Department of Social Systems and Management

Discussion Paper Series

No. 1192

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by

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February 2008

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JAPAN

Analysis of a Multivariate Counting Process Generated from an Age-dependent Non-homogeneous Poisson Process Defined on a Finite Semi-Markov Process

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Abstract. We consider a multivariate counting process generated from an age-dependent non-homogeneous Poisson process defined on a finite semi-Markov process, generalizing many existing counting processes of importance. The dynamic behavior of the multivariate counting process is captured through analysis of the underlying Laplace transform generating functions. Some asymptotic results are also obtained.

Keywords. Multivariate counting process, non-homogeneous Poisson process, semi-Markov process, asymptotic behavior.

1 Introduction

A stochastic process $\{N(t) : t \geq 0\}$ is called a counting process when $N(t)$ is non-negative, right continuous and monotone non-decreasing with $N(0) = 0$. The classic counting processes of importance include a Poisson process, a non-homogeneous Poisson process (NHPP) and a renewal process. More sophisticated counting processes have been developed in order to accommodate a wider range of applications. In Masuda and Sumita (1987), the number of entries of a semi-Markov process into a subset of the state space is analyzed. An age-dependent counting process generated from a renewal process is studied by Sumita and Shanthikumar (1988), which describes items arriving according to an NHPP which is interrupted and reset at random epochs governed by a renewal process. The purpose of this paper is to develop a multivariate counting process which would contain these two processes as special cases. The dynamic behavior of the proposed multivariate counting process is captured through analysis of the underlying Laplace transform generating functions. Some asymptotic results are also obtained.

2 A multivariate counting process $[\underline{M}(t),\ \underline{\underline{N}}(t)]$

We consider a system where items arrive according to an NHPP. This arrival stream is interrupted from time to time where the interruptions are governed by a finite semi-Markov process $J(t)$ on $\mathcal{J} = \{0, 1, 2, \cdots, J\}$. Whenever a state transition of the semi-Markov process occurs from i to j, the intensity function of the NHPP is switched from $\lambda_i(x)$ to $\lambda_j(x)$ with an initial value reset to $\lambda_j(0)$. In other words, the arrivals of items are generated by the NHPP with $\lambda_i(x)$ when the semi-Markov process is in state i with x denoting the time since the last entry into state i. Of particular interest in analysis of such systems are the multivariate counting processes $\underline{M}(t) = [M_i(t)]_{i\in\mathcal{J}}$ and $\underline{\underline{N}}(t) = [N_{ij}(t)]_{i,j\in\mathcal{J}}$ where $M_i(t)$ counts the cumulative number of items that have arrived in $[0, t]$ while the semi-Markov process is in state i and $N_{ij}(t)$ represents the cumulative number of the state transitions of the semi-Markov process from i to j in $[0, t]$. The two counting processes $\underline{M}(t)$ and $\underline{\underline{N}}(t)$ enable one to introduce a variety of interesting performance indicators as we will see.

Formally, let $\{J(t) : t \geq 0\}$ be a semi-Markov process on $\mathcal{J} = \{0, \cdots, J\}$ governed by a matrix cumulative distribution function (c.d.f.)

$$\underline{\underline{A}}(x) = [A_{ij}(x)] \,,$$

which is assumed to be absolutely continuous with the matrix probability density function (p.d.f.) $\underline{\underline{a}}(x) = [a_{ij}(x)] = \frac{d}{dx}\underline{\underline{A}}(x)$. It should be noted that, if we define $A_i(x)$ and $\bar{A}_i(x)$ by $A_i(x) = \sum_{j\in\mathcal{J}} A_{ij}(x)$ and $\bar{A}_i(x) = 1 - A_i(x)$ respectively, then $A_i(x)$ is an ordinary c.d.f. and $\bar{A}_i(x)$ is the corresponding survival function. The hazard rate functions associated with the semi-Markov process are defined as $\eta_{ij}(x) = a_{ij}(x)/\bar{A}_i(x),\ i, j \in \mathcal{J}$.

For notational convenience, the transition epochs of the semi-Markov process are denoted by $\tau_n, n \geq 0$, with $\tau_0 = 0$. The age process $X(t)$ associated with the semi-Markov process is then defined as

$$X(t) = t - \max\{\tau_n : 0 \leq \tau_n \leq t\}.$$

At time t with $J(t) = i$ and $X(t) = x$, the intensity function of the NHPP is given by $\lambda_i(x)$. For the cumulative arrival intensity function $L_i(x)$ in state i, one has

$$L_i(x) = \int_0^x \lambda_i(y)dy \ .$$

The probability of observing k arrivals in state i within the current age of x can then be obtained as

$$g_i(x, k) = e^{-L_i(x)}\frac{L_i(x)^k}{k!}, \quad k = 0, 1, 2, \cdots, \quad i \in \mathcal{J}.$$

Of interest are the multivariate counting processes

$$\underline{M}(t) = \left[M_0(t), \cdots, M_J(t)\right]^\top ,$$

where $M_i(t)$ represents the total number of items that have arrived by time t while the semi-Markov process stayed in state i, and

$$\underline{\underline{N}}(t) = \left[N_{ij}(t)\right] ,$$

with $N_{ij}(t)$ denoting the number of transitions of the semi-Markov process from state i to state j by time t. It is obvious that $N_i(t) \stackrel{\text{def}}{=} \sum_{\ell \in \mathcal{J}} N_{\ell i}(t)$ denotes the number of entries into state i by time t. The initial state is not included in $N_i(t)$ for any $i \in \mathcal{J}$. In other words, even if $J(0) = i$, $N_i(t)$ remains 0 until the first return of the semi-Markov process to state i. In the next section, we will analyze the dynamic behavior of $\left[\underline{M}(t), \underline{\underline{N}}(t)\right]$ based on analysis of the underlying Laplace transform generating functions. The associated asymptotic behavior as $t \to \infty$ would also be discussed in the subsequent section.

3 Dynamic analysis of $[\underline{M}(t),\ \underline{\underline{N}}(t)]$

The purpose of this section is to examine the dynamic behavior of the multivariate stochastic process $\left[\underline{M}(t), \underline{\underline{N}}(t)\right]$ by observing its probabilistic flow in its state space. Since $\left[\underline{M}(t), \underline{\underline{N}}(t)\right]$ is not Markov, we employ the method of supplementary variables. More specifically, the multivariate stochastic process $\left[\underline{M}(t),\ \underline{\underline{N}}(t), X(t), J(t)\right]$ is considered. This multivariate stochastic process is Markov and has the state space $S = \mathbb{Z}_+^{J+1} \times \mathbb{Z}_+^{(J+1)\times(J+1)} \times \mathbb{R}_+ \times \mathcal{J}$, where $\mathbb{Z}_+^{J+1}$ is the set of $(J+1)$ dimensional non-negative integer vectors, $\mathbb{Z}_+^{(J+1)\times(J+1)}$ is the set of $(J+1)\times(J+1)$ dimensional non-negative integer matrices, $\mathbb{R}_+$ is the set of non-negative real numbers and $\mathcal{J} = \{0, \cdots, J\}$. Let $F_{ij}(\underline{m}, \underline{\underline{n}}, x, t)$ be the joint probability distribution of $\left[\underline{M}(t), \underline{\underline{N}}(t), X(t), J(t)\right]$ defined by

$$F_{ij}(\underline{m}, \underline{\underline{n}}, x, t) = \\ \mathrm{P}\left[\underline{M}(t) = \underline{m}, \underline{\underline{N}}(t) = \underline{\underline{n}}, X(t) \le x, J(t) = j \mid \underline{M}(0) = \underline{0}, \underline{\underline{N}}(0) = \underline{\underline{0}}, J(0) = i\right] . \quad (1)$$

In order to assure the differenciability of $F_{ij}(\underline{m}, \underline{\underline{n}}, x, t)$ with respect to x, we assume that $X(0)$ has an absolutely continuous initial distribution function $D(x)$ with p.d.f. $d(x) = \frac{d}{dx}D(x)$. (If $X(0) = 0$ with probability 1, we consider a sequence of initial distribution functions $\{D_k(x)\}_{k=1}^{\infty}$ satisfying $D_k(x) \to U(x)$ as $k \to \infty$ where $U(x) = 1$ for $x \ge 0$ and $U(x) = 0$ otherwise. The desired results can be obtained through this limiting process.) One can then define

$$f_{ij}(\underline{m}, \underline{\underline{n}}, x, t) = \frac{\partial}{\partial x} F_{ij}(\underline{m}, \underline{\underline{n}}, x, t) \ . \quad (2)$$

By interpreting the probabilistic flow of the multivariate process $[\underline{M}(t), \underline{\underline{N}}(t), X(t), J(t)]$ in its state space, one can establish the following equations:

$$f_{ij}(\underline{m}, \underline{\underline{n}}, x, t) = \delta_{\{i=j\}}\delta_{\{\underline{m}=m_i\underline{1}_i\}}\delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}}d(x-t)\frac{\bar{A}_i(x)}{\bar{A}_i(x-t)}g_i(t, m_i) + \left(1-\delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}}\right)\sum_{k=0}^{m_j} f_{ij}(\underline{m}-k\underline{1}_j, \underline{\underline{n}}, 0+, t-x)\bar{A}_j(x)g_j(x,k) \ ; \tag{3}$$

$$f_{ij}(\underline{m}, \underline{\underline{n}}, 0+, t) = \left(1-\delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}}\right)\sum_{\ell\in\mathcal{J}}\int_0^\infty f_{i\ell}(\underline{m}, \underline{\underline{n}}-\underline{\underline{1}}_{\ell j}, x, t)\eta_{\ell j}(x)dx \ ; \tag{4}$$

$$f_{ij}(\underline{m}, \underline{\underline{n}}, x, 0) = \delta_{\{i=j\}}\delta_{\{\underline{m}=\underline{0}\}}\delta_{\{\underline{\underline{n}}=\underline{\underline{0}}\}}d(x) \ , \tag{5}$$

where $\underline{1}_i$ is the column vector whose i-th element is equal to 1 with all other elements being 0 , $\underline{\underline{1}}_{ij} = \underline{1}_i\underline{1}_j^\top$ and $f_{ij}(\underline{m}, \underline{\underline{n}}, 0+, t) = 0$ for $\underline{\underline{N}} \leq \underline{\underline{0}}$. In what follows, the dynamic behavior of the multivariate process $[\underline{M}(t), \underline{\underline{N}}(t), X(t), J(t)]$ would be captured by establishing the associated Laplace transform generating functions based on (3), (4) and (5). For notational convenience, the following matrix functions are employed.

$$\underline{\underline{\beta}}(\underline{u}, s) = [\beta_{ij}(u_i, s)] \ ; \ \beta_{ij}(u_i, s) = \sum_{k=0}^\infty \left(\int_0^\infty e^{-st}a_{ij}(t)g_i(t,k)dt\right)u_i^k \ ,$$

$$\underline{\underline{\beta}}_D^*(\underline{u}, s) = \left[\delta_{\{i=j\}}\beta_i^*(u_i, s)\right] \ ; \ \beta_i^*(u_i, s) = \sum_{k=0}^\infty \left(\int_0^\infty e^{-st}\bar{A}_i(t)g_i(t,k)dt\right)u_i^k \ ,$$

$$\hat{\underline{\underline{\xi}}}(\underline{u}, \underline{\underline{v}}, 0+, s) = \left[\hat{\xi}_{ij}(\underline{u}, \underline{\underline{v}}, 0+, s)\right] \ ;$$

$$\hat{\xi}_{ij}(\underline{u}, \underline{\underline{v}}, 0+, s) = \sum_{\underline{m}\in\mathbb{Z}_+^{J+1}} \sum_{\underline{\underline{n}}\in\mathbb{Z}_+^{(J+1)\times(J+1)}\setminus\{\underline{\underline{0}}\}} \xi_{ij}(\underline{m}, \underline{\underline{n}}, 0+, s)\underline{u}^{\underline{m}}\underline{\underline{v}}^{\underline{\underline{n}}} \ ,$$

$$\hat{\underline{\underline{\varphi}}}(\underline{u}, \underline{\underline{v}}, w, s) = \left[\hat{\varphi}_{ij}(\underline{u}, \underline{\underline{v}}, w, s)\right] \ ;$$

$$\hat{\varphi}_{ij}(\underline{u}, \underline{\underline{v}}, w, s) = \sum_{\underline{m}\in\mathbb{Z}_+^{J+1}} \sum_{\underline{\underline{n}}\in\mathbb{Z}_+^{(J+1)\times(J+1)}} \varphi_{ij}(\underline{m}, \underline{\underline{n}}, w, s)\underline{u}^{\underline{m}}\underline{\underline{v}}^{\underline{\underline{n}}} \ ,$$

where $\underline{u}^{\underline{m}}\underline{\underline{v}}^{\underline{\underline{n}}} = \prod_{i\in\mathcal{J}} u_i^{m_i} \prod_{(i,j)\in\mathcal{J}\times\mathcal{J}\setminus\{(0,0)\}} v_{ij}^{n_{ij}}$. Then the next theorem can be proven by taking Laplace transforms of (3), (4) and (5).

Theorem 1. *Let $X(0) = 0$. Then:*

$$\hat{\underline{\underline{\xi}}}(\underline{u}, \underline{\underline{v}}, 0+, s) = \tilde{\underline{\underline{\beta}}}(\underline{u}, \underline{\underline{v}}, s)\left\{\underline{\underline{I}} - \tilde{\underline{\underline{\beta}}}(\underline{u}, \underline{\underline{v}}, s)\right\}^{-1} \ ; \tag{6}$$

$$\hat{\underline{\underline{\varphi}}}(\underline{u}, \underline{\underline{v}}, w, s) = \left\{\underline{\underline{I}} - \tilde{\underline{\underline{\beta}}}(\underline{u}, \underline{\underline{v}}, s)\right\}^{-1}\underline{\underline{\beta}}_D^*(\underline{u}, w+s) \ , \tag{7}$$

where $\tilde{\underline{\underline{\beta}}}(\underline{u}, \underline{\underline{v}}, s) = [v_{ij}\cdot\beta_{ij}(u_i, s)]$.

4 Asymptotic behavior

Let $\mathcal{A}, \mathcal{M}$ and $\mathcal{N}$ be arbitrary subsets of the state space $\mathcal{J}$ of the underlying semi-Markov process, and define

$$M_{\mathcal{A}}(t) = \sum_{i \in \mathcal{A}} M_i(t) \; ; \quad N_{\mathcal{M}\mathcal{N}}(t) = \sum_{i \in \mathcal{M}} \sum_{j \in \mathcal{N}} N_{ij}(t) \; ,$$

where $M_{\mathcal{A}}(t)$ describes the total number of items that have arrived in $[0, t]$ according to the non-homogeneous Poisson processes within $\mathcal{A}$ and $N_{\mathcal{M}\mathcal{N}}(t)$ denotes the number of transitions from any state in $\mathcal{M}$ to any state in $\mathcal{N}$ in $[0, t]$. An appropriate choice of $\mathcal{A}, \mathcal{M}$ and $\mathcal{N}$ would then enable one to analyze processes of interest in a variety of applications. It may be also of interest to define

$$S(t) = cM_{\mathcal{A}}(t) + dN_{\mathcal{M}\mathcal{N}}(t) \; ,$$

often representing the total cost. The moment asymptotic behaviors of $M_{\mathcal{A}}(t)$, $N_{\mathcal{M}\mathcal{N}}(t)$ and $S(t)$ are now derived based on Theorem 1.

For notational simplicity, we introduce the following vectors and matrices. Let $\mathcal{A}, \mathcal{M}$ and $\mathcal{N} \subset \mathcal{J}$ with their compliments defined respectively by $\mathcal{A}^C = \mathcal{J} \setminus \mathcal{A}$, $\mathcal{M}^C = \mathcal{J} \setminus \mathcal{M}$ and $\mathcal{N}^C = \mathcal{J} \setminus \mathcal{N}$. The cardinality of a set A is denoted by $|A|$. Submatrices of $\underline{\underline{A}} \in \mathbb{R}^{(J+1)\times(J+1)}$ are denoted by

$$\underline{\underline{A}}_{\mathcal{A}\bullet} = \left[A_{ij}\right]_{i \in \mathcal{A}, j \in \mathcal{J}} \in \mathbb{R}^{|\mathcal{A}|\times(J+1)} \; ; \quad \underline{\underline{A}}_{\bullet\mathcal{A}} = \left[A_{ij}\right]_{i \in \mathcal{J}, j \in \mathcal{A}} \in \mathbb{R}^{(J+1)\times|\mathcal{A}|} \; ;$$
$$\underline{\underline{A}}_{\mathcal{M}\mathcal{N}} = \left[A_{ij}\right]_{i \in \mathcal{M}, j \in \mathcal{N}} \in \mathbb{R}^{|\mathcal{M}|\times|\mathcal{N}|} \; ,$$

so that one has

$$\underline{\underline{A}} = \begin{bmatrix} \underline{\underline{A}}_{\mathcal{M}\mathcal{N}} & \underline{\underline{A}}_{\mathcal{M}\mathcal{N}^C} \\ \underline{\underline{A}}_{\mathcal{M}^C\mathcal{N}} & \underline{\underline{A}}_{\mathcal{M}^C\mathcal{N}^C} \end{bmatrix} \; ,$$

with understanding that the states are arranged appropriately.

Let $\underline{\underline{A}}_k = \int_0^\infty x^k \underline{\underline{a}}(x) dx$, $k = 0, 1, 2, \cdots$. Throughout the paper, we assume that $||\underline{\underline{A}}_k|| < \infty$ for $0 \le k \le 2$. In particular, one has $\underline{\underline{A}}_0 = \underline{\underline{A}}(\infty)$ which is stochastic. Let $\underline{e}^\top$ be the normalized left eigenvector of $\underline{\underline{A}}_0$ associated with eigenvalue 1 so that $\underline{e}^\top \underline{\underline{A}}_0 = \underline{e}^\top$ and $\underline{e}^\top \underline{1} = 1$ where $\underline{1} = [1, \cdots, 1]^\top \in \mathbb{R}^{J+1}$. Similarly, we define $\underline{\underline{\Phi}}_{r:k} = \int_0^\infty t^k \underline{\underline{L}}_D^r(t) \underline{\underline{a}}(t) dt$ and $\underline{\underline{\Phi}}^*_{r:D:k} = \int_0^\infty t^k \underline{\underline{L}}_D^r(t) \underline{\underline{\bar{A}}}_D(t)$ dt, $r = 1, 2$. By using Keilson's Theorem (see, Keilson (1969)), one has the following theorem. The proof is omitted here.

Theorem 2. *Let $\underline{p}^\top(0)$ be an initial probability vector of the underlying semi-Markov process. As $t \to \infty$, one has*

$$\mathrm{E}\big[M_{\mathcal{A}}(t)\big] = \underline{p}^\top(0) \left\{ t \; \underline{\underline{P}}_1 + \underline{\underline{P}}_0 \right\} \underline{1} + o(1) \ , \tag{8}$$

$$\mathrm{Var}\Big[M_{\mathcal{A}}(t)\Big] = t \; \underline{p}^\top(0) \; \underline{\underline{U}}_0 \; \underline{1} + o(t) \ , \tag{9}$$

$$\mathrm{E}\big[N_{\mathcal{MN}}(t)\big] = \underline{p}^\top(0) \left\{ t \; \underline{\underline{Q}}_1 + \underline{\underline{Q}}_0 \right\} \underline{1} + o(1) \ , \tag{10}$$

$$\mathrm{Var}\Big[N_{\mathcal{MN}}(t)\Big] = t \; \underline{p}^\top(0) \; \underline{\underline{V}}_0 \; \underline{1} + o(t) \ , \tag{11}$$

$$\mathrm{E}\Big[S(t)\Big] = \underline{p}^\top(0) \left\{ t \left(c \; \underline{\underline{P}}_1 + d \; \underline{\underline{Q}}_1 \right) + c \; \underline{\underline{P}}_0 + d \; \underline{\underline{Q}}_0 \right\} \underline{1} + o(1) \ , \tag{12}$$

$$\mathrm{Var}\Big[S(t)\Big] = t \; \underline{p}^\top(0) \; \underline{\underline{W}}_0 \; \underline{1} + o(t) \ , \tag{13}$$

where

$$\underline{\underline{H}}_1 = \frac{1}{m} \; \underline{1} \; \underline{e}^\top, \quad m = \underline{e}^\top \underline{\underline{A}}_1 \underline{1} \ , \quad \underline{\underline{Z}}_0 = \left(\underline{\underline{I}} - \underline{\underline{A}}_0 + \underline{1} \cdot \underline{e}^\top \right)^{-1} \ ,$$

$$\underline{\underline{H}}_0 = \underline{\underline{H}}_1 \left(-\underline{\underline{A}}_1 + \frac{1}{2} \; \underline{\underline{A}}_2 \underline{\underline{H}}_1 \right) + \left(\underline{\underline{Z}}_0 - \underline{\underline{H}}_1 \underline{\underline{A}}_1 \underline{\underline{Z}}_0 \right) \left(\underline{\underline{A}}_0 - \underline{\underline{A}}_1 \underline{\underline{H}}_1 \right) + \underline{\underline{I}} \ ,$$

$$\underline{\underline{P}}_1 = \underline{\underline{H}}_{1:\bullet\mathcal{A}} \underline{\underline{\Phi}}_{1:0:\mathcal{A}\bullet} \ , \quad \underline{\underline{P}}_0 = \underline{\underline{H}}_{0:\bullet\mathcal{A}} \underline{\underline{\Phi}}_{1:0:\mathcal{A}\bullet} - \underline{\underline{H}}_{1:\bullet\mathcal{A}} \underline{\underline{\Phi}}_{1:1:\mathcal{A}\bullet} + \underline{\underline{H}}_{1:\bullet\mathcal{A}} \underline{\underline{\Phi}}^*_{1:D:0:\mathcal{A}\bullet} \ ,$$

$$\underline{\underline{P}}_2 = \underline{\underline{H}}_{1:\bullet\mathcal{A}} \underline{\underline{\Phi}}_{2:0:\mathcal{A}\bullet} \ , \quad \underline{\underline{\hat{P}}}_0 = \underline{\underline{H}}_{0:\bullet\mathcal{A}} \underline{\underline{\Phi}}_{1:0:\mathcal{A}\bullet} - \underline{\underline{H}}_{1:\bullet\mathcal{A}} \underline{\underline{\Phi}}_{1:1:\mathcal{A}\bullet} \ ,$$

$$\underline{\underline{Q}}_1 = \underline{\underline{H}}_{1:\bullet\mathcal{M}} \left[\underline{\underline{A}}_{0:\mathcal{MN}}, \; \underline{\underline{0}}_{\mathcal{MN}^C} \right] \ ,$$

$$\underline{\underline{Q}}_0 = \underline{\underline{H}}_{0:\bullet\mathcal{M}} \left[\underline{\underline{A}}_{0:\mathcal{MN}}, \; \underline{\underline{0}}_{\mathcal{MN}^C} \right] - \underline{\underline{H}}_{1:\bullet\mathcal{M}} \left[\underline{\underline{A}}_{1:\mathcal{MN}}, \; \underline{\underline{0}}_{\mathcal{MN}^C} \right] \ ,$$

$$\underline{\underline{U}}_0 = 2 \; \underline{\underline{P}}_1 \underline{\underline{P}}_0 + \underline{\underline{P}}_1 - \underline{\underline{P}}_1 \underline{1} \cdot \underline{p}^\top(0) \underline{\underline{P}}_0 + 2 \; \underline{\underline{\hat{P}}}_0 \underline{\underline{P}}_1 - \underline{\underline{P}}_0 \underline{\underline{P}}_1 + \underline{\underline{P}}_2 \ ;$$

$$\underline{\underline{V}}_0 = 2 \; \underline{\underline{Q}}_1 \underline{\underline{Q}}_0 + \underline{\underline{Q}}_1 - \underline{\underline{Q}}_1 \underline{1} \cdot \underline{p}^\top(0) \underline{\underline{Q}}_0 + \underline{\underline{Q}}_0 \underline{\underline{Q}}_1 \ ,$$

$$\underline{\underline{T}}_0 = \underline{\underline{\hat{P}}}_0 \underline{\underline{Q}}_1 + \underline{\underline{P}}_1 \underline{\underline{Q}}_0 + \underline{\underline{H}}_{1:\bullet,(\mathcal{A}\cap\mathcal{M})} \left[\underline{\underline{\Phi}}_{1:0:(\mathcal{A}\cap\mathcal{M}),\mathcal{N}} \ , \; \underline{\underline{0}}_{(\mathcal{A}\cap\mathcal{M}),\mathcal{N}^C} \right] + \underline{\underline{Q}}_0 \underline{\underline{P}}_1 + \underline{\underline{Q}}_1 \underline{\underline{P}}_0 \ ,$$

$$\underline{\underline{W}}_0 = c^2 \; \underline{\underline{U}}_0 + d^2 \; \underline{\underline{V}}_0 + 2cd \; \underline{\underline{T}}_0 - 2cd \left(\underline{\underline{P}}_1 \underline{1} \cdot \underline{p}^\top(0) \underline{\underline{Q}}_0 + \underline{\underline{P}}_0 \underline{1} \cdot \underline{p}^\top(0) \underline{\underline{Q}}_1 \right) \ .$$

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