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and Related Limit Theorems**

by

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STRUCTURAL ANALYSIS OF FIRST PASSAGE TIMES OF SKIP-FREE SEMI-MARKOV PROCESSES AND RELATED LIMIT THEOREMS

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ABSTRACT. A skip-free semi-Markov process is considered, which moves through its state space $\mathcal{N} = \{0, 1, 2, \dots\}$ only in a lattice continuous manner. Such processes can be considered to generalize a class of birth-death processes. Of interest, then, is to see how the first passage time structure of birth-death processes may or may not be inherited in that of skip-free semi-Markov processes. It is shown that a recursion formula satisfied by the first passage times T_n^+ from n to $n+1$ for skip-free semi-Markov processes is quite similar to that for birth-death processes. While the distributional properties of the first passage times of birth-death processes are not necessarily present for skip-free semi-Markov processes, the limit theorems hold true almost in an identical form.

Keywords skip-free semi-Markov processes, first passage times, distributional properties, limit theorems.

0. INTRODUCTION

The study of semi-Markov processes dates back to the middle of 1950's represented by the original papers by Léby[11], Smith[18] and Takács[26]. Subsequently various aspects of these processes were studied in a series of papers by Pyke[13, 14], Pyke and Schaufele[15, 16] and Moore and Pyke[12]. Since the early 1960's, the field attracted many researchers, resulting in a collection of quite extensive results. A bibliography on semi-Markov processes in 1976 by Teugels[27] contains about 600 papers by some 300 authors. The reader is referred to two excellent survey papers by Çinlar[1, 2] for the summary of these results. To the best knowledge of the authors, however, there exists no literature specifically focusing on the first passage time structure of skip-free semi-Markov processes, which can be considered as a generalization of a class of birth-death processes.

A Markov chain in continuous time defined on $\{0, 1, 2, \dots\}$ is called a birth-death process when transitions occur only in a lattice continuous manner. Because of this skip-free property, the first passage times of birth-death processes possess a variety of peculiar results as shown in Keilson[6, 8, 9, 10], Sumita[21, 25] and Sumita and Masuda[22] to name only a few. The purpose of this paper is to analyze the first passage time structure of skip-free semi-Markov processes, and to see how the first passage time structure of birth-death processes can be carried over to that of skip-free

semi-Markov processes. It is also examined whether certain limit theorems involving first passage times of birth-death processes still hold true for skip-free semi-Markov processes.

The structure of this paper is as follows. In Section 1, we formally define a semi-Markov process and summarize some well known results. Using these results, the Laplace transforms of the first passage times of skip-free semi-Markov processes are explicitly derived in Section 2. Moments, a recursion formula and its probabilistic interpretation are also discussed. Section 3 establishes a sufficient condition under which the first passage time T_n^+ from n to $n + 1$ of a skip-free semi-Markov process is a mixture of independent exponential random variables. Finally in Section 4, it is shown under certain conditions that $T_n^+/\mathbb{E}[T_n^+]$ for skip-free semi-Markov processes converges in law to the exponential variate of mean one as $n \rightarrow \infty$, as for birth-death processes. Other related limit theorems are also discussed.

Throughout the paper, the following notation is employed. Vectors and matrices are distinguished by a single underline and double underlines respectively. For an $(N + 1)$ dimensional vector \underline{a}^T and an $(N + 1) \times (N + 1)$ matrix \underline{a} , subvectors and submatrices are denoted by $\underline{a}_G^T = [a_i]_{i \in G}$, $\underline{a}_{GB} = [a_{ij}]_{i \in G, j \in B}$, etc. where $G, B \subset \{0, 1, \dots, N\}$. We also define $\underline{1}$ as a column vector having all components equal to 1, and $\underline{0}$ as a matrix having all components equal to zero.

1. FINITE SEMI-MARKOV PROCESS $J(t)$ AND FIRST PASSAGE TIMES

Let $\{(J_n, T_n) : n = 0, 1, \dots\}$ be a Markov renewal process where J_n is a Markov chain in discrete time on $\mathcal{N} = \{0, 1, \dots, N\}$ and T_n is the n -th transition epoch with $T_0 = 0$. The behavior of the Markov renewal process is governed by a semi-Markov matrix $\underline{A}(x) = [A_{ij}(x)]$, where

$$(1.1) \quad A_{ij}(x) = \mathbb{P}[J_{n+1} = j, T_{n+1} - T_n \leq x | J_n = i] .$$

If $\sup_n T_n = +\infty$, then the process $\{J(t) : t \geq 0\}$ where $J(t) = J_n$ for $T_n \leq t \leq T_{n+1}$ is called the minimal semi-Markov process associated with the Markov renewal process $\{(J_n, T_n) : n = 0, 1, \dots\}$, see Çinlar[1].

We assume that the stochastic matrix $\underline{A}_0 = [A_{0:ij}] = \underline{A}(\infty)$ governing the embedded Markov chain $\{J_n : n = 0, 1, \dots\}$ is ergodic, having the ergodic vector $\underline{\check{e}}^T$, i.e.

$$(1.2) \quad \underline{\check{e}}^T \underline{A}_0 = \underline{\check{e}}^T ; \underline{\check{e}}^T > \underline{0}^T ; \underline{\check{e}}^T \cdot \underline{1} = 1 .$$

For notational convenience, let $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, and define

$$(1.3) \quad \underline{\underline{A}}_D(x) = [\delta_{ij}A_i(x)] ; A_i(x) = \sum_{j \in \mathcal{N}} A_{ij}(x) ,$$

$$(1.4) \quad \bar{\underline{\underline{A}}}_D(x) = [\delta_{ij}\bar{A}_i(x)] ; \bar{A}_i(x) = 1 - A_i(x) ,$$

$$(1.5) \quad \underline{\underline{A}}_k = [A_{k:ij}] ; A_{k:ij} = \int_0^\infty x^k dA_{ij}(x) ,$$

$$(1.6) \quad \underline{\underline{A}}_{D:k} = [\delta_{ij}A_{k:i}] ; A_{k:i} = \int_0^\infty x^k dA_i(x) , k \geq 0 .$$

We note that $A_i(x) = P[T_{n+1} - T_n \leq x | J_n = i]$ is the c.d.f of the dwell time of the semi-Markov process at state i and $\bar{A}_i(x)$ is the corresponding survival function. Throughout the paper, we assume that $A_{k:i} < \infty$ for all $i \in \mathcal{N}$ for $0 \leq k \leq 2$. It should be noted that $\underline{\underline{A}}_{D:0} = \underline{\underline{I}}$.

The transform of $\underline{\underline{A}}(x)$ is denoted by

$$(1.7) \quad \underline{\underline{\alpha}}(s) = [\alpha_{ij}(s)] ; \alpha_{ij}(s) = \int_0^\infty e^{-sx} dA_{ij}(x) .$$

Laplace-Stieltjes transforms $\alpha_i(s)$, $\underline{\underline{\alpha}}_D(s)$, etc. are defined similarly. The transition probability matrix of $J(t)$ is written by $\underline{\underline{P}}(t)$, i.e.,

$$(1.8) \quad \underline{\underline{P}}(t) = [P_{ij}(t)] ; P_{ij}(t) = P [J(t) = j | J(0) = i] .$$

Correspondingly, the state probability vector $\underline{p}^T(t) = [p_0(t), \dots, p_N(t)]$ at time t is given by

$$(1.9) \quad \underline{p}^T(t) = \underline{p}^T(0)\underline{\underline{P}}(t) .$$

The Laplace transforms of $\underline{\underline{P}}(t)$ and $\underline{p}^T(t)$ are denoted by

$$(1.10) \quad \underline{\underline{\Pi}}(s) = \int_0^\infty e^{-st} \underline{\underline{P}}(t) dt , \quad \underline{\pi}^T(s) = \int_0^\infty e^{-st} \underline{p}^T(t) dt .$$

From the classical renewal argument, one has (see e.g. Çinlar[1, 2])

$$(1.11) \quad P_{ij}(t) = \delta_{ij}\bar{A}_i(t) + \sum_{k \in \mathcal{N}} \int_0^t P_{kj}(t - \tau) dA_{ik}(\tau) , \quad i, j \in \mathcal{N} .$$

By taking the Laplace transform on both sides of (1.11), it follows that

$$(1.12) \quad \underline{\underline{\Pi}}(s) = \frac{1}{s} [\underline{\underline{I}} - \underline{\underline{\alpha}}(s)]^{-1} [\underline{\underline{I}} - \underline{\underline{\alpha}}_D(s)] .$$

It has been shown in Keilson[7] that

$$(1.13) \quad \underline{\underline{\alpha}}(s) [\underline{\underline{I}} - \underline{\underline{\alpha}}(s)]^{-1} = \frac{1}{s} \underline{\underline{H}}_1 + \underline{\underline{H}}_0 + \underline{o}(1) \text{ as } s \rightarrow 0+ ,$$

where the two matrices $\underline{\underline{H}}_1$ and $\underline{\underline{H}}_0$ are given by

$$(1.14) \quad \underline{\underline{H}}_1 = \frac{1}{m} \underline{\underline{J}} ; \underline{\underline{J}} = \underline{\underline{1}} \cdot \underline{\check{e}}^T ; m = \underline{\check{e}}^T \underline{\underline{A}}_1 \underline{\underline{1}} ,$$

with \check{e}^T as given in (1.2) and

$$(1.15) \quad \underline{H}_0 = \underline{H}_1 \left(-\underline{A}_1 + \frac{1}{2} \underline{A}_2 \underline{H}_1 \right) + \left(\underline{Z} - \underline{H}_1 \underline{A}_1 \underline{Z} \right) \left(\underline{A}_0 - \underline{A}_1 \underline{H}_1 \right) .$$

Here, \underline{Z} is the fundamental matrix associated with \underline{A}_0 defined by

$$(1.16) \quad \underline{Z} = \left[\underline{I} - \underline{A}_0 + \underline{J} \right]^{-1} .$$

Since $[\underline{I} - \underline{\alpha}(s)]^{-1} = \underline{I} + \underline{\alpha}(s) [\underline{I} - \underline{\alpha}(s)]^{-1}$ and $\underline{I} - \underline{\alpha}_D(s) = s \underline{A}_{D:1} + \underline{\alpha}(s)$, it can be readily seen from (1.12) and (1.13) that

$$(1.17) \quad \underline{\Pi}(s) = \frac{1}{s} \underline{H}_1 \cdot \underline{A}_{D:1} + \underline{A}_{D:1} + \underline{\alpha}(1) \text{ as } s \rightarrow 0+ .$$

It then follows from (1.14) that

$$(1.18) \quad \lim_{t \rightarrow \infty} \underline{P}(t) = \lim_{s \rightarrow 0} s \underline{\Pi}(s) = \underline{H}_1 \cdot \underline{A}_{D:1} = \frac{1}{m} \underline{1} \cdot \check{e}^T \underline{A}_{D:1} .$$

Accordingly, $J(t)$ is ergodic and the ergodic probability vector \underline{e}^T of $J(t)$ can be written as

$$(1.19) \quad \underline{e}^T = \frac{1}{m} \check{e}^T \underline{A}_{D:1} = \left[\frac{\check{e}_0 A_{1:0}}{\sum_{i \in \mathcal{N}} \check{e}_i A_{1:i}}, \dots, \frac{\check{e}_N A_{1:N}}{\sum_{i \in \mathcal{N}} \check{e}_i A_{1:i}} \right] .$$

From an application point of view, it may be useful to decompose the state space \mathcal{N} into two subsets G and B where the good set G is the set of desirable states and the bad set B is the set of undesirable states. Then of interest is the first passage time of $J(t)$ from $i \in G$ to the bad set B . More formally, let this first passage time be denoted by T_{iB} so that

$$(1.20) \quad T_{iB} = \inf\{t : J(t) \in B \mid J(0) = i \in G\} .$$

When the bad set B is a singleton set, the Laplace transform $\sigma_{iB}(s) = \mathbb{E}[e^{-sT_{iB}}]$ is given in (5.17) of Çinlar[2]. A more general result has been obtained by Sumita and Masuda[21] and we follow their proof here so as to facilitate further discussions.

For the study of the first passage time T_{iB} of $J(t)$ from $i \in G$ to any state in B , we consider the absorbing process $\tilde{J}(t)$ obtained from the original process by making all states in B absorbing. Let $\tilde{p}^T(t) = [\tilde{p}_0(t), \dots, \tilde{p}_N(t)]$ where $\tilde{p}_j(t) = \mathbb{P}[\tilde{J}(t) = j \mid \tilde{J}(0) = i]$ with the Laplace transform $\tilde{\pi}^T(s)$. It should be noted that $\tilde{p}_j(t)$ for $j \in B$ is the probability that the process $J(t)$ hits the bad set B at $j \in B$ for the first time in $[0, t)$. Without loss of generality, we assume that $G = \{0, 1, \dots, K\}$ and $B = \{K+1, \dots, N\}$. It is easy to see that

$$(1.21) \quad \mathbb{P}[T_{iB} \leq t] = \tilde{p}_B^T(t) \underline{1}_B .$$

Let

$$(1.22) \quad \tilde{\underline{\alpha}}(s) = [\tilde{\alpha}_{ij}(s)] = \begin{bmatrix} \underline{\alpha}_{GG}(s) & \underline{\alpha}_{GB}(s) \\ \underline{0}_{BG} & \underline{0}_{BB} \end{bmatrix} ; \quad \tilde{\underline{\alpha}}_D(s) = \left[\delta_{ij} \sum_{k \in \mathcal{N}} \tilde{\alpha}_{ik}(s) \right] .$$

If we denote the i -th unit vector by \underline{u}_i , it can be seen from (1.10), with $\underline{\Pi}(s)$, $\underline{\alpha}(s)$ and $\underline{\alpha}_D(s)$ replaced by $\tilde{\underline{\Pi}}(s)$, $\tilde{\underline{\alpha}}(s)$ and $\tilde{\underline{\alpha}}_D(s)$ respectively, that

$$(1.23) \quad \tilde{\underline{\pi}}_G^T(s) = \frac{1}{s} \underline{u}_{i:G}^T [\underline{I}_{GG} - \underline{\alpha}_{GG}(s)]^{-1} [\underline{I}_{GG} - \underline{\alpha}_{D:GG}(s)] ,$$

and

$$(1.24) \quad \tilde{\underline{\pi}}_B^T(s) = \tilde{\underline{\pi}}_G^T(s) [\underline{I}_{GG} - \underline{\alpha}_{D:GG}(s)]^{-1} \underline{\alpha}_{GB}(s) .$$

Substituting (1.23) into (1.24), one finally has

$$(1.25) \quad \tilde{\underline{\pi}}_B^T(s) = \frac{1}{s} \underline{u}_{i:G}^T [\underline{I}_{GG} - \underline{\alpha}_{GG}(s)]^{-1} \underline{\alpha}_{GB}(s) .$$

From (1.21), one easily sees that $\sigma_{iB}(s) = s \tilde{\underline{\pi}}_B^T(s) \underline{1}_B$. The following theorem then holds true from (1.25).

Theorem 1.1 (Theorem 4.1 of Sumita and Masuda[23])

Let $\sigma_{iB}(s) = \mathbb{E}[e^{-sT_{iB}}]$ and define $\underline{\sigma}_G(s) = [\sigma_{iB}(s)]_{i \in G}$. Then

$$(1.26) \quad \underline{\sigma}_G(s) = [\underline{I}_{GG} - \underline{\alpha}_{GG}(s)]^{-1} \underline{\alpha}_{GB}(s) \underline{1}_B .$$

By differentiating $\underline{\sigma}_G(s)$ in Theorem 1.1 with respect to s at $s = 0$, the next corollary is immediate.

Corollary 1.2 (Sumita and Masuda[23])

Let $\mathbb{E}[\underline{T}_G^k] = [\mathbb{E}[T_{iB}^k]]_{i \in G}$. Then for $k = 1, 2, \dots$

$$\mathbb{E}[\underline{T}_G^k] = [\underline{I}_{GG} - \underline{A}_{0:GG}]^{-1} \left[\underline{A}_{k:GB} \underline{1}_B + \sum_{l=1}^k \binom{k}{l} \underline{A}_{l:GG} \mathbb{E}[\underline{T}_G^{k-l}] \right] .$$

2. FIRST PASSAGE TIME STRUCTURE OF SKIP-FREE SEMI-MARKOV PROCESSES

In this section, we focus on the first passage time structure of skip-free semi-Markov processes, and examine to what extent the basic properties of the first passage time structure of birth-death processes can be carried over. A semi-Markov process $J(t)$ on $\mathcal{N} = \{0, 1, \dots\}$ governed by a semi-Markov matrix $\underline{A}(x) = [A_{ij}(x)]$ is said to be skip-free if

$$(2.1) \quad A_{ij}(x) = \begin{cases} q_i A_i^-(x) & j = i - 1, \quad i \geq 1, \\ r_i A_i^0(x) & j = i, \quad i \geq 0, \\ p_i A_i^+(x) & j = i + 1, \quad i \geq 0, \\ 0 & \text{else,} \end{cases}$$

where $q_i, r_i, p_i \geq 0$, $r_0 + p_0 = 1$ and $q_i + r_i + p_i = 1$, $i \geq 1$. Here, $A_i^-(x)$, $A_i^0(x)$ and $A_i^+(x)$ are probability distribution functions having Laplace-Stieltjes transforms $\alpha_i^-(s) = \int_0^\infty e^{-sx} dA_i^-(x)$, $\alpha_i^0(s) = \int_0^\infty e^{-sx} dA_i^0(x)$ and $\alpha_i^+(s) = \int_0^\infty e^{-sx} dA_i^+(x)$

respectively. For notational convenience, we define $q_0 = 0$. Then, the dwell time of $J(t)$ in state i is given by

$$(2.2) \quad A_i(x) = q_i A_i^-(x) + r_i A_i^0(x) + p_i A_i^+(x), \quad i \geq 0,$$

and the corresponding Laplace transform is denoted by $\alpha_i(s) = \int_0^\infty e^{-sx} dA_i(x)$. The semi-Markov process $J(t)$ is skip-free in that it moves only in a lattice continuous manner in either direction.

Let T_n^+ be the first passage time of $J(t)$ from n to $n+1$. In what follows, we establish some structural properties of $\sigma_n^+(s) = \mathbb{E}[e^{-sT_n^+}]$ based on Theorem 1.1. The following matrix is frequently employed in this study.

$$(2.3) \quad \underline{\alpha}_n(s) = \begin{bmatrix} r_0 \alpha_0^0(s) & p_0 \alpha_0^+(s) & & \underline{0} \\ q_1 \alpha_1^-(s) & r_1 \alpha_1^0(s) & p_1 \alpha_1^+(s) & \\ \ddots & \ddots & \ddots & \\ & q_{n-1} \alpha_{n-1}^-(s) & r_{n-1} \alpha_{n-1}^0(s) & p_{n-1} \alpha_{n-1}^+(s) \\ \underline{0} & & q_n \alpha_n^-(s) & r_n \alpha_n^0(s) \end{bmatrix}.$$

For convenience, we define $\underline{\alpha}_n(s) = \underline{0}$ for $n < 0$. Our first theorem describes $\sigma_n^+(s)$ in terms of determinants of matrices of the form specified in (2.3).

Theorem 2.1

$$\sigma_n^+(s) = p_n \alpha_n^+(s) \frac{\det \left(\underline{I} - \underline{\alpha}_{n-1}(s) \right)}{\det \left(\underline{I} - \underline{\alpha}_n(s) \right)}, \quad n \geq 0.$$

Proof. Let $G = \{0, 1, 2, \dots, n\}$ and $B = \{n+1, n+2, \dots\}$. Since $J(t)$ is skip-free, an entry into B is possible only through $n+1$. Hence we can redefine B as a singleton set $B = \{n+1\}$. Let $T_{j,n+1}$ be the first passage time of $J(t)$ from j to $n+1$, and define $\underline{\sigma}_G^T(s) = [\sigma_{0,n+1}(s), \dots, \sigma_{n,n+1}(s)]$ where $\sigma_{j,n+1}(s) = \mathbb{E}[e^{-sT_{j,n+1}}]$. We note that $\sigma_n^+(s) = \sigma_{n,n+1}(s)$. From Theorem 1.1, one sees that

$$(2.4) \quad \left[\underline{I} - \underline{\alpha}_n(s) \right] \underline{\sigma}_G(s) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ p_n \alpha_n^+(s) \end{pmatrix}.$$

Since $\sigma_n^+(s)$ is the last element in $\underline{\sigma}_G(s)$, the theorem follows from (2.3) by applying Cramer's rule to (2.4). \square

Theorem 2.1 then leads to the following recursion formula for $\sigma_n^+(s)$.

Theorem 2.2

$$\sigma_n^+(s) = \frac{p_n \alpha_n^+(s)}{1 - r_n \alpha_n^0(s) - q_n \alpha_n^-(s) \sigma_{n-1}^+(s)}, \quad n \geq 0,$$

where $\alpha_n^-(s) = \sigma_n^+(s) = 0$ for $n < 0$.

Proof. From (2.3), it can be readily seen that

$$(2.5) \quad \det \left(\underline{I} - \underline{\alpha}_n(s) \right) = (1 - r_n \alpha_n^0(s)) \det \left(\underline{I} - \underline{\alpha}_{n-1}(s) \right) \\ - q_n \alpha_n^-(s) p_{n-1} \alpha_{n-1}^+(s) \det \left(\underline{I} - \underline{\alpha}_{n-2}(s) \right).$$

Substituting (2.5) into Theorem 2.1, and then dividing both numerator and denominator by $\det \left(\underline{I} - \underline{\alpha}_{n-1}(s) \right)$, the theorem follows. \square

The recursion formula given in Theorem 2.2 can be rewritten as

$$(2.6) \quad \sigma_n^+(s) = p_n \alpha_n^+(s) + r_n \alpha_n^0(s) \sigma_n^+(s) + q_n \alpha_n^-(s) \sigma_{n-1}^+(s) \sigma_n^+(s).$$

The probabilistic interpretation of (2.6) is now clear. Namely, suppose that the process $J(t)$ has entered state n at time $t = 0$. It goes to state $n + 1$ directly with probability p_n . In this case the first passage time from n to $n + 1$ is given by the dwell time of $J(t)$ at state n given that the upward transition occurs before any self- or downward transition, characterized by $\alpha_n^+(s)$. The first transition of $J(t)$ is a self-transition with probability r_n . The time required to reach $n + 1$ is then the sum of the dwell time of $J(t)$ given that the self-transition occurs before any downward or upward transition, having the Laplace transform $\alpha_n^0(s)$, and the first passage time from n to $n + 1$. If $J(t)$ goes down to $n - 1$ first, which occurs with probability q_n , the dwell time of $J(t)$ given that the downward transition occurs before any self- or upward transition, characterized by the Laplace transform $\alpha_n^-(s)$, has to be added to the first passage time from $n - 1$ to n and that from n to $n + 1$.

We next derive the mean and variance of T_n^+ . For $k \geq 0$, let $\mu_{n:k}^-, \mu_{n:k}^0, \mu_{n:k}^+$ and $\mu_{n:k}$ be defined by

$$(2.7) \quad \begin{cases} \mu_{n:k}^- &= \int_0^\infty x^k dA_n^-(x); \\ \mu_{n:k}^0 &= \int_0^\infty x^k dA_n^0(x); \\ \mu_{n:k}^+ &= \int_0^\infty x^k dA_n^+(x); \\ \mu_{n:k} &= \int_0^\infty x^k dA_n(x) = q_n \mu_{n:k}^- + r_n \mu_{n:k}^0 + p_n \mu_{n:k}^+. \end{cases}$$

Theorem 2.3

$$(a) \quad \mathbb{E}[T_n^+] = \frac{1}{p_n} \{ \mu_{n:1} + q_n \mathbb{E}[T_{n-1}^+] \}, \quad n \geq 0,$$

where $\mathbb{E}[T_n^+] = 0$ for $n < 0$.

$$(b) \quad \text{Var}[T_n^+] = \frac{q_n}{p_n} \text{Var}[T_{n-1}^+] + \frac{q_n}{p_n} \mathbb{E}^2[T_{n-1}^+] + \mathbb{E}^2[T_n^+] \\ + \frac{1}{p_n} \{ \mu_{n:2} - 2\mu_{n:1}\mu_{n:1}^+ + 2q_n(\mu_{n:1}^- - \mu_{n:1}^+) \mathbb{E}[T_{n-1}^+] \}, \quad n \geq 0,$$

where $\text{Var}[T_n^+] = 0$ for $n < 0$.

Proof. Let

$$(2.8) \quad D_n(s) = 1 - r_n \alpha_n^0(s) - q_n \alpha_n^-(s) \sigma_{n-1}^+(s) ,$$

so that $\sigma_n^+(s) = p_n \alpha_n^+(s) / D_n(s)$. We easily see that

$$(2.9) \quad D_n(0) = p_n ,$$

$$(2.10) \quad \left. \frac{d}{ds} D_n(s) \right|_{s=0} = \mu_{n:1} - p_n \mu_{n:1}^+ + q_n \mathbb{E}[T_{n-1}^+] ,$$

and

$$(2.11) \quad \left(\frac{d}{ds} \right)^2 D_n(s) \Big|_{s=0} = p_n \mu_{n:2}^+ - \left(\mu_{n:2} + 2q_n \mu_{n:1}^- \mathbb{E}[T_{n-1}^+] + q_n \mathbb{E}[T_{n-1}^{+2}] \right) .$$

The theorem follows by differentiating $\log \sigma_n^+(s) = \log p_n + \log \alpha_n^+(s) - \log D_n(s)$ twice at $s = 0$. \square

Remark 2.4

We note that, with $\nu_n = \lambda_n + \mu_n$, $r_0 = 0$, $p_n = \lambda_n / \nu_n$, $q_n = \mu_n / \nu_n$ and $\alpha_n(s) = \alpha_n^+(s) = \alpha_n^-(s) = \nu_n / (s + \nu_n)$, the semi-Markov process $J(t)$ is reduced to a birth-death process governed by upward transition rates λ_n ($n \geq 0$) and downward transition rates μ_n ($n \geq 1$). Theorems 2.2 and 2.3 are then rewritten as

$$(2.12) \quad \sigma_n^+(s) = \frac{\lambda_n}{s + \nu_n - \mu_n \sigma_{n-1}^+(s)} ,$$

$$(2.13) \quad \mathbb{E}[T_n^+] = \frac{1}{\lambda_n} \{1 + \mu_n \mathbb{E}[T_{n-1}^+]\} , \quad n \geq 0 ,$$

where $\mathbb{E}[T_n^+] = 0$ for $n < 0$, and

$$(2.14) \quad \text{Var}[T_n^+] = \frac{\mu_n}{\lambda_n} \text{Var}[T_{n-1}^+] + \frac{\mu_n}{\lambda_n} \mathbb{E}^2[T_{n-1}^+] + \mathbb{E}^2[T_n^+] , \quad n \geq 0 ,$$

where $\text{Var}[T_n^+] = 0$ for $n < 0$. While (2.12) and (2.13) are more or less similar to the counterparts of the semi-Markov process, the variance formula of (2.14) is simpler than Theorem 2.3 (b) where the last term in Theorem 2.3 (b) vanishes.

One sees that the first passage time of $J(t)$ from 0 to n , denoted by $T_{0,n}$, can be expressed as the sum of T_j^+ for $j = 0, \dots, n-1$, i.e.

$$(2.15) \quad T_{0,n} \stackrel{\text{def}}{=} \sum_{j=0}^{n-1} T_j^+ .$$

Let $s_{0,n}(\tau)$ be the p.d.f. of $T_{0,n}$ with $\sigma_{0,n}(s) = \int_0^\infty e^{-s\tau} s_{0,n}(\tau) d\tau$. From (2.15), it follows that

$$(2.16) \quad \sigma_{0,n}(s) = \prod_{j=0}^{n-1} \sigma_j^+(s) , \quad n \geq 1 .$$

It is obvious that the expectation and the variance of $T_{0,n}$ would be the sums of those of T_j^+ for $j = 0, \dots, n-1$. Namely, one has

$$(2.17) \quad \mathbb{E}[T_{0,n}] = \sum_{j=0}^{n-1} \mathbb{E}[T_j^+] ,$$

and

$$(2.18) \quad \text{Var}[T_{0,n}] = \sum_{j=0}^{n-1} \text{Var}[T_j^+] .$$

3. DISTRIBUTIONAL PROPERTIES OF FIRST PASSAGE TIMES OF SKIP-FREE SEMI-MARKOV PROCESSES

For any birth-death process on $\{0, 1, \dots\}$, Keilson[8, 9] shows that the first passage time T_n^+ from n to $n+1$ is always a mixture of $n+1$ independent exponential random variables and the first passage time $T_{0,n+1}$ from 0 to $n+1$ is always a sum of $n+1$ independent exponential random variables. It has been also shown in Keilson[10], Rösler[17], and Sumita and Masuda[24] that the p.d.f. of the first passage time $T_{m,n}$ from state m to state n is always unimodal. For skip-free semi-Markov processes, however, since $\sigma_n^-(s)$, $\sigma_n^0(s)$ and $\sigma_n^+(s)$ can be Laplace transforms of arbitrary p.d.f.'s, it is virtually impossible to establish general structural properties concerning zeros of $\det\left(\underline{I} - \underline{\alpha}_n(s)\right)$ in (2.3). Consequently, the study of distributional properties of first passage times of skip-free semi-Markov processes is quite difficult. In order to demonstrate this difficulty, we report here some results under rather restrictive conditions.

Let CM be the class of completely monotone p.d.f.'s on $[0, \infty)$ defined by

Definition 3.1

$$CM = \left\{ f \left| f(x) \geq 0, \int_0^\infty f(x) dx = 1, (-1)^k \left(\frac{d}{dx} \right)^k f(x) \geq 0, k = 0, 1, \dots \right. \right\} .$$

Of related interest is a class of p.d.f.'s each of which is a finite mixture of exponential densities. Formally, we define

$$CM_n = \left\{ f \left| f(x) = \sum_{i=1}^n p_i \theta_i e^{-\theta_i x}, 0 < n < \infty, \theta_i, p_i > 0, \sum_{i=1}^n p_i = 1, \theta_i \neq \theta_j, \text{ for } i \neq j \right. \right\} .$$

The union of CM_n is denoted by

$$(3.1) \quad CM^* = \bigcup_{n=0}^{\infty} CM_n .$$

This class is contained in the class CM as a subset. Following Sumita and Masuda[24], we also introduce SCM defined by

$$(3.2) \quad SCM = \{ f | f = f_1 * f_2 * \dots * f_n, f_i \in CM_{m_i}, m_i > 0, 1 \leq i \leq n \} ,$$

where the asterisk denotes convolution, i.e. $f * g(x) = \int_0^x f(x-y)g(y)dy$. The next proposition is well known, see e.g. Lemma 2.12.1 of Steutel [19].

Proposition 3.2

$$[f \in CM_n] \Leftrightarrow \left[\phi_f(s) = \frac{\prod_{i=1}^{n-1}(1+s/\eta_i)}{\prod_{j=1}^n(1+s/\theta_j)}, 0 < \theta_1 < \eta_1 < \theta_2 < \eta_2 < \dots < \eta_{n-1} < \theta_n \right].$$

One sees from Theorem 2.2 that

$$(3.3) \quad \sigma_0^+(s) = \frac{p_0 \alpha_0^+(s)}{1 - r_0 \alpha_0^0(s)}.$$

The next theorem then directly follows from Theorem 3.2 of Sumita and Masuda[24]. The p.d.f. of T_n^+ is denoted by $s_n^+(\tau)$ so that $\sigma_n^+(s) = \int_0^\infty e^{-s\tau} s_n^+(\tau) d\tau$.

Theorem 3.3

Let $a_0^+ \in CM_n$ and $a_0^0 \in CM_m$. Let $\rho(s) = (1 - r_0 \alpha_0^0(s))^{-1}$. Then $s_0^+ \in CM_r$ for some r with $0 < r < m + n$ if and only if $[\rho(s) \leq 0 \Rightarrow \alpha_0^+(s) \geq 0]$.

It may be worth noting that even when both a_0^+ and a_0^0 are exponential, the p.d.f. s_0^+ still may not be a mixture of exponential p.d.f.'s. As an example, let

$$(3.4) \quad \alpha_0^+(s) = \frac{\lambda}{s + \lambda}; \quad \alpha_0^0(s) = \frac{\mu}{s + \mu}, \quad \lambda \neq \mu.$$

One then easily sees that

$$(3.5) \quad \sigma_0^+(s) = \frac{p_0(\lambda - \mu)}{\lambda - p_0\mu} \cdot \frac{\lambda}{s + \lambda} + \frac{\lambda(1 - p_0)}{\lambda - p_0\mu} \cdot \frac{p_0\mu}{s + p_0\mu}.$$

Hence $s_0^+(\tau)$ is a mixture of exponential p.d.f.'s if and only if $\lambda > \mu$.

In general, sufficient conditions under which $s_n^+(\tau)$ is a mixture of exponential p.d.f.'s are hard to come by. We therefore consider skip-free semi-Markov processes with a special structure where the dwell times of the process at any state j ($0 \leq j \leq n$) do not depend on the next destination. This is equivalent to saying that

$$(3.6) \quad \begin{cases} a_0(\tau) = a_0^0(\tau) = a_0^+(\tau); \\ a_j(\tau) = a_j^-(\tau) = a_j^0(\tau) = a_j^+(\tau), \quad 1 \leq j \leq n. \end{cases}$$

We next show that, under the conditions of (3.6), if $a_j(\tau)$ are finite mixtures of exponential p.d.f.'s for $0 \leq j \leq n$, so are the first passage time p.d.f.'s $s_j^+(\tau)$ for $0 \leq j \leq n$.

Theorem 3.4

Under the conditions of (3.6), let $a_j(\tau) \in CM^*$ for $0 \leq j \leq n$. Then $s_j^+(\tau) \in CM^*$ for $0 \leq j \leq n$.

Proof. We prove the theorem by induction. For $j = 0$, one sees from (3.3) that $\sigma_0^+(s) = p_0\alpha_0(s)/\{1 - r_0\alpha_0(s)\}$. Hence from Theorem 3.3, $s_0^+(\tau) \in CM^*$. Suppose $s_{j-1}^+(\tau) \in CM^*$ for $j \leq n$ and consider s_j^+ . We note from Theorem 2.2 that

$$(3.7) \quad \sigma_j^+(s) = p_j\alpha_j^+(s)/D_j(s)$$

where $D_j(s)$ is defined in (2.8). One easily sees that

$$(3.8) \quad \frac{d}{ds}D_j(s) = - \left[r_j \frac{d}{ds}\alpha_j(s) + q_j \left\{ \sigma_{j-1}^+(s) \frac{d}{ds}\alpha_j(s) + \alpha_j(s) \frac{d}{ds}\sigma_{j-1}^+(s) \right\} \right].$$

It then follows from (3.7) and (3.8) that

$$(3.9) \quad \frac{d}{ds}\sigma_j^+(s) = \frac{p_j}{D_j(s)^2} \left[\frac{d}{ds}\alpha_j(s) + q_j\alpha_j^2(s) \frac{d}{ds}\sigma_{j-1}^+(s) \right].$$

By the assumption, for $0 \leq j \leq n$, one has $a_j \in CM^*$ so that $\frac{d}{ds}\alpha_j(s) < 0$ for real s apart from singularities. For $j \leq n$, the induction hypothesis assures that $s_{j-1}^+ \in CM^*$ and hence $\frac{d}{ds}\sigma_{j-1}^+(s) < 0$ for real s apart from singularities. Consequently $\frac{d}{ds}\sigma_j^+(s) < 0$ for real s apart from singularities. This means that $\sigma_j^+(s)$ is strictly decreasing in real s apart from singularities.

Since both $a_j(\tau)$ and $s_{j-1}^+(\tau)$ belong to CM^* , from Proposition 3.2 there exist polynomials $f_{j:K-1}$, $f_{j:K}$ and $g_{j-1:L-1}$, $g_{j-1:L}$ such that

$$(3.10) \quad \alpha_j(s) = \frac{f_{j:K-1}(s)}{f_{j:K}(s)}; \quad \sigma_{j-1}^+(s) = \frac{g_{j-1:L-1}(s)}{g_{j-1:L}(s)}.$$

Here K and L denote the order of polynomials. All of the four polynomials have only zeros of multiplicity one on the negative real axis. Furthermore zeros of $f_{j:K}(s)$ ($g_{j-1:L}(s)$) interleave those of $f_{j:K-1}(s)$ ($g_{j-1:L-1}(s)$). From (3.7), it can be readily seen that

$$(3.11) \quad \sigma_j^+(s) = \frac{p_j \frac{f_{j:K-1}(s)}{f_{j:K}(s)}}{1 - r_j \frac{f_{j:K-1}(s)}{f_{j:K}(s)} - q_j \frac{f_{j:K-1}(s)g_{j-1:L-1}(s)}{f_{j:K}(s)g_{j-1:L}(s)}}.$$

Any common factors between $f_{j:K-1}(s)$ and $g_{j-1:L}(s)$ should be cancelled in the last term of the denominator. Let l be the number of common factors which the two polynomials share where $0 \leq l \leq \min\{K-1, L\}$. Then the last term of the denominator can be rewritten as

$$\frac{f_{j:K-1}(s)g_{j-1:L-1}(s)}{f_{j:K}(s)g_{j-1:L}(s)} = \frac{\tilde{f}_{j:K-1-l}(s)g_{j-1:L-1}(s)}{f_{j:K}(s)\tilde{g}_{j-1:L-l}(s)}.$$

Multiplying both the numerator and the denominator of (3.11) by $f_{j:K}(s)\tilde{g}_{j-1:L-l}(s)$, one finds that

$$(3.12) \quad \sigma_j^+(s) = \frac{g_{j:M-1}(s)}{g_{j:M}(s)}$$

where

$$(3.13) \quad g_{j:M-1}(s) = p_j f_{j:K-1}(s) \tilde{g}_{j-1:L-l}(s)$$

and

$$(3.14) \quad g_{j:M}(s) = f_{j:K}(s)\tilde{g}_{j-1:L-l}(s) \\ - r_j f_{j:K-1}(s)\tilde{g}_{j-1:L-l}(s) - q_j \tilde{f}_{j:K-1-l}(s)g_{j-1:L-1}(s) .$$

We note that $M = K + L - l$ and $\sigma_j^+(s)$ has exactly $M - 1$ zeros of multiplicity one, all of which are located on the negative real axis. We have seen that $\frac{d}{ds}\sigma_j^+(s) < 0$ for real s apart from singularities. Hence as a real function of s , $\sigma_j^+(s)$ is strictly decreasing between singularities. Furthermore from Theorem 1.1 of Sumita and Masuda[24], $\sigma_j^+(s)$ cannot have a singular point in the positive right half plane. Hence $g_{j:M}(s)$ has M distinct zeros on the negative real axis which interleave zeros of $g_{j:M-1}(s)$. The theorem then follows from Proposition 3.2. \square

The next corollary follows immediately from Theorem 3.4 and the definition of SCM in (3.2).

Corollary 3.5

Under the conditions of (3.6), let the p.d.f. $s_j^+(\tau)$ of T_j^+ for $0 \leq j \leq n$ be as in Theorem 3.4. Then the p.d.f. $s_{0,n+1}(\tau)$ of $T_{0,n+1}$ satisfies $s_{0,n+1} \in SCM$.

Proof. Since $T_{0,n+1}$ is defined by $T_{0,n+1} = T_0^+ + T_1^+ + \dots + T_n^+$, the p.d.f. $s_{0,n+1}(\tau)$ of $T_{0,n+1}$ is given by $s_{0,n+1}(\tau) = s_0^+ * s_1^+ * \dots * s_n^+(\tau)$, where the asterisk denotes convolution. From Theorem 3.4, for each j , $0 \leq j \leq n$, there exists $m(j)$ such that $s_j^+ \in CM_{m(j)}$. Hence $s_{0,n+1}(\tau) \in SCM$ from (3.2), completing the proof. \square

The discussions in this section reveal that the distributional properties of first passage times for birth-death processes are not necessarily inherited in those for skip-free semi-Markov processes. Their limiting behaviors, however, seem to be quite similar as we show in the next section.

4. LIMITING BEHAVIOR OF FIRST PASSAGE TIMES FOR SKIP-FREE SEMI-MARKOV PROCESSES

Let $N(t)$ be a birth-death process governed by upward transition rates $\lambda_n > 0$, $n \geq 0$ and downward transition rates $\mu_n > 0$, $n \geq 1$ satisfying $\lambda_n \rightarrow \lambda > 0$ and $\mu_n \rightarrow \mu > 0$. When $\rho = \lambda/\mu < 1$, $N(t)$ is ergodic. In this case, Keilson[9] has shown the following limit theorems concerning the first passage times T_n^+ and $T_{0,n}$.

Theorem 4.1 (Theorem 8.2B of Keilson[9])

If $0 < \rho < 1$, then

- (a) $\frac{T_{0,n}}{\mathbb{E}[T_{0,n}]} \xrightarrow{d} E$ as $n \rightarrow \infty$, where $\mathbb{P}[E > x] = e^{-x}$;
- (b) $\frac{T_n^+}{\mathbb{E}[T_n^+]} \xrightarrow{d} X$ as $n \rightarrow \infty$, where $\mathbb{P}[X > x] = (1 - \rho)e^{-(1-\rho)x}$.

It should be noted that the random variable X has a mass ρ at $X = 0$, describing the jitter effect. This means that, when n is large, the sum of the ergodic probabilities from state 0 up to state n is very close to 1 and $E[T_{0,n+1}]$ becomes very large. If the process enters the state n , one observes clustering of the epochs at which the process crosses from n to $n + 1$ within a time interval much smaller compared to $E[T_{0,n+1}]$. As this scaling factor $E[T_{0,n+1}]$ goes to infinity as $n \rightarrow \infty$, these multiple crossings amount to the mass ρ of the limiting distribution at $X = 0$.

When $\rho > 1$, $N(t)$ is not ergodic. For this non-ergodic case, Sumita[25] has proven that the following limit theorems hold true.

Theorem 4.2 (Theorem 0.1 (2) of Sumita[25])

If $\rho > 1$, then

$$\begin{aligned} \text{(a)} \quad & T_n^+ \xrightarrow{d} T_{BP(\mu,\lambda)} ; \\ \text{(b)} \quad & \frac{T_{0,n}}{E[T_{0,n}]} \rightarrow 1 \text{ with probability 1 as } n \rightarrow \infty . \end{aligned}$$

Here, $T_{BP(\mu,\lambda)}$ is the server busy period of $M/M/1$ queueing system with Poisson arrivals of intensity μ and the exponential service rate λ .

The purpose of this section is to show that the limit theorems in Theorem 4.1 and Theorem 4.2 for birth-death processes can be more or less carried over to skip-free semi-Markov processes. Throughout this section, we assume that, for $n \in \mathcal{N} = \{0, 1, 2, \dots\}$,

$$(4.1) \quad \begin{aligned} & p_n > 0, \quad \lim_{n \rightarrow \infty} p_n = p > 0, \quad q_n > 0, \quad \lim_{n \rightarrow \infty} q_n = q > 0; \text{ and} \\ & \lim_{n \rightarrow \infty} \alpha_n^-(s) = \alpha^-(s), \quad \lim_{n \rightarrow \infty} \alpha_n^0(s) = \alpha^0(s), \quad \lim_{n \rightarrow \infty} \alpha_n^+(s) = \alpha^+(s). \end{aligned}$$

For notational convenience, we also define

$$(4.2) \quad \hat{\rho} \stackrel{\text{def}}{=} \frac{p}{q} .$$

We first show that the counterpart of Theorem 4.1 for skip-free semi-Markov processes is present under the ergodic condition $0 < \hat{\rho} < 1$.

Theorem 4.3

Let $T_{0,n}$ be the first passage time of a skip-free semi-Markov process from 0 to n satisfying (4.1). If $0 < \hat{\rho} < 1$, then

$$\frac{T_{0,n}}{E[T_{0,n}]} \xrightarrow{d} E \text{ as } n \rightarrow \infty, \text{ where } P[E > x] = e^{-x} .$$

Proof. We consider the regenerative process representing the return of the underlying skip-free semi-Markov process from state 0 to state 0. Let $\underline{V}_k = [Y_k, Z_k]$ where Z_k is the length of the k -th regenerative cycle and Y_k is the largest state visited within this regenerative cycle. It should be noted that Y_k and Z_k are correlated but \underline{V}_k ($k =$

1, 2, 3, \dots) constitute a sequence of independently and identically distributed (i.i.d.) random vectors.

Let the domain of \underline{V}_k be decomposed into $\mathcal{N} \times R^+ = G(n) \cup B(n)$ where $G(n) = \{(m, x) : 0 \leq m < n, x \geq 0\}$ and $B(n) = \{(m, x) : n \leq m, x \geq 0\}$. We now consider the following experiment. If $\underline{V}_k \in G(n)$, then the experiment continues and \underline{V}_{k+1} is chosen. The experiment stops when a random vector falls in $B(n)$. In other words, the experiment stops with probability $\eta_n = P[\underline{V}_k \in B(n)]$. It should be noted that the first passage time $T_{0,n}$ from state 0 to state n is the sum of Z_k 's until the experiment stops, and η_n is the probability that the state n is visited before returning to state 0 within a regenerative cycle. We next show that $\eta_n \rightarrow 0$ as $n \rightarrow \infty$ under the conditions of (4.1). The limit theorem then follows by Theorem 1.A4 of Shanthikumar and Sumita[20].

Let $\xi_{m,n}$ be the probability that the process $J(t)$ reaches state n before state 0, given that $J(0) = m$, $1 \leq m < n$. One then easily sees that

$$(4.3) \quad \begin{cases} \xi_{1,n} &= r_1 \xi_{1,n} + p_1 \xi_{2,n} ; \\ \xi_{m,n} &= q_m \xi_{m-1,n} + r_m \xi_{m,n} + p_m \xi_{m+1,n} , \quad 2 \leq m \leq n-2 ; \\ \xi_{n-1,n} &= q_{n-1} \xi_{n-2,n} + r_{n-1} \xi_{n-1,n} + p_{n-1} . \end{cases}$$

Let $\Delta \xi_{m,n+1} \stackrel{\text{def}}{=} \xi_{m,n+1} - \xi_{m-1,n+1}$ and define $\hat{\rho}_j \stackrel{\text{def}}{=} p_j/q_j$. Substituting $r_m = 1 - q_m - p_m$ for $1 \leq m < n$ into (4.3), it follows that

$$(4.4) \quad \begin{cases} \xi_{1,n} &= \hat{\rho}_1 \Delta \xi_{2,n} ; \\ \Delta \xi_{m,n} &= \hat{\rho}_m \Delta \xi_{m+1,n}, \quad 2 \leq m \leq n-2 ; \\ \Delta \xi_{n-1,n} &= \hat{\rho}_{n-1} (1 - \xi_{n-1,n}) , \end{cases}$$

which in turn leads to

$$(4.5) \quad \xi_{1,n} = (1 - \xi_{n-1,n}) \prod_{j=1}^{n-1} \hat{\rho}_j .$$

From (4.1), for arbitrarily small $\varepsilon > 0$, there exists $M(\varepsilon)$ such that $|q_n - q| < \varepsilon$ and $|p_n - p| < \varepsilon$ for $n > M(\varepsilon)$. One then sees from (4.5) that, for $n > M(\varepsilon) + 1$,

$$(4.6) \quad \left(\prod_{j=1}^{M(\varepsilon)} \hat{\rho}_j \right) \left(\frac{p - \varepsilon}{q + \varepsilon} \right)^{n-M(\varepsilon)-2} < \prod_{j=1}^{n-1} \hat{\rho}_j < \left(\prod_{j=1}^{M(\varepsilon)} \hat{\rho}_j \right) \left(\frac{p + \varepsilon}{q - \varepsilon} \right)^{n-M(\varepsilon)-2} .$$

Under the assumption that $0 < \hat{\rho} < 1$, one sees for sufficiently small $\varepsilon > 0$ that $(p - \varepsilon)/(q + \varepsilon) < 1$ and $(p + \varepsilon)/(q - \varepsilon) < 1$, so that $\xi_{1,n} \rightarrow 0$ as $n \rightarrow \infty$ from (4.5) and (4.6). Since $\eta_n = r_0 \eta_n + p_0 \xi_{1,n}$ and therefore $\eta_n = p_0 \xi_{1,n}/(1 - r_0)$, one concludes that $\eta_n \rightarrow 0$ as $n \rightarrow \infty$, completing the proof. \square

We next turn our attention to the exponential limit theorem of Theorem 4.1 (b) with the jitter effect. As for birth-death processes, this limit theorem still holds true for skip-free semi-Markov processes. A preliminary lemma is needed.

Lemma 4.4

Let T_n^+ be the first passage time of a skip-free semi-Markov process from n to $n+1$ satisfying (4.1). If $0 < \hat{\rho} < 1$, then

$$(a) \quad \frac{\mathbb{E}[T_{n-1}^+]}{\mathbb{E}[T_n^+]} \rightarrow \hat{\rho} \text{ as } n \rightarrow \infty ;$$

$$(b) \quad \frac{\mathbb{E}[T_n^+]}{\mathbb{E}[T_{0,n+1}]} \rightarrow 1 - \hat{\rho} \text{ as } n \rightarrow \infty .$$

Proof. From the recursion formula in Theorem 2.3 (a), one sees for $n \geq 1$ that

$$(4.7) \quad \frac{\mathbb{E}[T_{n-1}^+]}{\mathbb{E}[T_n^+]} = \frac{p_n}{q_n} - \frac{\mu_{n:1}}{q_n \mathbb{E}[T_n^+]} \leq \hat{\rho}_n .$$

We next show that $\frac{\mu_{n:1}}{q_n \mathbb{E}[T_n^+]}$ vanishes as $n \rightarrow \infty$. To do so, we write

$$\frac{\mu_{n:1}}{q_n \mathbb{E}[T_n^+]} = \frac{\mu_{n:1}}{q_n \mathbb{E}[T_0^+]} \cdot \frac{\mathbb{E}[T_0^+]}{\mathbb{E}[T_1^+]} \cdot \frac{\mathbb{E}[T_1^+]}{\mathbb{E}[T_2^+]} \cdots \frac{\mathbb{E}[T_{n-2}^+]}{\mathbb{E}[T_{n-1}^+]} \cdot \frac{\mathbb{E}[T_{n-1}^+]}{\mathbb{E}[T_n^+]} .$$

From (4.7), this leads to

$$(4.8) \quad \frac{\mu_{n:1}}{q_n \mathbb{E}[T_n^+]} \leq \frac{p_n \mu_{n:1}}{q_n \mathbb{E}[T_0^+]} \cdot \prod_{j=1}^n \hat{\rho}_j .$$

For arbitrarily small $\varepsilon > 0$, let $M(\varepsilon)$ be as in the proof of Theorem 4.3. From (4.6) and (4.7), it then follows that $\mu_{n:1}/q_n \mathbb{E}[T_n^+] \rightarrow 0$ as $n \rightarrow \infty$. This in turn implies from (4.7) that $\mathbb{E}[T_{n-1}^+]/\mathbb{E}[T_n^+] \rightarrow \hat{\rho}$ as $n \rightarrow \infty$, proving (a). For (b), it can be seen from (2.17) that

$$\frac{\mathbb{E}[T_{0,n+1}]}{\mathbb{E}[T_n^+]} = \frac{\sum_{j=0}^n \mathbb{E}[T_j^+]}{\mathbb{E}[T_n^+]} = 1 + \sum_{j=0}^{n-1} \frac{\mathbb{E}[T_j^+]}{\mathbb{E}[T_n^+]} = 1 + \sum_{k=1}^n \prod_{j=k}^n \frac{\mathbb{E}[T_{j-1}^+]}{\mathbb{E}[T_j^+]} .$$

From (a), one has $\mathbb{E}[T_{j-1}^+]/\mathbb{E}[T_j^+] \rightarrow \hat{\rho}$ as $j \rightarrow \infty$. From Lemma 8.3B of Keilson[9], it then follows that

$$(4.9) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{E}[T_{0,n+1}]}{\mathbb{E}[T_n^+]} = 1 + \frac{\hat{\rho}}{1 - \hat{\rho}} = \frac{1}{1 - \hat{\rho}} ,$$

completing the proof □

We are now in a position to prove the exponential limit theorem with the jitter effect for skip-free semi-Markov processes.

Theorem 4.5

If $0 < \hat{\rho} < 1$, then

$$\frac{T_n^+}{\mathbb{E}[T_n^+]} \xrightarrow{d} X \text{ as } n \rightarrow \infty , \text{ where } \mathbb{P}[X > x] = (1 - \hat{\rho})e^{-(1-\hat{\rho})x} .$$

Proof. From (2.16), one has

$$\sigma_n^+(s) = \frac{\sigma_{0,n+1}(s)}{\sigma_{0,n}(s)},$$

so that

$$(4.10) \quad \sigma_n^+ \left(\frac{s}{\mathbf{E}[T_n^+]} \right) = \frac{\sigma_{0,n+1} \left(\frac{s}{\mathbf{E}[T_{0,n+1}]} \frac{\mathbf{E}[T_{0,n+1}]}{\mathbf{E}[T_n^+]} \right)}{\sigma_{0,n} \left(\frac{s}{\mathbf{E}[T_{0,n}]} \frac{\mathbf{E}[T_{0,n}]}{\mathbf{E}[T_n^+]} \right)}.$$

Let E be the exponential variate of mean 1 as in Theorem 4.3. It should be noted from Lemma 4.4 that

$$(4.11) \quad \lim_{n \rightarrow \infty} \frac{\mathbf{E}[T_{0,n}]}{\mathbf{E}[T_n^+]} = \lim_{n \rightarrow \infty} \frac{\mathbf{E}[T_{0,n+1}] - \mathbf{E}[T_n^+]}{\mathbf{E}[T_n^+]} = \frac{1}{1 - \hat{\rho}} - 1 = \frac{\hat{\rho}}{1 - \hat{\rho}}.$$

It then follows from (4.10), Theorem 4.3, Lemma 4.4 and (4.11) that

$$\sigma_n^+ \left(\frac{s}{\mathbf{E}[T_n^+]} \right) \rightarrow \frac{\mathbf{E}[e^{-s(\frac{1}{1-\hat{\rho}})E}]}{\mathbf{E}[e^{-s(\frac{\hat{\rho}}{1-\hat{\rho}})E}]} = \frac{\left[\frac{1}{1-\hat{\rho}}s + 1 \right]^{-1}}{\left[\frac{\hat{\rho}}{1-\hat{\rho}}s + 1 \right]^{-1}} = \hat{\rho} + (1 - \hat{\rho}) \frac{(1 - \hat{\rho})}{s + (1 - \hat{\rho})},$$

as $n \rightarrow \infty$, completing the proof. \square

Unlike a birth-death process satisfying $\lambda_n \rightarrow \lambda > 0$ and $\mu_n \rightarrow \mu > 0$ as $n \rightarrow \infty$, a skip-free semi-Markov process satisfying (4.1), in general, cannot be related to a queueing system easily. When $\hat{\rho} > 1$, the counterpart of the limit theorem of Sumita[21] in Theorem 4.2 (a) for skip-free semi-Markov processes, therefore, has to take a different form.

Theorem 4.6

If $\hat{\rho} > 1$, then T_n^+ converges in distribution to T^+ where $\sigma^+(s) = \mathbf{E}[e^{-sT^+}]$ is given by

$$(4.12) \quad \sigma^+(s) = \frac{1 - r\alpha^0(s) - \sqrt{\{1 - r\alpha^0(s)\}^2 - 4pq\alpha^+(s)\alpha^-(s)}}{2q\alpha^-(s)}.$$

Proof. Let $g_n(x, s)$ be defined as

$$(4.13) \quad g_n(x, s) \stackrel{\text{def}}{=} \frac{p_n\alpha_n^+(s)}{1 - r_n\alpha_n^0(s) - q_n\alpha_n^-(s)x}, \quad 0 \leq x \leq 1.$$

From Theorem 2.2, one sees that $\sigma_n^+(s) = g(\sigma_{n-1}^+(s), s)$ so that

$$(4.14) \quad \sigma_n^+(s) - \sigma_{n-1}^+(s) = \int_{\sigma_{n-2}^+(s)}^{\sigma_{n-1}^+(s)} \frac{d}{dx} g(x, s) dx.$$

By differentiating (4.13) with respect to x , one has

$$(4.15) \quad \frac{d}{dx} g_n(x, s) = \frac{p_n\alpha_n^+(s)q_n\alpha_n^-(s)}{\{1 - r_n\alpha_n^0(s) - q_n\alpha_n^-(s)x\}^2}.$$

Since the domain of $\frac{d}{dx}g_n(x, s)$ is $0 \leq x \leq 1$, its range is given by

$$(4.16) \quad \frac{p_n \alpha_n^+(s) q_n \alpha_n^-(s)}{\{1 - r_n \alpha_n^0(s)\}^2} \leq \frac{d}{dx} g_n(x, s) \leq \frac{p_n \alpha_n^+(s) q_n \alpha_n^-(s)}{\{1 - r_n \alpha_n^0(s) - q_n \alpha_n^-(s)\}^2} \stackrel{\text{def}}{=} \theta_n(s).$$

It can be readily seen that $\theta_n(s)$ is monotone decreasing for real $s > 0$, and therefore $\theta_n(s) < \theta_n(0) \stackrel{\text{def}}{=} \theta_n$. From (4.14) it then follows that

$$\begin{aligned} |\sigma_n^+(s) - \sigma_{n-1}^+(s)| &= \left| \int_{\sigma_{n-2}^+(s)}^{\sigma_{n-1}^+(s)} \frac{d}{dx} g(x, s) dx \right| \\ &\leq \theta_n(s) |\sigma_{n-1}^+(s) - \sigma_{n-2}^+(s)| \\ &< \theta_n |\sigma_{n-1}^+(s) - \sigma_{n-2}^+(s)|. \end{aligned}$$

Under the conditions of (4.1), one has $\theta_n = \theta_n(0) = p_n q_n / (1 - r_n - q_n)^2 = q_n / p_n \rightarrow 1 / \hat{\rho} \stackrel{\text{def}}{=} \theta$ as $n \rightarrow \infty$. Since $\hat{\rho} > 1$, for sufficiently small $\varepsilon > 0$, there exists $N(\varepsilon)$ such that $\theta - \varepsilon < \theta_n < \theta + \varepsilon < 1$ for $n > N(\varepsilon)$. This implies, for $n > N(\varepsilon)$, that

$$|\sigma_n^+(s) - \sigma_{n-1}^+(s)| < (\theta + \varepsilon) |\sigma_{n-1}^+(s) - \sigma_{n-2}^+(s)|.$$

From the contraction mapping theorem, it then follows that $\sigma_n^+(s)$ converges uniquely to a function of s denoted by $\sigma^+(s)$ as $n \rightarrow \infty$. It remains to show that $\sigma^+(s)$ satisfies (4.12).

From Theorem 2.2, the unique convergence of $\sigma_n^+(s)$ to $\sigma^+(s)$ implies that $\sigma^+(s)$ has to satisfy

$$\sigma^+(s) = \frac{p \alpha^+(s)}{1 - r \alpha^0(s) - q \alpha^-(s) \sigma^+(s)},$$

which in turn leads to the following functional equation.

$$(4.17) \quad q \alpha^-(s) \sigma^+(s)^2 - \{1 - r \alpha^0(s)\} \sigma^+(s) + p \alpha^+(s) = 0.$$

This quadratic equation has two solutions given by

$$\frac{1 - r \alpha^0(s) \pm \sqrt{\{1 - r \alpha^0(s)\}^2 - 4 p q \alpha^+(s) \alpha^-(s)}}{2 q \alpha^-(s)}.$$

Since $\sigma^+(0) = 1$, one concludes that

$$\sigma^+(s) = \frac{1 - r \alpha^0(s) - \sqrt{\{1 - r \alpha^0(s)\}^2 - 4 p q \alpha^+(s) \alpha^-(s)}}{2 q \alpha^-(s)},$$

completing the proof. \square

We next derive the mean and variance of T^+ based on Theorem 4.6. For notational convenience, we denote the limits of the first and second moments given in (2.7) as

$$(4.18) \quad \begin{aligned} \hat{\mu}_k^- &\stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \mu_{n:k}^-; \quad \hat{\mu}_k^0 \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \mu_{n:k}^0; \quad \hat{\mu}_k^+ \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \mu_{n:k}^+; \\ \hat{\mu}_k &= \hat{\mu}_k^- + \hat{\mu}_k^0 + \hat{\mu}_k^+, \quad \text{for } k = 1, 2. \end{aligned}$$

It is assumed that

$$(4.19) \quad \hat{\mu}_k < \infty, \text{ for } k = 1, 2.$$

Theorem 4.7

$$(4.20) \text{ (a)} \quad \mathbb{E}[T^+] = \frac{\hat{\mu}_1}{p - q}$$

$$(4.21) \text{ (b)} \quad \text{Var}[T^+] = \frac{\hat{\mu}_2}{p - q} + \left\{ \frac{2p(q\hat{\mu}_1^+ + r\hat{\mu}_1^0 + p\hat{\mu}_1^-)}{(p - q)^3} - \frac{\hat{\mu}_1}{(p - q)^2} + \frac{2\hat{\mu}_1^-}{p - q} \right\} \hat{\mu}_1$$

Proof. Let $C(s)$ be defined as

$$(4.22) \quad C(s) = 1 - r\alpha^0(s) - \sqrt{\{1 - r\alpha^0(s)\}^2 - 4pq\alpha^+(s)\alpha^-(s)},$$

so that $\sigma^+(s) = C(s)/2q\alpha^-(s)$. One easily sees that

$$(4.23) \quad C(0) = -2q,$$

$$(4.24) \quad \left. \frac{d}{ds} C(s) \right|_{s=0} = \frac{-2q(p\hat{\mu}_1^+ + r\hat{\mu}_1^0 + p\hat{\mu}_1^-)}{p - q},$$

and

$$(4.25) \quad \left(\frac{d}{ds} \right)^2 C(s) \Big|_{s=0} = \frac{-2q(p\hat{\mu}_2^+ + r\hat{\mu}_2^0 + p\hat{\mu}_2^-)}{p - q} + \frac{4pq\hat{\mu}_1(q\hat{\mu}_1^+ + r\hat{\mu}_1^0 + p\hat{\mu}_1^-)}{(p - q)^3}.$$

The theorem now follows by differentiating $\log \sigma^+(s) = \log C(s) - \log 2q\alpha^-(s)$ twice at $s = 0$. \square

In order to prove the counterpart of Theorem 4.2 (b) for skip-free semi-Markov processes, a preliminary lemma is needed.

Lemma 4.8

Under the condition of (4.19), the following statement holds true.

$$\frac{1}{n} \mathbb{E}[T_{0,n}] \rightarrow \frac{\hat{\mu}_1}{p - q} \text{ as } n \rightarrow \infty.$$

Proof. From Theorem 4.7 (a), for arbitrarily small $\varepsilon > 0$, there exists $M(\varepsilon)$ such that $|\mathbb{E}[T_n^+] - \hat{\mu}_1/(p - q)| < \varepsilon$ for $n > M(\varepsilon)$. From (2.17), it can be seen that

$$\frac{1}{n} \mathbb{E}[T_{0,n}] = \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{E}[T_j^+] = \frac{1}{n} \sum_{j=0}^{M(\varepsilon)} \mathbb{E}[T_j^+] + \frac{1}{n} \sum_{j=M(\varepsilon)+1}^{n-1} \mathbb{E}[T_j^+],$$

so that one has, for $n > M(\varepsilon) + 1$,

$$(4.26) \quad \frac{1}{n} \sum_{j=0}^{M(\varepsilon)} E[T_j^+] + \frac{n - M(\varepsilon) - 2}{n} \left(\frac{\hat{\mu}_1}{p - q} - \varepsilon \right) < \frac{1}{n} \sum_{j=0}^{n-1} E[T_j^+] \\ < \frac{1}{n} \sum_{j=0}^{M(\varepsilon)} E[T_j^+] + \frac{n - M(\varepsilon) - 2}{n} \left(\frac{\hat{\mu}_1}{p - q} + \varepsilon \right),$$

and hence $(1/n)E[T_{0,n}] \rightarrow \hat{\mu}_1/(p - q)$ as $n \rightarrow \infty$, completing the proof. \square

The following theorem can now be proven.

Theorem 4.9

$$\frac{T_{0,n}}{E[T_{0,n}]} \rightarrow 1 \text{ with probability 1 as } n \rightarrow \infty.$$

Proof. Let $X_n \stackrel{\text{def}}{=} T_n^+ - E[T_n^+]$ for $n \geq 0$. Clearly X_n are mutually independent and $E[X_n] = 0$ so that $E[X_n^2] = \text{Var}[T_n^+]$ for $n \geq 0$. Under the condition of (4.19), it can be seen from Theorem 4.7 (b) that $\lim_{n \rightarrow \infty} \text{Var}[T_n^+] < \infty$. This then implies that

$$\sum_{m=0}^{\infty} \frac{1}{m^2} E[X_{m-1}^2] < \infty.$$

Hence from VII.8 Theorem 3 of Feller[3], one has

$$\frac{1}{n} \sum_{m=1}^n X_m = \frac{E[T_{0,n}]}{n} \left\{ \frac{T_{0,n}}{E[T_{0,n}]} - 1 \right\} \rightarrow 0$$

with probability 1 as $n \rightarrow \infty$. The theorem now follows from Lemma 4.8. \square

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