Department of Social Systems and Management

Discussion Paper Series

No. 1194

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February 2008

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JAPAN

Analysis of a Correlated Multivariate Shock Model Generated from a Renewal Sequence

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Abstract. A correlated multivariate random shock model is considered, where a system is subject to a sequence of K different shocks triggered by a common renewal process. Let $(Y_j)_{j=1}^{\infty}$ be a sequence of independently and identically distributed (i.i.d.) nonnegative random variables associated with the renewal process. For the magnitudes of the jth shock denoted by a random vector $\underline{X}(j)$, it is assumed that $[\underline{X}(j), Y_j](j = 1, 2, \cdots)$ constitute a sequence of i.i.d. random vectors with respect to j while $\underline{X}(j)$ and Y_j may be correlated. The system fails as soon as the historical maximum of the magnitudes of any component of the random vector exceeds a prespecified level of that component. The Laplace transform of the probability density function of the system lifetime is derived, and its mean and variance are obtained explicitly. The model is applied for analyzing the browsing behavior of users of the Internet.
Keywords. Correlated multivariate random shock model, System lifetime, Browsing behavior.

1 Introduction

A general shock model is studied by Shanthikumar and Sumita (1982,1983), where a system is subject to a sequence of random shocks generated by a renewal sequence. More specifically, the model is characterized by a correlated pairs of nonnegative random variables $[X_j, Y_j](j = 1, 2, \cdots)$ where X_j is the magnitude of the jth shock and Y_j describes the time interval between two consecutive shocks. The variates $[X_j, Y_j](j = 1, 2, \cdots)$ are i.i.d. pairwise, while X_j and Y_j may be correlated. The underlying system fails as soon as the magnitude of a shock exceeds a prespecified level. The transform results, an exponential limit theorem and properties of the associated renewal processes of the system failure times are obtained with an application to a stochastic clearing system. The model is extended subsequently by Sumita and Shanthikumar (1984) to incorporate the system lifetime based on the cumulative

shock.
While the general shock model has widened the application areas much beyond the traditional Poisson shock model, it is still limited in that the model accepts only one type of shocks. In some applications, it is important to deal with multiple types of shocks generated by a common renewal sequence. In analyzing the browsing behavior of users of the Internet, for example, it is common to find a user moving from one website to another in order to gather information about a specific product of his/her interest. Assuming that dwell times at different websites constitute a renewal sequence, the first type of shocks may correspond to the values of information gathered from various websites concerning the product produced by company $C1$, while the second type of shocks may describe those concerning the product produced by company $C2$. The internet search would be terminated when the user obtains enough information to decide which company's product should be purchased.
The purpose of this paper is to extend the general shock model of Shanthikumar and Sumita (1982) so as to incorporate such multiple different random shocks generated from a common renewal sequence.
The structure of the paper is as follows. The correlated multivariate shock model is described in Section 2 and the system lifetime is analyzed in Section 3. An application to analysis of the browsing behavior of users of the Internet is discussed in Section 4. Some numerical examples are also presented.

2 Model description

We consider a system where a sequence of K different shocks are triggered by a common renewal process characterized by a sequence of i.i.d. nonnegative random varibles $(Y_j)_{j=1}^{\infty}$. Let $\underline{X}(j) = [X_1(j), \cdots, X_K(j)]$ be the random vector describing the magnitudes of K different shocks occured at the j-th renewal epoch. Throughout the paper, we assume that all random variables are absolutely continuous with $\underline{X}(j) \in R_+^K$ and $Y_j \in R_+$, where R_+^K is the set of K dimensional nonnegative vectors and R_+ denotes the set of nonnegative real numbers. For notational convenience, we define $\mathcal{K} = \{1, 2, , \cdots, K\}$ and its power set denoted by $\mathcal{B}(\mathcal{K}) = \{A : A \subset \mathcal{K}\}$. In addition, while $\underline{X}(j)$ and Y_j may be correlated, it is assumed that $[\underline{X}(j), Y_j](j = 1, 2, \cdots)$ constitute a sequence of i.i.d. random vectors with respect to j. The joint distribution function and the joint probability density function of $[\underline{X}(j), Y_j]$ are denoted by

$$F_{\underline{X},Y}(\underline{x}, y) = P[\underline{X}(j) < \underline{x}, Y_j \leq y] \tag{1}$$

and

$$f_{\underline{X},Y}(\underline{x}, y) = \frac{\partial^K}{\partial \underline{x}} \frac{\partial}{\partial y} F_{\underline{X},Y}(\underline{x}, y) \stackrel{\text{def}}{=} \frac{\partial^K}{\partial x_1 \cdots \partial x_K} \frac{\partial}{\partial y} F_{\underline{X},Y}(\underline{x}, y) \quad , \tag{2}$$

respectively. We note that the inequality associated with $\underline{X}(j)$ in $F_{\underline{X},Y}(\underline{x}, y)$ is taken to be strict. Since the historical maximum processes are of our main

concern, equalities are attached to tail probabilities for $\underline{X}(j)$ as a general rule in this paper.
Let $N(t)$ be the counting process associated with the renewal sequence $(Y_j)_{j=1}^{\infty}$ and define the historical maximum process $\underline{M}(t)$ by

$$\underline{M}(t) = [M_1(t), \cdots, M_K(t)] \quad ; \quad M_k(t) = \max_{0 \le j \le N(t)} \{X_k(j)\} \quad , \tag{3}$$

where $\underline{X}(0) = \underline{0}$ is employed for notational convenience. The system fails as soon as the historical maximum of any component exceeds a prespecified level of that compontent. More specifically, we define, for $\underline{z} = [z_1, \cdots, z_K] > \underline{0}$,

$$T_{\underline{z}} = inf\{t : M_k(t) > z_k, \quad \text{for any } k \in \mathcal{K}\} \quad . \tag{4}$$

In what follows, we analyze $T_{\underline{z}}$, deriving the transform results and it's mean and variance.

3 Analysis of $T_{\underline{z}}$

Let $[\underline{X}(j), Y_j]$ be a correlated pair of renewal sequences with the common joint distribution function $F_{\underline{X},Y}(\underline{x}, y)$ and the joint probability density function $f_{\underline{X},Y}(\underline{x}, y)$ as in (1) and (2) respectively. Correspondingly, the marginal distribution functions and the marginal density functions are given by

$$F_{\underline{X}}(\underline{x}) = F_{\underline{X},Y}(\underline{x}, \infty) \qquad ; \quad F_Y(y) = F_{\underline{X},Y}(\underline{\infty}, y) \quad , \tag{5}$$

$$f_{\underline{X}}(\underline{x}) = \int_0^{\infty} f_{\underline{X},Y}(\underline{x}, y) dy \quad , \tag{6}$$

$$f_Y(y) = \int_{\underline{0}}^{\underline{\infty}} f_{\underline{X},Y}(\underline{x}, y) d\underline{x} \stackrel{\text{def}}{=} \int_0^{\infty} \cdots \int_0^{\infty} f(\underline{\tau}, y) d\tau_1 \cdots \tau_K \quad . \tag{7}$$

Throughout the paper, we assume that the first two moments of both $\underline{X}(j)$ and Y_j are finite. For notational convenience, the following functions are also introduced:

$$G_{\underline{X}}(\underline{x}, y) = \int_{\underline{0}}^{\underline{x}} f_{\underline{X},Y}(\underline{\tau}, y) d\underline{\tau} \quad ; \quad \overline{G}_{\underline{X}}(\underline{x}, y) = \int_{\underline{x}}^{\underline{\infty}} f_{\underline{X},Y}(\underline{\tau}, y) d\underline{\tau} \quad , \tag{8}$$

$$G_Y(\underline{x}, y) = \int_0^{y} f_{\underline{X},Y}(\underline{x}, w) dw \quad ; \quad \overline{G}_Y(\underline{x}, y) = \int_y^{\infty} f_{\underline{X},Y}(\underline{x}, w) dw \quad . \tag{9}$$

We now define the distribution functions of $\underline{M}(t)$ and $T_{\underline{z}}$ denoted by

$$V(t, \underline{z}) = P[\underline{M}(t) < \underline{z}] \quad ; \quad W_{\underline{z}}(t) = P[T_{\underline{z}} \le t] \quad . \tag{10}$$

Lapalace transforms with respect to t are denoted by a circumflex, i.e.,

$$\hat{V}(s, \underline{z}) = \int_0^{\infty} e^{-st} V(t, \underline{z}) dt \quad ; \quad \hat{w}_{\underline{z}}(s) = \int_0^{\infty} e^{-st} dW_{\underline{z}}(t) \quad . \tag{11}$$

One then easily sees that there exists a dual relationship between $\underline{M}(t)$ and $T_{\underline{z}}$ specified by

$$V(t, \underline{z}) = P[\underline{M}(t) < \underline{z}] = P[T_{\underline{z}} > t] = \overline{W}_{\underline{z}}(t) \quad , \tag{12}$$

where $\overline{W}_{\underline{z}}(t) = 1 - W_{\underline{z}}(t)$ is the survival function of $T_{\underline{z}}$. In this section, we derive $\hat{w}_{\underline{z}}(s)$ explicitly based on (12).
In this Model, the system starts anew at time $t = 0$. The magnitudes $\underline{X}(j)$ of the j-th shocks are correlated only to the time interval Y_j since the $(j-1)$st shocks and do not affect the future events. One then has the following theorem.

Theorem 1. *Let $\hat{\varphi}_Y(s)$ be the Laplace transform of $f_Y(t)$ in (7), i.e. $\hat{\varphi}_Y(s) \stackrel{def}{=} \int_0^\infty e^{-st} f_Y(t)dt$. One then has*

$$\hat{V}(\underline{z}, s) = \frac{1 - \hat{\varphi}_Y(s)}{s\{1 - \hat{G}_{\underline{X}}(\underline{z}, s)\}} \quad , \quad Re(s) \geq 0 \quad .$$

Proof. Since $V(\underline{z}, t)$ is the probability that the maximum magnitude of $X_k(j)$ has not exceeded the level z_k for $0 \leq j \leq N(t)$ for all $k \in \mathcal{K}$, by conditioning on the first renewal time Y_1 and using the regenerative property of the paired process $[\underline{X}(j), Y_j]$ at Y_1, one sees that

$$V(\underline{z}, t) = \overline{F}_Y(t) + \int_0^t G_{\underline{X}}(\underline{z}, y) V(\underline{z}, t - y) dy \quad . \tag{13}$$

By taking the Laplace transform of both sides of (13) with respect to t, it can be seen that

$$\hat{V}(\underline{z}, s) = \frac{1 - \hat{\varphi}_Y(s)}{s} + \hat{G}_{\underline{X}}(\underline{z}, s)\hat{V}(\underline{z}, s) \quad .$$

This equation can be solved for $\hat{V}(\underline{z}, s)$ as $\hat{V}(\underline{z}, s) = \frac{1-\hat{\varphi}_Y(s)}{s\{1-\hat{G}_{\underline{X}}(\underline{z},s)\}}$, completing the proof.

The system failure time $T_{\underline{z}}$ has the dual relation to $\underline{M}(t)$ given in (12). The Laplace transform $\hat{w}_{\underline{z}}(s) = E[e^{-sT_{\underline{z}}}]$ is then easily found from Theorem 1.

Theorem 2.

$$\hat{w}_{\underline{z}}(s) = \frac{\hat{\varphi}_Y(s) - \hat{G}_{\underline{X}}(\underline{z}, s)}{1 - \hat{G}_{\underline{X}}(\underline{z}, s)} = \frac{\hat{\overline{G}}_{\underline{X}}(\underline{z}, s)}{1 - \hat{G}_{\underline{X}}(\underline{z}, s)} \quad , \quad Re(s) \geq 0 \quad .$$

Proof. From (12), one finds that $\hat{V}(\underline{z}, s) = \frac{1-\hat{w}_{\underline{z}}(s)}{s}$, so that $\hat{w}_{\underline{z}}(s) = 1 - s\hat{V}(\underline{z}, s)$. The theorem now follows from Theorem 1.

By differentiating $\hat{w}_{\underline{z}}(s)$ at $s = 0$, the mean and the variance of $T_{\underline{z}}$ can be obtained.

Corollary 1.

$a)\ E[T_{\underline{z}}] = \dfrac{E[Y]}{1 - F_{\underline{X}}(\underline{z})}$

$b)\ Var[T_{\underline{z}}] = \dfrac{E[Y^2]}{1 - F_{\underline{X}}(\underline{z})} + \dfrac{E[Y]}{(1 - F_{\underline{X}}(\underline{z}))^2}\{2F_{\underline{X}}(\underline{z})E[Y|\underline{X} < \underline{z}] - E[Y]\}$

4 Application to analysis of the browsing behavior of users of the Internet

We suppose that a consumer visits various websites in order to gather information about a product. Let $X_{1:j}$ be the value of information about the product produced by company $C1$ that the consumer gains from the jth search with length of Y_j, and $X_{2:j}$ is defined similarly for the product produced by company $C2$. We assume that both $X_{1:j}$ and $X_{2:j}$ consist of two parts: a part independent of Y_j and another part proportional to Y_j. The former parts for $X_{1:j}$ and $X_{2:j}$ are denoted by $\hat{X}_{1:j}$ and $\hat{X}_{2:j}$ respectively. More formally, we define

$$X_1 = \hat{X}_1 + \alpha_1 Y \quad ; \quad X_2 = \hat{X}_2 + \alpha_2 Y \quad . \tag{14}$$

We assume that $\hat{X}_{1:j}$, $\hat{X}_{2:j}$ and Y_j constitute three independent renewal sequences with respect to j having the following exponential density functions:

$$f_{\hat{X}_1}(\hat{x}_1) = \mu_1 e^{-\mu_1 \hat{x}_1}; \quad f_{\hat{X}_2}(\hat{x}_2) = \mu_1 e^{-\mu_2 \hat{x}_2}; \quad f_Y(y) = \lambda e^{-\lambda y}. \tag{15}$$

While $\hat{X}_{1:j}$ and $\hat{X}_{2:j}$ are assumed to be independent, $X_{1:j}$ and $X_{2:j}$ are not independent because of sharing the common value of Y_j. Let $F_{\underline{X},Y}(\underline{x}, y) = P[\underline{X}(j) < \underline{x}, Y_j \leq y]$. By conditioning on Y, one finds that

$$F_{\underline{X},Y}(\underline{x}, y) = \int_0^y F_{\hat{X}_1}(x_1 - \alpha_1\tau)F_{\hat{X}_2}(x_2 - \alpha_2\tau)f_Y(\tau)d\tau \cdot I, \tag{16}$$

where $I = I\{0 \leq y \leq min\{\frac{x_1}{\alpha_1}, \frac{x_2}{\alpha_2}\}\}$, and $F_{\hat{X}_1}(x)$ and $F_{\hat{X}_2}(x)$ are the distribution functions of $\hat{X}_1$ and $\hat{X}_2$ respectively. From (2), it then follows that

$$f_{\underline{X},Y}(\underline{x}, y) = f_{\hat{X}_1}(x_1 - \alpha_1 y)f_{\hat{X}_2}(x_2 - \alpha_2 y)f_Y(y) \cdot I. \tag{17}$$

Suppose that the consumer will stop the search process whenever the desired information for either company, specified by z_1 or z_2, is obtained. Then from Theorem 2 and Equations (14)$\sim$ (17), we have

$$\hat{w}_{\underline{z}}(s) = \frac{\hat{\overline{G}}_{\underline{X}}(\underline{z}, s)}{1 - \hat{G}_{\underline{X}}(\underline{z}, s)} = \left\{1 + \frac{\frac{s}{s+\lambda}}{A + B - C + D}\right\}^{-1}, \tag{18}$$

where, by using $z^* \stackrel{\text{def}}{=} min\{\frac{z_1}{\alpha_1}, \frac{z_2}{\alpha_2}\}$, A,B,C and D are given by

$$A = \frac{\lambda e^{-\mu_2 z_2}(1 - e^{-(s+\lambda-\mu_1\alpha_1)z^*})}{s+\lambda-\mu_1\alpha_1}; \quad B = \frac{\lambda e^{-\mu_1 z_1}(1 - e^{-(s+\lambda-\mu_2\alpha_2)z^*})}{s+\lambda-\mu_2\alpha_2}$$
$$C = \frac{\lambda e^{-(\mu_1 z_1+\mu_2 z_2)}(1 - e^{-(s+\lambda-\mu_1\alpha_1-\mu_2\alpha_2)z^*})}{s+\lambda-\mu_1\alpha_1-\mu_2\alpha_2}; \quad D = \frac{\lambda e^{-(s+\lambda)z^*}}{s+\lambda}.$$

The mean $E[T_{\underline{z}}]$ can be computed from Corollary 1, as depicted in Figure 1. We note that the monotonicity of $E[T_{\underline{z}}]$ in $\underline{z}$ can be observed.

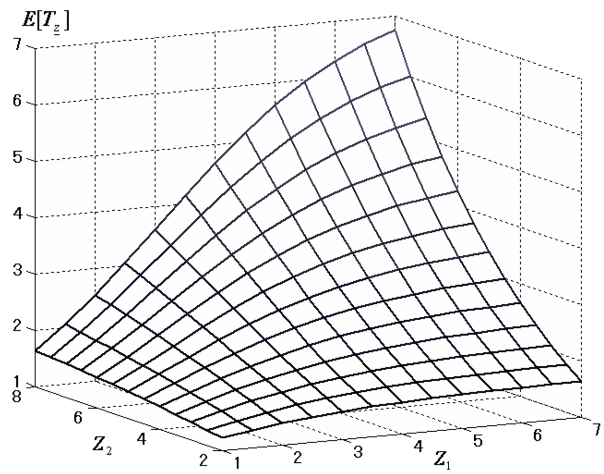

Fig. 1. Mean Search Time:$\alpha_1 = 1, \alpha_2 = 1, \mu_1 = 0.6, \mu_2 = 0.25, \lambda = 1$

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