}}_1 - \underline{\underline{P}}_1\underline{1}\cdot\underline{p}^\top(0)\underline{\underline{P}}_0 + 2\,\underline{\underline{\hat{P}}}_0\underline{\underline{P}}_1 - \underline{\underline{P}}_0\underline{\underline{P}}_1 + \underline{\underline{P}}_2\ ;$$
$$\underline{\underline{V}}_0 = 2\,\underline{\underline{Q}}_1\underline{\underline{Q}}_0 + \underline{\underline{Q}}_1 - \underline{\underline{Q}}_1\underline{1}\cdot\underline{p}^\top(0)\underline{\underline{Q}}_0 + \underline{\underline{Q}}_0\underline{\underline{Q}}_1\ .$$

Proof. It can be readily seen that

$$\text{(5.30)}\qquad \mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2 = \mathrm{E}[X(X-1)] + \mathrm{E}[X] - \mathrm{E}[X]^2\ .$$

Substituting the results of Theorems 3 and 4 into this equation, one see that

$$\text{(5.31)}\qquad \mathrm{Var}\Big[M_{\mathcal{A}}(t)\Big] = \underline{p}^\top(0)\Big\{t^2\,\underline{\underline{U}}_1 + t\,\underline{\underline{U}}_0\Big\}\underline{1} + o(t)\ ,$$

and

$$\text{(5.32)}\qquad \mathrm{Var}\Big[N_{\mathcal{MN}}(t)\Big] = \underline{p}^\top(0)\Big\{t^2\,\underline{\underline{V}}_1 + t\,\underline{\underline{V}}_0\Big\}\underline{1} + o(t)\ ,$$

where

$$\underline{\underline{U}}_1 = \underline{\underline{P}}_1\Big(\underline{\underline{P}}_1 - \underline{1}\cdot\underline{p}^\top(0)\underline{\underline{P}}_1\Big)\ ,$$
$$\underline{\underline{U}}_0 = 2\,\underline{\underline{P}}_1\underline{\underline{P}}_0 + \underline{\underline{P}}_1 - \underline{\underline{P}}_1\underline{1}\cdot\underline{p}^\top(0)\underline{\underline{P}}_0 + 2\,\underline{\underline{\hat{P}}}_0\underline{\underline{P}}_1 - \underline{\underline{P}}_0\underline{1}\cdot\underline{p}^\top(0)\underline{\underline{P}}_1 + \underline{\underline{P}}_2\ ;$$
$$\underline{\underline{V}}_1 = \underline{\underline{Q}}_1\Big(\underline{\underline{Q}}_1 - \underline{1}\cdot\underline{p}^\top(0)\underline{\underline{Q}}_1\Big)\ ,$$
$$\underline{\underline{V}}_0 = 2\,\underline{\underline{Q}}_1\underline{\underline{Q}}_0 + \underline{\underline{Q}}_1 - \underline{\underline{Q}}_1\underline{1}\cdot\underline{p}^\top(0)\underline{\underline{Q}}_0 + 2\,\underline{\underline{Q}}_0\underline{\underline{Q}}_1 - \underline{\underline{Q}}_0\underline{1}\cdot\underline{p}^\top(0)\underline{\underline{Q}}_1\ .$$

From Theorem 2, one has $\underline{\underline{H}}_1 = \frac{1}{m}\,\underline{1}\,\underline{e}^\top$ which is of rank one having identical row. As can be seen from Theorem 3, $\underline{\underline{P}}_1$ and $\underline{\underline{Q}}_1$ have the same property, which in turn leads to $\underline{1}\cdot\underline{p}^\top(0)\underline{\underline{P}}_1 = \underline{\underline{P}}_1$, $\underline{1}\cdot\underline{p}^\top(0)\underline{\underline{Q}}_1 = \underline{\underline{Q}}_1$, and the theorem follows.

The asymptotic behavior of $\mathrm{E}\big[S(t)\big]$ can be easily found from Equation (5.2) and Theorem 3. The asymptotic expansion of $\mathrm{Var}\big[S(t)\big]$, however, requires a little precaution because it involves the joint expectation of $M_{\mathcal{A}}(t)$ and $N_{\mathcal{MN}}(t)$. More specifically, one has

$$\begin{aligned}
\mathrm{Var}\Big[S(t)\Big] &= \mathrm{E}\Big[S(t)^2\Big] - \mathrm{E}\Big[S(t)\Big]^2 \\
&= \mathrm{E}\Big[\big\{cM_{\mathcal{A}}(t) + dN_{\mathcal{MN}}(t)\big\}^2\Big] - \mathrm{E}\Big[cM_{\mathcal{A}}(t) + dN_{\mathcal{MN}}(t)\Big]^2 \\
&= c^2\mathrm{E}\Big[M_{\mathcal{A}}^2(t)\Big] + 2cd\,\mathrm{E}\Big[M_{\mathcal{A}}(t)N_{\mathcal{MN}}(t)\Big] + d^2\mathrm{E}\Big[N_{\mathcal{MN}}^2(t)\Big] \\
&\quad - c^2\mathrm{E}\Big[M_{\mathcal{A}}(t)\Big]^2 - 2cd\,\mathrm{E}\Big[M_{\mathcal{A}}(t)\Big]\mathrm{E}\Big[N_{\mathcal{MN}}(t)\Big] - d^2\mathrm{E}\Big[N_{\mathcal{MN}}(t)\Big]^2\ ,
\end{aligned}$$

so that

$$\text{(5.33)}\qquad \operatorname{Var}\Big[S(t)\Big] = c^2\operatorname{Var}\Big[M_{\mathcal{A}}(t)\Big] + d^2\operatorname{Var}\Big[N_{\mathcal{MN}}(t)\Big] \\ + 2cd\ \mathrm{E}\Big[M_{\mathcal{A}}(t)N_{\mathcal{MN}}(t)\Big] - 2cd\ \mathrm{E}\Big[M_{\mathcal{A}}(t)\Big]\mathrm{E}\Big[N_{\mathcal{MN}}(t)\Big]\ .$$

In order to evaluate $\mathrm{E}\big[M_{\mathcal{A}}(t)N_{\mathcal{MN}}(t)\big]$, we note from (5.8) that

$$\text{(5.34)}\qquad \mathcal{L}\Big\{\mathrm{E}\Big[M_{\mathcal{A}}(t)N_{\mathcal{MN}}(t)\Big]\Big\} \\ = \underline{p}^{\top}(0)\left[\frac{\partial^2}{\partial u\partial v}\underline{\underline{\hat{\varphi}}}(\underline{u}(\mathcal{A}),\underline{v}(\mathcal{M},\mathcal{N}),0+,s)\Big|_{u=v=1}\right]\underline{1}\ .$$

The asymptotic expansion of $\mathrm{E}\big[M_{\mathcal{A}}(t)N_{\mathcal{MN}}(t)\big]$ can then be obtained as for the previous theorems.

Theorem 6. *As $t\to\infty$, one has*

$$\text{(5.35)}\qquad \mathrm{E}\Big[M_{\mathcal{A}}(t)N_{\mathcal{MN}}(t)\Big] = \underline{p}^{\top}(0)\Big\{\frac{t^2}{2}\underline{\underline{T}}_1 + t\ \underline{\underline{T}}_0\Big\}\underline{1} + o(t)\ ,$$

where

$$\underline{\underline{T}}_1 = \underline{\underline{P}}_1\underline{\underline{Q}}_1 + \underline{\underline{Q}}_1\underline{\underline{P}}_1\ ,\quad \underline{\underline{T}}_0 = \underline{\underline{\hat{P}}}_0\underline{\underline{Q}}_1 + \underline{\underline{P}}_1\underline{\underline{Q}}_0 + \underline{\underline{R}} + \underline{\underline{Q}}_0\underline{\underline{P}}_1 + \underline{\underline{Q}}_1\underline{\underline{P}}_0\ ,$$
$$\underline{\underline{R}} = \underline{\underline{H}}_{1:\bullet,(\mathcal{A}\cap\mathcal{M})}\Big[\underline{\underline{\Phi}}_{1:0:(\mathcal{A}\cap\mathcal{M}),\mathcal{N}}\ ,\ \underline{\underline{0}}_{(\mathcal{A}\cap\mathcal{M}),\mathcal{N}^C}\Big]\ .$$

Now the key theorem of this section is given from Equation (5.33), Theorems 3, 5 and 6.

Theorem 7. *As $t\to\infty$, one has*

$$\text{(5.36)}\qquad \mathrm{E}\Big[S(t)\Big] = \underline{p}^{\top}(0)\Big\{t\left(c\ \underline{\underline{P}}_1 + d\ \underline{\underline{Q}}_1\right) + c\ \underline{\underline{P}}_0 + d\ \underline{\underline{Q}}_0\Big\}\underline{1} + o(1)\ ,$$

$$\text{(5.37)}\qquad \operatorname{Var}\Big[S(t)\Big] = t\ \underline{p}^{\top}(0)\ \underline{\underline{W}}_0\ \underline{1} + o(t)\ ,$$

where $\underline{\underline{W}}_0 = c^2\ \underline{\underline{U}}_0 + d^2\ \underline{\underline{V}}_0 + 2cd\ \underline{\underline{T}}_0 - 2cd\left(\underline{\underline{P}}_1\underline{1}\cdot\underline{p}^{\top}(0)\underline{\underline{Q}}_0 + \underline{\underline{P}}_0\underline{1}\cdot\underline{p}^{\top}(0)\underline{\underline{Q}}_1\right)\ .$

Proof. Equation (5.36) follows trivially from Theorem 3. For (5.37), we note from Theorems 3, 5 and 6 together with (5.33) that

$$\text{(5.38)}\qquad \operatorname{Var}\Big[S(t)\Big] = \underline{p}^{\top}(0)\Big\{t^2\ \underline{\underline{W}}_1 + t\ \underline{\underline{W}}_0\Big\}\underline{1} + o(t)\ ,$$

where

$$\underline{\underline{W}}_1 = cd\ \underline{\underline{T}}_1 - 2cd\ \underline{\underline{P}}_1 \underline{1} \cdot \underline{p}^\top(0) \underline{\underline{Q}}_1\ ,$$
$$\underline{\underline{W}}_0 = c^2\ \underline{\underline{U}}_0 + d^2\ \underline{\underline{V}}_0 + 2cd\ \underline{\underline{T}}_0 - 2cd \left(\underline{\underline{P}}_1 \underline{1} \cdot \underline{p}^\top(0) \underline{\underline{Q}}_0 + \underline{\underline{P}}_0 \underline{1} \cdot \underline{p}^\top(0) \underline{\underline{Q}}_1 \right)\ .$$

Since $\underline{\underline{P}}_1 = \underline{1} \cdot \underline{p}^\top(0) \underline{\underline{P}}_1$ and $\underline{\underline{Q}}_1 = \underline{1} \cdot \underline{p}^\top(0) \underline{\underline{Q}}_1$ as we shown in the proof of Theorem 5, the coefficient of the first term at right hand side of Equation (5.38) can be rewritten as

$$\begin{aligned}
& cd\ \underline{p}^\top(0)\ \underline{\underline{T}}_1\ \underline{1} - 2cd\ \underline{p}^\top(0)\ \underline{\underline{P}}_1 \underline{1} \cdot \underline{p}^\top(0)\ \underline{\underline{Q}}_1\ \underline{1} \\
= \ & cd\ \underline{p}^\top(0)\ \left(\underline{\underline{P}}_1 \underline{\underline{Q}}_1 + \underline{\underline{Q}}_1 \underline{\underline{P}}_1 \right)\ \underline{1} - 2cd\ \left(\underline{p}^\top(0)\ \underline{\underline{P}}_1 \underline{1} \right) \left(\underline{p}^\top(0)\ \underline{\underline{Q}}_1\ \underline{1} \right) \\
= \ & cd\ \left(\underline{p}^\top(0)\ \underline{\underline{P}}_1 \underline{\underline{Q}}_1\ \underline{1} + \ \underline{p}^\top(0)\ \underline{\underline{Q}}_1 \underline{\underline{P}}_1\ \underline{1} \right) - 2cd\ \left(\underline{p}^\top(0)\ \underline{\underline{P}}_1 \underline{1} \right) \left(\underline{p}^\top(0)\ \underline{\underline{Q}}_1\ \underline{1} \right) \\
= \ & cd\ \left\{ \left(\underline{p}^\top(0)\ \underline{\underline{P}}_1 \underline{1} \right) \left(\underline{p}^\top(0)\ \underline{\underline{Q}}_1\ \underline{1} \right) + \left(\underline{p}^\top(0)\ \underline{\underline{Q}}_1\ \underline{1} \right) \left(\underline{p}^\top(0)\ \underline{\underline{P}}_1 \underline{1} \right) \right\} \\
& \qquad\qquad -2cd\ \left(\underline{p}^\top(0)\ \underline{\underline{P}}_1 \underline{1} \right) \left(\underline{p}^\top(0)\ \underline{\underline{Q}}_1\ \underline{1} \right) \\
= \ & 0\ ,
\end{aligned}$$

completing the proof.

6. Dynamic Analysis of a Manufacturing System for Determining Optimal Maintenance Policy

As an application of the unified multivariate counting process, in this section, we consider a manufacturing system with a certain maintenance policy, where the system starts anew at time $t = 0$, and tends to generate product defects more often as time goes by. When the system reaches a certain state, the manufacturing system would be overhauled completely and the system returns to the fresh state. More specifically, let $J(t)$ be a semi-Markov process on $\mathcal{J} = \{0, 1, 2, \cdots, J\}$ governed by $\underline{\underline{A}}(x)$, describing the system state at time t where state 0 is the fresh state and state J is the maintenance state. When the system is in state j, $0 \leq j \leq J - 1$, product defects are generated according to an NHPP with intensity $\lambda_j(x)$. It is assumed that the system deteriorates monotonically and accordingly $\lambda_j(x)$ increases as a function of both x and j. When the system reaches state J, the manufacturing operation is stopped and the system is overhauled completely. The maintenance time increases stochastically as a function of

J. In other words, the further the maintenance is delayed, the longer the maintenance time would tend to be. Upon finishing the overhaul, the system is brought back to the fresh state 0. Of interest, then, is to determine the optimal maintenance policy concerning how to set J.

In order to determine the optimal maintenance policy, it is necessary to define the objective function precisely. Let ψ_d be the cost associated with each defect and let ψ_m be the cost for each of the maintenance operation. If we define two counting processes $M(t)$ and $N(t)$ as the total number of defects generated by time t and the number of the maintenance operations occurred by time t respectively, the total cost in $[0, T]$ can be described as

$$C_J(T) = \psi_d \cdot \mathrm{E}\big[M(T)\big] + \psi_m \cdot \mathrm{E}\big[N(T)\big] \ . \tag{6.1}$$

Let $\mathbb{N}$ be the set of natural numbers. The optimal maintenance policy J^* is then determined by

$$C_{J^*}(T) = \min_{J \in \mathbb{N}} \quad C_J(T) \ . \tag{6.2}$$

In what follows, we present a numerical example by letting $\mathcal{J} = \{0, 1, \cdots, J\}$ for $1 \le J \le 9$. For the underlying semi-Markov process, we define the matrix Laplace transform $\underline{\underline{\alpha}}(s)$ having IFR (Increasing Failure Rate) and DFR (Decreasing Failure Rate) dwell time distributions as described below. By introducing matrices $\underline{\underline{\theta}}, \underline{\underline{\hat{\theta}}}$ and $\underline{p}$, for which the details are given in Table 1 along with other parameter values, we define

$$\underline{\underline{\alpha}}_{IFR}(s) = \left[\frac{\theta_{ii}}{s + \theta_{ii}} \cdot \frac{\theta_{ij}}{s + \sum_{j=0}^{J} \theta_{ij}} \right] \ , \tag{6.3}$$

and

$$\underline{\underline{\alpha}}_{DFR}(s) = \left[p_i \cdot \frac{\theta_{ij}}{s + \sum_{j=0}^{J} \theta_{ij}} + (1 - p_i) \cdot \frac{\hat{\theta}_{ij}}{s + \sum_{j=0}^{J} \hat{\theta}_{ij}} \right] \ , \tag{6.4}$$

where

$$p_i = \left(\frac{\theta_{ii} + \sum_{j=0}^{J} \theta_{ij}}{\theta_{ii} \cdot \sum_{j=0}^{J} \theta_{ij}} - \frac{1}{\sum_{j=0}^{J} \hat{\theta}_{ij}} \right) \Bigg/ \left(\frac{1}{\sum_{j=0}^{J} \theta_{ij}} - \frac{1}{\sum_{j=0}^{J} \hat{\theta}_{ij}} \right) \ ,$$

$$\sum_{j=0}^{J} \theta_{ij} \neq \sum_{j=0}^{J} \hat{\theta}_{ij} \ .$$

Table 1: Parameters of the numerical example

Parameter	**Value**
J	$1, 2, \cdots, 9$
$\underline{\lambda}(x)$	$\left[\cdots, 3j^2x^2, \cdots\right]^\top$
$\mathcal{A}$	$\{0, 1, \cdots, J-1\}$
$\mathcal{M}$	$\{J-1\}$
$\mathcal{N}$	$\{J\}$
$M(t)$	$\sum_{i\in\mathcal{A}} M_i(t)$
$N(t)$	$\sum_{i\in\mathcal{M}, j\in\mathcal{N}} N_{ij}(t)$
$\Theta(i,j)$	$\sqrt{\exp(i-7.9)+\exp(7.9-j)}$
$\hat{\Theta}(i,j)$	$\dfrac{1}{\exp(i+j-5)}+\dfrac{1}{2}$
$\underline{\underline{\theta}}$	$\begin{bmatrix} \Theta(0,0) & \Theta(0,1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & 0 & \Theta(i,i) & \Theta(i,i+1) & 0 \\ 0 & \ddots & & \ddots & \ddots \\ \Theta(J,J+1) & 0 & \cdots & 0 & \Theta(J,J) \end{bmatrix}$
$\underline{\underline{\hat{\theta}}}$	$\begin{bmatrix} \hat{\Theta}(0,0) & \hat{\Theta}(0,1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & 0 & \hat{\Theta}(i,i) & \hat{\Theta}(i,i+1) & 0 \\ 0 & \ddots & & \ddots & \ddots \\ \hat{\Theta}(J,J+1) & 0 & \cdots & 0 & \hat{\Theta}(J,J) \end{bmatrix}$
ψ_d	10
ψ_m	1000
T	1000

The asymptotic behaviors of the mean of $M(t)$ and $N(t)$ per unit time with maintenance policy $J = 1, \cdots, 9$ are depicted in Figures 3 and 4. One could see that both the mean of $M(t)$ and $N(t)$ converges to a positive value as time t increases, confirming Theorem 3. In order to determine the optimal maintenance policy, for $J \in \{1, 2, \cdots, 9\}$, the corresponding total cost $C_J(T)$ can be computed as shown in

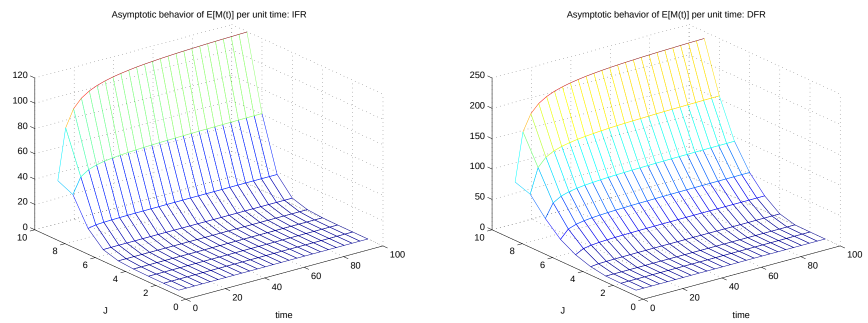

FIGURE 3: Mean of $M(t)$ per unit time

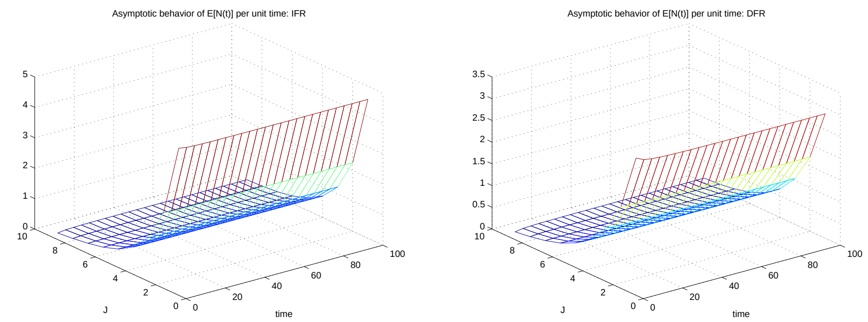

FIGURE 4: Mean of $N(t)$ per unit time

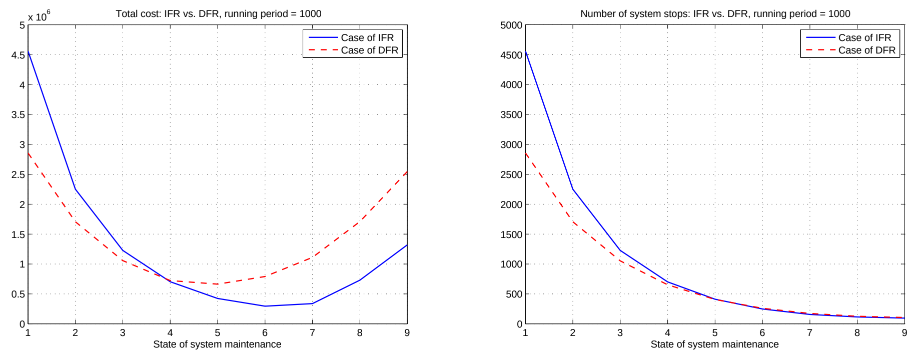

FIGURE 5: Optimal maintenance policy: IFR vs. DFR with $T = 1000$

Table 2 which is depicted in Figure 5. For the case of IFR, the optimal maintenance policy is at $J^* = 6$, while $J^* = 5$ for the DFR case, where the running period T is taken to be $T = 1000$ hours.

TABLE 2: Results of the problem of determining optimal maintenance policy

	Maintenance Policy J	Total Cost	Mean of $N(t)$
IFR	1	4558439.7050	4558.4281
	2	2249788.3693	2249.6796
	3	1226609.0459	1225.9839
	4	703523.0084	700.5894
	5	423335.3246	411.0384
	6	**294670.0741**	**246.5977**
	7	337391.4590	156.3405
	8	729034.3960	113.8727
	9	1319510.6644	95.4171
DFR	1	2855729.2399	2855.7277
	2	1706004.8495	1705.8080
	3	1055886.1726	1051.3492
	4	722164.3163	651.5757
	5	**663418.1912**	**406.685**
	6	790848.6824	258.8155
	7	1110782.8637	172.8299
	8	1708966.7374	125.8760
	9	2544519.3472	103.9733

7. Conclusion

In this paper, a unified multivariate counting process $[\underline{M}(t), \underline{\underline{N}}(t)]$ is proposed with non-homogeneous Poisson processes lying on a finite semi-Markov process. Here the vector process $\underline{M}(t)$ counts the cumulative number of such non-homogeneous Poisson arrivals at every state and the matrix process $\underline{\underline{N}}(t)$ counts the cumulative number of state transitions of the semi-Markov process in $[0, t]$. This unified multivariate counting process contains many existing counting processes as special cases. The dynamic analysis of the unified multivariate counting process is given, demonstrating the fact that the existing counting processes can be treated as special cases of the unified multivariate counting process. The asymptotic behaviors of the mean and the variance of the unified multivariate counting process are analyzed. As an application, a manufacturing system with certain maintenance policies is considered. The unified multivariate counting process enables one to determine the optimal maintenance policy minimizing the total cost. Numerical examples are given with IFR and DFR dwell times of the underlying semi-Markov process. As for the future agenda, the impact of such distributional properties on the optimal maintenance policy would be pursued theoretically. Other possible theoretical extensions include: 1) analysis of the reward process associated with the unified multivariate counting process; and 2) exploration of further applications in the areas of modern communication networks and credit risk analysis such as CDOs (collateralized debt obligations) for financial engineering.

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