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A New Approach for Computing Option Prices of the Hull-White Type with Stepwise Reversion and Volatility Functions

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A New Approach for Computing Option Prices of the Hull-White Type with Stepwise Reversion and Volatility Functions

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Abstract

The Hull-White model is a one factor Markov model well known for its capability to capture the current term structure of interest rates. Analytical results are available for pricing zero-coupon discount bonds and associated European options when both reversion and volatility functions are constant. In reality, however, these functions do vary over time. It is then of practical interest to develop efficient computational algorithms that can deal with time dependent reversion and volatility functions. The purpose of this paper is to achieve this goal via the Ehrenfest approximation of the underlying O-U process, where the time dependent structure is represented by step functions. Based on the convergence theorem by Sumita, Gotoh and Jin [8] and the uniformization procedure of Keilson [5], a novel approach is proposed to evaluate the prices of zero-coupon discount bonds and associated European options for stepwise reversion and volatility functions. The ordinary Hull-White trinomial tree approach is also modified to cope with this case for comparison purposes. However, it is shown that the modified trinomial tree approach is not applicable for certain step functions, while the Ehrenfest approach can always be used for any step functions. Numerical results are given, demonstrating the excellent speed and accuracy of the Ehrenfest approach.

Keywords Hull-White model, stepwise reversion and volatility functions, uniformization procedure, Ehrenfest approach, modified trinomial tree approach

0 Introduction

The Hull-White model is a one factor term structure model characterized by a stochastic differential equation of the form

$$\mathrm{d}R(t) = (\phi(t) - \alpha(t)R(t))\mathrm{d}t + \sigma(t)\mathrm{d}W(t), \tag{0.1}$$

where $R(t)$ is a random short rate, $W(t)$ is the standard Wiener process, $\phi(t)$ is a market fitting function, $\alpha(t) \geq 0$ is a reversion function and $\sigma(t) \geq 0$ is a volatility function. Its probability density function $f(r,t) = \frac{\mathrm{d}}{\mathrm{d}r}\mathrm{P}\left[R(t) \leq r\right]$ satisfies the Kolmogorov forward diffusion equation given by

$$\frac{\partial}{\partial t}f(r,t) = \frac{1}{2}\sigma^2(t)\frac{\partial^2}{\partial r^2}f(r,t) - (\phi(t) - \alpha(t)r)\frac{\partial}{\partial r}f(r,t) + \alpha(t)f(r,t). \tag{0.2}$$

A basic function describing this process is the conditional transition probability density $g(r_0, r, t) \stackrel{\text{def.}}{=} \frac{\mathrm{d}}{\mathrm{d}r}\mathrm{P}\left[R(t) \le r | R(0) = r_0\right]$ given by

$$g(r_0, r, t) = \frac{1}{\sqrt{2\pi\sigma^2(t|r_0)}} \exp\left\{\frac{(r - \mu(t|r_0))^2}{2\sigma(t|r_0)}\right\}, \tag{0.3}$$

where

$$\begin{cases} \mu(t|r_0) & = E[R(t)|R(0) = r_0] & = r_0 \exp\{-\int_0^t \alpha(y)dy\} + \int_0^t \phi(\tau) \exp\{-\int_\tau^t \alpha(y)dy\}d\tau, \\ \sigma^2(t|r_0) & = Var[R(t)|R(0) = r_0] & = \int_0^t \sigma(\tau)^2 \exp\{-2\int_\tau^t \alpha(y)dy\}d\tau. \end{cases} \tag{0.4}$$

The Hull-White model is well known for its capability to capture the current term structure of interest rates, e.g. the current yield curve when the yield curve is flat. Because of this reason, the Hull-White model is considered to be one of the most reasonable models for pricing interest rate options. When both the reversion function $\alpha(t)$ and the volatility function $\sigma(t)$ are constant, Hull and White [1] show explicit formulas for evaluating the prices of zero-coupon discount bonds and associated European options. However, when $\alpha(t)$ and $\sigma(t)$ are time-dependent, $R(t)$ cannot be evaluated explicitly and it becomes impossible to obtain the desired prices directly from those formulas.

In reality, the reversion function $\alpha(t)$ and the volatility function $\sigma(t)$ are dependent on time t and often nonlinear. As in the theory of integrals, one may then approximate such nonlinear functions by step functions. The purpose of this paper is to develop a new computational algorithm for evaluating the prices of zero-coupon discount bonds and associated European options when both $\alpha(t)$ and $\sigma(t)$ are step functions. The parameter $\phi(t)$ relates to the yield curve. The time dependent structure of $\phi(t)$, however, does not affect the proposed approach in any essential way. Accordingly, in order to focus on the stepwise reversion and volatility functions, and avoid the yield curve fitting, only the case of constant $\phi(t)$ is considered in this paper. A remark will be provided, regarding how the described numerical procedures should be altered when $\phi(t)$ is not constant.

For the case of constant $\phi(t)$, $\alpha(t)$ and $\sigma(t)$, the Hull-White model is reduced to the Vasicek model [9] specified by

$$\mathrm{d}\widehat{X}_{\mathrm{OU}}(t) = (\phi - \alpha\widehat{X}_{\mathrm{OU}}(t))\mathrm{d}t + \sigma\mathrm{d}W(t). \tag{0.5}$$

The associated probability density function $\hat{f}(x,t) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}\left[\widehat{X}_{\mathrm{OU}}(t) \le x\right]$ satisfies

$$\frac{\partial}{\partial t}\hat{f}(x,t) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\hat{f}(x,t) - (\phi - \alpha x)\frac{\partial}{\partial x}\hat{f}(x,t) + \alpha\hat{f}(x,t), \tag{0.6}$$

and the conditional transition probability density $\hat{g}(\hat{x}_0, x, t) \stackrel{\text{def.}}{=} \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}\left[\widehat{X}_{\mathrm{OU}}(t) \le x | \widehat{X}_{\mathrm{OU}}(0) = \hat{x}_0\right]$ is given by

$$\hat{g}(\hat{x}_0, x, t) = \frac{1}{\sqrt{2\pi\hat{\sigma}^2(t|\hat{x}_0)}} \exp\left\{\frac{(x - \hat{\mu}(t|\hat{x}_0))^2}{2\hat{\sigma}(t|\hat{x}_0)}\right\}, \tag{0.7}$$

where

$$\begin{cases} \hat{\mu}(t|\hat{x}_0) & = E[\widehat{X}_{\mathrm{OU}}(t)|\widehat{X}_{\mathrm{OU}}(0) = \hat{x}_0] & = \frac{\phi}{\alpha}(1 - e^{-\alpha t}) + \hat{x}_0 e^{-\alpha t}, \\ \hat{\sigma}(t|\hat{x}_0) & = Var[\widehat{X}_{\mathrm{OU}}(t)|\widehat{X}_{\mathrm{OU}}(0) = \hat{x}_0] & = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}). \end{cases} \tag{0.8}$$

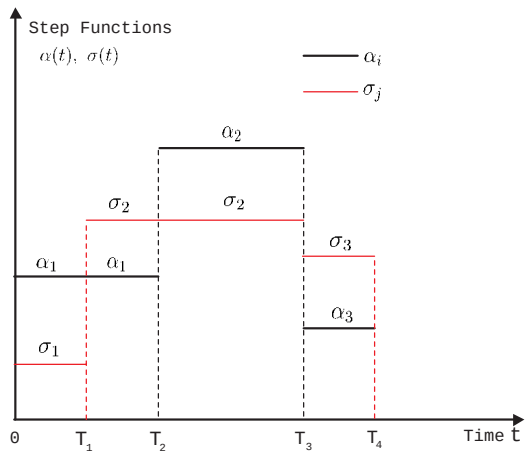


Figure 0.1: Illustration of Stepwise Reversion and Volatility Functions

If $\alpha(t)$ and $\sigma(t)$ are step functions, the process $R(t)$ can be expressed by patching a sequence of $\widehat{X}_{\mathrm{OU}}(t)$'s, each defined in a time interval in which both $\alpha(t)$ and $\sigma(t)$ are constant. Figure 0.1 illustrates a stepwise reversion function and a stepwise volatility function treated in this paper. The idea is to define the time intervals in an exclusive and exhaustive manner so that both the reversion function and the volatility function remain constant within each interval. For the example given in Figure 0.1, the reversion function takes three different values α_i $(i = 1, 2, 3)$ while the volatility function assumes another three different values σ_j $(j = 1, 2, 3)$, resulting in four time intervals. Consequently, the process $R(t)$ can be constructed by patching four different $\widehat{X}^i_{\mathrm{OU}}(t)$ $(i = 1, 2, 3, 4)$. More specifically, one has

$$R(t) = \begin{cases} \widehat{X}^1_{\mathrm{OU}}(t): & \mathrm{d}\widehat{X}^1_{\mathrm{OU}}(t) = (\phi - \alpha_1 \widehat{X}^1_{\mathrm{OU}}(t))\mathrm{d}t + \sigma_1 \mathrm{d}W(t), \quad 0 \le t < T_1, \\ \widehat{X}^2_{\mathrm{OU}}(t): & \mathrm{d}\widehat{X}^2_{\mathrm{OU}}(t) = (\phi - \alpha_1 d\widehat{X}^2_{\mathrm{OU}}(t))\mathrm{d}t + \sigma_2 \mathrm{d}W(t), \quad T_1 \le t < T_2, \\ \widehat{X}^3_{\mathrm{OU}}(t): & \mathrm{d}\widehat{X}^3_{\mathrm{OU}}(t) = (\phi - \alpha_2 d\widehat{X}^3_{\mathrm{OU}}(t))\mathrm{d}t + \sigma_2 \mathrm{d}W(t), \quad T_2 \le t < T_3, \\ \widehat{X}^4_{\mathrm{OU}}(t): & \mathrm{d}\widehat{X}^4_{\mathrm{OU}}(t) = (\phi - \alpha_3 d\widehat{X}^4_{\mathrm{OU}}(t))\mathrm{d}t + \sigma_3 \mathrm{d}W(t), \quad T_3 \le t \le T_4, \end{cases} \tag{0.9}$$

where the conditional time dependent probability density at the end of one interval becomes the initial probability density of the following interval.

In order to facilitate the idea above, it is essential to develop efficient numerical procedures for evaluating the dynamic behavior of $\widehat{X}_{\mathrm{OU}}(t)$ of the Vasicek model with speed and accuracy. In addition, the whole computational procedures ought to be automated because of repeated patching. The proposed approach relies upon the convergence in law of a sequence of Ehrenfest processes to an O-U process proven by Sumita, Gotoh and Jin [8] (hereafter SGJ [8]). By employing the uniformization procedure of Keilson [5] for the underlying Ehrenfest processes, the proposed approach enables one to evaluate the prices of zero-coupon discount bonds and relevant European options associated with a short rate process $R(t)$ with stepwise reversion and volatility functions.

While the trinomial tree procedure of Hull and White [2, 3] is originally designed for constant reversion and volatility functions, it is possible to modify it so as to deal with stepwise reversion and volatility functions, which we call the modified trinomial tree approach. (A succinct summary of the Hull-White trinomial tree approach and its modification is given in

Appendix A.) In comparison with the proposed approach, however, the modified trinomial tree approach has two essential pitfalls.

1) For both the proposed approach and the modified trinomial tree approach, it is necessary to discretize the time axis, say with a unit of Δt. Because of the use of Ehrenfest processes, our approach enables one to evaluate the conditional time dependent probability density where the state transition during Δt could occur from any state to any other state governed by the conditional probability density. On the other hand, in the modified trinomial tree approach, the state change in Δt involves only adjacent states. Accordingly, when the volatility is rather high, the modified trinomial approach may require much smaller Δt, possibly causing a substantial computational burden.

2) In the proposed approach, the level of discretization of the state space can be chosen arbitrarily. While more accurate discretization does increase the computational burden, it does not affect the automated mechanism of the computational procedures in any way. In contrast, the modified trinomial tree approach requires the level of discretization of the state space to be determined by satisfying a certain inequality so that transition probabilities to adjacent states remain positive. As discussed above, when both $\alpha(t)$ and $\sigma(t)$ are step functions, the conditional time dependent probability density at the end of one interval becomes the initial probability density of the following interval. Hence, the level of discretization of the state space has to be the same throughout all the intervals. This may hinder the modified trinomial tree approach substantially since there is no guarantee that the necessary inequalities are satisfied simultaneously throughout all the intervals, while making the level of discretization the same. This limitation makes it difficult to automate the whole numerical procedures based on the modified trinomial tree approach.

As we will see, numerical experiments reveal that the proposed approach provides a better accuracy in a consistent manner than the Hull-White trinomial tree approach for the case of constant $\alpha(t)$ and $\sigma(t)$ for which the prices of zero-coupon discount bonds and associated European options are known. For the case of stepwise reversion and volatility functions, the proposed approach also seems to be more attractive for practical use than the modified trinomial tree approach because of the two reasons stated above. Indeed, an example will be given for which the modified trinomial tree approach is not applicable while the proposed approach can handle it without any difficulty.

The structure of this paper is as follows. In Section 1, the O-U process $\widehat{X}_{\mathrm{OU}}(t)$ associated with the Vasicek model is first expressed using another O-U process $\widetilde{X}_{\mathrm{OU}}(t)$ and a deterministic shift function $\theta(t)$. Based on the result of SGJ [8], it is shown that a sequence of certain Ehrenfest processes converges in law to the process $\{\widetilde{X}_{\mathrm{OU}}(t) : t \geq 0\}$ with appropriate shifting and scaling as the state space size of the Ehrenfest processes goes to infinity. Using the uniformization procedure of Keilson [5], a numerical algorithm is then developed to compute the time dependent transition probability matrix of the Ehrenfest process, which in turn enables one to approximate the conditional transition probability density of $\widehat{X}_{\mathrm{OU}}(t)$ systematically.

Section 2 is devoted to development of computational procedures for evaluating the prices of zero-coupon discount bonds and associated European options for the Vasicek model with $\alpha(t) = \alpha$ and $\sigma(t) = \sigma$ based on the new approach. Computational results are then compared with the numerical results of the trinomial tree approach and the analytically known results.

It is found that the computational results by both the Ehrenfest approach and the Hull-White trinomial tree approach converge to the analytical solution monotonically in K where the discount bond maturity is $T = K\Delta t = 5$. For the prices of zero-coupon discount bonds, the accuracy of the absolute relative errors for both the Ehrenfest approach and the Hull-White trinomial tree approach is around 0.3% at $K = 120$. Although the former performs slightly but consistently better than the latter, and this performance difference is enlarged when α and σ become large, the actual difference is minimal. In case of the prices of the associated European call options, the performance characteristics of the two approaches remain to be similar with the accuracy of the absolute relative errors at $K = 120$ being around 0.4%.

Section 3 deals with the case of stepwise reversion and volatility functions. Computational algorithms are developed by patching the numerical procedures of Section 2 based on the idea discussed in this section. The modified trinomial tree approach is also constructed for accommodating this case. Numerical results of the two different approaches are presented, where the absolute relative errors of the two approaches are contained within 0.4% for the range of parameter values as well as different levels of discretization of time axis and the state space considered in this paper. While the exact solutions are not available, the fact that the relative errors are bounded by 0.4% indicates the trustworthiness of the new proposed approach and the modified trinomial tree approach. A numerical example is given for which the modified trinomial tree approach collapses but our proposed approach can handle without any difficulty. Because of this reason and the ease for automating the patching procedure, the proposed approach seems to be more attractive for practical use than the modified trinomial tree approach. Finally in Section 4, some concluding remarks are given.

For notational convenience, throughout the paper, we denote a vector by attaching a single underline as $\underline{x}$, and a matrix by attaching double underlines as $\underline{\underline{a}}$. Moreover, $\underline{1}$ means a vector whose elements are all 1 and the vector $\underline{u}_m$ means that its element corresponding to state m is 1 and all other elements are 0.

1 Development of Computational Algorithms for Analyzing Dynamic Behaviors of O-U Processes Associated with the Vasicek Model

Let $\{\widetilde{X}_{\mathrm{OU}}(t) : t \geq 0\}$ be an O-U process associated with the Vasicek model with $\phi(t) = 0$. From (0.5), it can be readily seen that

$$\mathrm{d}\widetilde{X}_{\mathrm{OU}}(t) = -\alpha\widetilde{X}_{\mathrm{OU}}(t)\mathrm{d}t + \sigma\mathrm{d}W(t). \tag{1.1}$$

The relationship between $\widehat{X}_{\mathrm{OU}}(t)$ with $\phi(t) = \phi \neq 0$ and $\widetilde{X}_{\mathrm{OU}}(t)$ with $\phi(t) = 0$ can be given as

$$\widehat{X}_{\mathrm{OU}}(t) = \widetilde{X}_{\mathrm{OU}}(t) + \theta(t), \text{ with } \widetilde{X}_{\mathrm{OU}}(0) = 0, \tag{1.2}$$

where

$$\theta(t) \stackrel{\mathrm{def}}{=} \frac{\phi}{\alpha}(1 - \mathrm{e}^{-\alpha t}) + \widehat{X}_{\mathrm{OU}}(0)\mathrm{e}^{-\alpha t}. \tag{1.3}$$

The relationship in (1.2) is exploited by Kijima and Nagayama [6] where a better version of the Hull-White trinomial tree approach is proposed by using the shift function $\theta(t)$ explicitly.

Because of the two reasons stated in Section 0, this paper proposes a new approach for capturing dynamic behavior of $\widetilde{X}_{\mathrm{OU}}(t)$ based on the Ehrenfest approximation of SGJ [8]. Through (1.2), the proposed approach enables one to deal with dynamic behavior of $\widetilde{X}_{\mathrm{OU}}(t)$, thereby providing a computational vehicle to evaluate the prices of zero-coupon discount bonds and associated European options. This section describes the computational algorithms necessary for this purpose.

The conditional probability density function $\tilde{g}(\tilde{x}_0, x, t) \stackrel{\text{def.}}{=} \dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}\left[\widetilde{X}_{\mathrm{OU}}(t) \le x | \widetilde{X}_{\mathrm{OU}}(0) = \tilde{x}_0\right]$ of $\{\widetilde{X}_{\mathrm{OU}}(t) : t \ge 0\}$ is given by

$$\tilde{g}(\tilde{x}_0, x, t) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2(t|\tilde{x}_0)}} \exp\left\{\frac{(x - \tilde{\mu}(t|\tilde{x}_0)^2}{2\tilde{\sigma}(t|\tilde{x}_0)}\right\}, \tag{1.4}$$

where

$$\begin{cases} \tilde{\mu}(t|\tilde{x}_0) & = E[\widetilde{X}_{\mathrm{OU}}(t)|\widetilde{X}_{\mathrm{OU}}(0) = \tilde{x}_0] & = \tilde{x}_0 \mathrm{e}^{-\alpha t}, \\ \tilde{\sigma}^2(t|\tilde{x}_0) & = Var[\widetilde{X}_{\mathrm{OU}}(t)|\widetilde{X}_{\mathrm{OU}}(0) = \tilde{x}_0] & = \frac{\sigma^2}{2\alpha}(1 - \mathrm{e}^{-2\alpha t}). \end{cases} \tag{1.5}$$

The corresponding Laplace transform with respect to x is

$$\tilde{\gamma}_{\mathrm{OU}}(\tilde{x}_0, w, t) = \exp\left\{-\tilde{\mu}(t|\tilde{x}_0)w + \frac{1}{2}\tilde{\sigma}^2(t|\tilde{x}_0)\, w^2\right\}. \tag{1.6}$$

The first step for establishing the necessary computational algorithms is to show the convergence in law of a certain sequence of Ehrenfest processes to $\{\widetilde{X}_{\mathrm{OU}}(t) : t \ge 0\}$ based on SGJ [8].

Let $\{\, N_{2V}(t) : t \ge 0 \,\}$ be a continuous time Markov chain on $\mathcal{N}_V = \{0, 1, ..., 2V\}$ governed by the upward transition rates λ_n and the downward transition rates μ_n, where

$$\lambda_n = \frac{1}{2}(2V - n)\,; \quad \mu_n = \frac{n}{2}, \quad n \in \mathcal{N}_V. \tag{1.7}$$

It is shown in SGJ [8] that a sequence of Ehrenfest processes with appropriate scaling and shifting given by

$$X_V(t) \stackrel{\text{def}}{=} \sqrt{\frac{2}{V}}\, N_{2V}(t) - \sqrt{2V}, \quad V = 1, 2, 3, \cdots \tag{1.8}$$

converges in law to the O-U process $\{X_{\mathrm{OU}}(t) : t \ge 0\}$ as $V \to \infty$, where $X_{\mathrm{OU}}(t)$ is a special case of $\widetilde{X}_{\mathrm{OU}}(t)$ with $\alpha = 1$ and $\sigma = \sqrt{2}$. More specifically, one has

$$\varphi_{X_V}(x_0, w, t) \longrightarrow \gamma_{\mathrm{OU}}(x_0, w, t) \text{ as } V \to \infty \text{ for all } \ t \ge 0, \tag{1.9}$$

where

$$\begin{cases} \varphi_{X_V}(x_0, w, t) = E[\mathrm{e}^{-wX_V(t)}|X_V(0) = \sqrt{\frac{2}{V}}\eta_V(x_0)], \\ \gamma_{\mathrm{OU}}(x_0, w, t) = E[\mathrm{e}^{-wX_{\mathrm{OU}}(t)}|X_{\mathrm{OU}}(0) = x_0] = \exp\{-\mu(t|x_0)w + \frac{1}{2}\sigma^2(t|x_0)w^2\}, \end{cases} \tag{1.10}$$

$$\mu(t|x_0) = x_0\mathrm{e}^{-t};\ \sigma^2(t|x_0) = 1 - \mathrm{e}^{-2t} \text{ and } \eta_V(x) = \left\lceil \sqrt{\frac{V}{2}}x \right\rceil, \tag{1.11}$$

and $\lceil a \rceil$ denotes the minimum integer which is greater than or equal to a.

For $\alpha > 0$, let $\left\{ \widetilde{N}_{2V}(t) \,:\, t \geq 0 \right\}$ be defined by

$$\widetilde{N}_{2V}(t) \stackrel{\text{def}}{=} N_{2V}(\alpha t). \tag{1.12}$$

Corresponding to (1.8), for $\alpha > 0$ and $\sigma > 0$, we define $\left\{ \widetilde{X}_V(t) \,:\, t \geq 0 \right\}$ by

$$\widetilde{X}_V(t) \stackrel{\text{def}}{=} \frac{\sigma}{\sqrt{\alpha V}} \widetilde{N}_{2V}(t) - \sigma\sqrt{\frac{V}{\alpha}}\,, \quad V = 1, 2, 3, \cdots. \tag{1.13}$$

It should be noted that the process $\left\{ \widetilde{X}_V(t) \,:\, t \geq 0 \right\}$ has a discrete support on $\{ \tilde{x}_V(0), ..., \tilde{x}_V(2V) \}$ where

$$\tilde{x}_V(n) = \frac{\sigma}{\sqrt{\alpha V}}\, n - \sigma\sqrt{\frac{V}{\alpha}}, \quad n = 0, 1, 2, \cdots, 2V. \tag{1.14}$$

When the initial conditions are adjusted appropriately, $\widetilde{X}_V(t)$ converges in law to $\widetilde{X}_{\text{OU}}(t)$ as $V \to \infty$ as we prove next.

Theorem 1.1 *Let $\left\{ \widetilde{X}_{\text{OU}}(t) : t \geq 0 \right\}$ be the O-U process specified by (1.1) with $\widetilde{X}_{\text{OU}}(0) = \tilde{x}_0$, $-\infty < \tilde{x}_0 < \infty$ and let $\left\{ \widetilde{X}_V(t) : t \geq 0 \right\}$ be as in (1.13) with $\widetilde{X}_V(0) = \frac{\sigma}{\sqrt{\alpha V}}\, \tilde{\eta}_V(\tilde{x}_0)$ where $\tilde{\eta}_V(x) \stackrel{\text{def}}{=} \left\lceil \frac{\sqrt{\alpha V}}{\sigma} x \right\rceil$ and V is chosen large enough so that $-\sigma\sqrt{\frac{V}{\alpha}} \leq \widetilde{X}_V(0) \leq \sigma\sqrt{\frac{V}{\alpha}}$. Then $\widetilde{X}_V(t)$ converges in law to $\widetilde{X}_{\text{OU}}(t)$ for all $t \geq 0$, as $V \to \infty$.*

Proof Let $\tilde{\varphi}_{\widetilde{X}_V}(\tilde{x}_0, w, t) \stackrel{\text{def}}{=} E[\mathrm{e}^{-w\widetilde{X}_V(t)} | \widetilde{X}_V(0) = \frac{\sigma}{\sqrt{\alpha V}}\, \tilde{\eta}_V(\tilde{x}_0)]$. We wish to show that $\tilde{\varphi}_{\widetilde{X}_V}(\tilde{x}_0, w, t) \to \tilde{\gamma}_{\text{OU}}(\tilde{x}_0, w, t)$ as $V \to \infty$, where $\tilde{\gamma}_{\text{OU}}(\tilde{x}_0, w, t)$ is as given in (1.6). From (1.8), (1.12) and (1.13), one sees that $\widetilde{X}_V(t) = \dfrac{\sigma}{\sqrt{2\alpha}} X_V(\alpha t)$ so that

$$\begin{aligned} \tilde{\varphi}_{\widetilde{X}_V}(\tilde{x}_0, w, t) &= E[\mathrm{e}^{-w\frac{\sigma}{\sqrt{2\alpha}} X_V(\alpha t)} | X_V(0) = \sqrt{\frac{2}{V}}\eta_V(\frac{\sqrt{2\alpha}}{\sigma}\tilde{x}_0)] \\ &= \varphi_{X_V}(\frac{\sqrt{2\alpha}}{\sigma}\tilde{x}_0, \frac{\sigma}{\sqrt{2\alpha}} w, \alpha t). \end{aligned} \tag{1.15}$$

Accordingly, one has

$$\left\{ \begin{array}{ll} E[\widetilde{X}_V(t) | \widetilde{X}_V(0) = \frac{\sigma}{\sqrt{\alpha V}}\, \tilde{\eta}_V(\tilde{x}_0)] & = \frac{\sigma}{\sqrt{2\alpha}} E[X_V(\alpha t) | X_V(0) = \sqrt{\frac{2}{V}}\eta_V(\frac{\sqrt{2\alpha}}{\sigma}\tilde{x}_0)], \\ Var[\widetilde{X}_V(t) | \widetilde{X}_V(0) = \frac{\sigma}{\sqrt{\alpha V}}\, \tilde{\eta}_V(\tilde{x}_0)] & = \frac{\sigma^2}{2\alpha} Var[X_V(\alpha t) | X_V(0) = \sqrt{\frac{2}{V}}\eta_V(\frac{\sqrt{2\alpha}}{\sigma}\tilde{x}_0)]. \end{array} \right. \tag{1.16}$$

From the convergence in law of (1.9), it can be seen that

$$\left\{ \begin{array}{lcl} \frac{\sigma}{\sqrt{2\alpha}} E[X_V(\alpha t) | X_V(0) = \sqrt{\frac{2}{V}}\eta_V(\frac{\sqrt{2\alpha}}{\sigma}\tilde{x}_0)] & \to & \frac{\sigma}{\sqrt{2\alpha}} \mu(\alpha t | \frac{\sqrt{2\alpha}}{\sigma}\tilde{x}_0), \\ \frac{\sigma^2}{2\alpha} Var[X_V(\alpha t) | X_V(0) = \sqrt{\frac{2}{V}}\eta_V(\frac{\sqrt{2\alpha}}{\sigma}\tilde{x}_0)] & \to & \frac{\sigma^2}{2\alpha} \sigma^2(\alpha t | \frac{\sqrt{2\alpha}}{\sigma}\tilde{x}_0), \end{array} \right. \tag{1.17}$$

as $V \to \infty$. On the other hand, one finds from (1.5) and (1.11) that the mean and the variance of $\{\widetilde{X}_{\text{OU}}(t) : t \geq 0\}$ can be related to those of $\{X_{\text{OU}}(\alpha t) : t \geq 0\}$ with $x_0 = \frac{\sqrt{2\alpha}}{\sigma}\tilde{x}_0$ as

$$\left\{ \begin{array}{ll} \tilde{\mu}(t | \tilde{x}_0) & = \frac{\sigma}{\sqrt{2\alpha}} \mu(\alpha t | \frac{\sqrt{2\alpha}}{\sigma}\tilde{x}_0), \\ \tilde{\sigma}^2(t | \tilde{x}_0) & = \frac{\sigma^2}{2\alpha} \sigma^2(\alpha t | \frac{\sqrt{2\alpha}}{\sigma}\tilde{x}_0). \end{array} \right. \tag{1.18}$$

It then follows from (1.16), (1.17) and (1.18) that

$$(1.19)\qquad \begin{cases} E[\widetilde{X}_V(t)|\widetilde{X}_V(0) = \frac{\sigma}{\sqrt{\alpha V}}\,\tilde{\eta}_V(\tilde{x}_0)] & \to \quad \tilde{\mu}(t|\tilde{x}_0), \\ Var[\widetilde{X}_V(t)|\widetilde{X}_V(0) = \frac{\sigma}{\sqrt{\alpha V}}\,\tilde{\eta}_V(\tilde{x}_0)] & \to \quad \tilde{\sigma}^2(t|\tilde{x}_0). \end{cases}$$

Consequently, one sees that

$$\tilde{\varphi}_{\widetilde{X}_V}(\tilde{x}_0, w, t) \to \exp\{-\tilde{\mu}(t|\tilde{x}_0)w + \frac{1}{2}\tilde{\sigma}^2(t|\tilde{x}_0)w^2\} = \tilde{\gamma}_{\mathrm{OU}}(\tilde{x}, w, t)$$

as $V \to \infty$, completing the proof. □

Based on Theorem 1.1, for sufficiently large V, the O-U process $\left\{\widehat{X}_{\mathrm{OU}}(t) : t \geq 0\right\}$ can be approximated by

$$(1.20)\qquad \widehat{X}_{\mathrm{OU}}(t) \approx \widetilde{X}_V(t) + \theta(t),$$

where

$$\begin{cases} \widetilde{X}_V(t) & = & \frac{\sigma}{\sqrt{\alpha V}}\,\widetilde{N}_{2V}(t) - \sigma\sqrt{\frac{V}{\alpha}}, \\ \theta(t) & = & \frac{\phi}{\alpha}(1 - \mathrm{e}^{-\alpha t}) + \widehat{X}_{\mathrm{OU}}(0)\mathrm{e}^{-\alpha t}. \end{cases}$$

The state conversions among $\{\widetilde{N}_{2V}(t) : t \geq 0\}$, $\{\widetilde{X}_V(t) : t \geq 0\}$, $\{\widetilde{X}_{\mathrm{OU}}(t) : t \geq 0\}$ and $\{\widehat{X}_{\mathrm{OU}}(t) : t \geq 0\}$ are summarized in Table 1.1.

Table 1.1: State Conversions

Process	State Conversion		State Space
	$x \in \mathbb{R} \to m \in \mathcal{N}_V$	$m \in \mathcal{N}_V \to x \in \mathbb{R}$	
$\widetilde{N}_{2V}(t) = N_{2V}(\alpha t)$	$m = \tilde{\eta}_V(\tilde{x}) + V$	m	$\mathcal{N}_V = \{0, 1, ..., 2V\}$
$\widetilde{X}_V(t)$	$\frac{\sigma}{\sqrt{\alpha V}}\tilde{\eta}_V(\tilde{x})$	$\tilde{x}_V(m) = \frac{\sigma}{\sqrt{\alpha V}}m - \sigma\sqrt{\frac{V}{\alpha}}$	$\{-\sigma\sqrt{\frac{V}{\alpha}}, ..., \sigma\sqrt{\frac{V}{\alpha}}\}$
$\widetilde{X}_{\mathrm{OU}}(t)$	$\tilde{x}$	$\tilde{x} = \tilde{x}_V(m)$	$\mathbb{R} = (-\infty, \infty)$
$\widehat{X}_{\mathrm{OU}}(t)$	$x = \tilde{x} + \theta(t)$	$x = \tilde{x}_V(m) + \theta(t)$	$\mathbb{R} = (-\infty, \infty)$

Remark : $\tilde{\eta}_V(\tilde{x}) = \left\lceil \frac{\sqrt{\alpha V}}{\sigma}\tilde{x} \right\rceil$; $\theta(t) = \frac{\phi}{\alpha}(1 - \mathrm{e}^{-\alpha t}) + \widehat{X}_{\mathrm{OU}}(0)\mathrm{e}^{-\alpha t}$.

When $\theta(t)$ is a one-to-one mapping, from (1.2), the transition probabilities of $\widehat{X}_{\mathrm{OU}}(t)$ can be constructed from those of $\widetilde{X}_{\mathrm{OU}}(t)$, which in turn can be approximated by those of $\widetilde{N}_{2V}(t)$ from (1.20). Let $\underline{\underline{\widetilde{\mathrm{P}}}}_{2V}(t) = [\tilde{p}_{2V:mn}(t)]$ $m, n \in \mathcal{N}_V$ be the transition probability matrix of $\{\widetilde{N}_{2V}(t) : t \geq 0\}$ and let $\underline{\tilde{p}}^{\top}_{2V:m}(t) = [\tilde{p}_{2V:m0}(t), \tilde{p}_{2V:m1}(t), \cdots, \tilde{p}_{2V:m,2V}(t)]$ be the corresponding state probability vector starting from state m. In what follows, the uniformization procedure of Keilson [5] is employed for computing $\underline{\underline{\widetilde{\mathrm{P}}}}_{2V}(t)$ and $\underline{\tilde{p}}^{\top}_{2V:m}(t)$ in an efficient manner.

For a continuous time Markov chain $\{N(t) : t \geq 0\}$ defined on $\mathcal{N} = \{0, 1, 2, \cdots, N\}$ governed by $\underline{\underline{\nu}} = \{\nu_{mn}\}$ with $\nu_m \stackrel{\text{def.}}{=} \sum_{n \in \mathcal{N}} \nu_{mn} < \infty$, the transition probability matrix $\underline{\underline{\mathrm{P}}}(t)$ can be given in terms of the infinitesimal generator $\underline{\underline{Q}}$ as

$$(1.21)\qquad \underline{\underline{P}}(t) = \mathrm{e}^{\underline{\underline{Q}}t};\ \underline{\underline{Q}} = -\underline{\underline{\nu}}_D + \underline{\underline{\nu}};\ \underline{\underline{\nu}}_D = diag\{\nu_m\}.$$

For $\nu \geq \max_m \nu_m$, let $\underline{\underline{a}}_\nu$ be a stochastic matrix defined by

$$\underline{\underline{a}}_\nu \stackrel{\text{def}}{=} \underline{\underline{I}} - \frac{1}{\nu}\underline{\underline{\nu}}_D + \frac{1}{\nu}\underline{\underline{\nu}} = \underline{\underline{I}} + \frac{1}{\nu}\underline{\underline{Q}}. \tag{1.22}$$

By substituting (1.22) into (1.21), one then finds that

$$\underline{\underline{P}}(t) = e^{-\nu t[\underline{\underline{I}} - \underline{\underline{a}}_\nu]} = \sum_{k=0}^{\infty} e^{-\nu t}\frac{(\nu t)^k}{k!}\underline{\underline{a}}_\nu^k, \tag{1.23}$$

where $\underline{\underline{a}}_\nu^0 = \underline{\underline{I}}$. For $\{\widetilde{N}_{2V}(t) : t \geq 0\}$, it should be noted that the state space is $\mathcal{N}_V = \{0, 1, ..., 2V\}$ governed by the transition rates λ_n and μ_n, $n \in \mathcal{N}_V$ as given in (1.7). From (1.12), by setting $\nu = V$, Equation (1.23) then becomes

$$\underline{\underline{\widetilde{P}}}_{2V}(t) = \sum_{k=0}^{\infty} e^{-V\alpha t}\frac{(V\alpha t)^k}{k!}\underline{\underline{a}}_V^k, \tag{1.24}$$

where

$$\underline{\underline{a}}_V = \frac{1}{V}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & 2V-1 & 2V \end{array} \\ \begin{pmatrix} 0 & \lambda_0 & 0 & \cdots & 0 & 0 \\ \mu_1 & 0 & \lambda_1 & \cdots & 0 & 0 \\ 0 & \mu_2 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & \lambda_{2V-1} \\ 0 & 0 & 0 & \cdots & \mu_{2V} & 0 \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ 2V-1 \\ 2V \end{array}. \tag{1.25}$$

The time dependent state probability vector $\underline{\tilde{p}}_{2V:m}^\top(t)$ can be evaluated accordingly as

$$\begin{aligned} \underline{\tilde{p}}_{2V:m}^\top(t) &= \underline{u}_m^\top \underline{\underline{\tilde{P}}}_{2V}(t) \\ &= \sum_{k=0}^{\infty} \underbrace{e^{-\alpha V t}\frac{(\alpha V t)^k}{k!}}_{A} \underbrace{(\underline{u}_m^\top \underline{\underline{a}}_V^k)}_{B}, \end{aligned} \tag{1.26}$$

where $\underline{u}_m^\top$ is the unit vector corresponding to state m. For each k in (1.26), the underbraced parts A and B may be calculated separately. The matrix-vector multiplication denoted as B in (1.26) can be reduced to vector-based operations. For any $(2V+1)$-dimensional real vector

$$\underline{x}^\top \stackrel{\text{def.}}{=} [x_0, x_1, x_2, \cdots, x_{2V-2}, x_{2V-1}, x_{2V}],$$

let $\underline{x}^0$ and $\underline{x}^1 \in \mathbb{R}^{2V}$ be defined by

$$\underline{x}^{0\top} \stackrel{\text{def.}}{=} [x_0, x_1, x_2, \cdots, x_{2V-2}, x_{2V-1}] \text{ and } \underline{x}^{1\top} \stackrel{\text{def.}}{=} [x_1, x_2, x_3, \cdots, x_{2V-1}, x_{2V}].$$

For the transition rates described in the form of

$$\underline{\lambda}^\top \stackrel{\text{def.}}{=} [\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{2V-2}, \lambda_{2V-1}] \text{ and } \underline{\mu}^\top \stackrel{\text{def.}}{=} [\mu_1, \mu_2, \mu_3, \cdots, \mu_{2V-1}, \mu_{2V}],$$

one has

$$\underline{x}^\top \underline{\underline{a}}_V = \frac{1}{V}[0, \quad \underline{x}^{0\top} \otimes \underline{\lambda}^\top] + \frac{1}{V}[\underline{x}^{1\top} \otimes \underline{\mu}^\top, \quad 0], \tag{1.27}$$

where $\otimes$ denotes the componentwise multiplications. For the computation of A in (1.26), by taking logarithm as

$$b(V,k,\alpha t) = \log \frac{e^{-\alpha V t}(\alpha V t)^k}{k!} = -\alpha V t + k \log(\alpha V t) - \sum_{m=1}^{k} \log m, \tag{1.28}$$

the following recurrence equation may be exploited:

$$b(V,k,\alpha t) = \log(\alpha V t) - \log k + b(V,k-1,\alpha t). \tag{1.29}$$

The above computational procedure is summarized as Algorithm $B.1$ in Appendix B.

Remark 1.2 It should be noted from (1.20) that the market fitting function $\phi(t)$ appears only in the shift function $\theta(t)$. Hence whether or not $\phi(t)$ is constant does not affect the above computational procedure for evaluating the transition probabilities of $\widehat{X}_{\mathrm{OU}}(t)$ in any essential way, provided that $\theta(t)$ remains a one-to-one mapping. Accordingly, Algorithm B.1 remains intact even when $\phi(t)$ is not constant.

2 Development of Computational Procedures for Evaluating Prices of Discount Bonds and European Options for the Vasicek Model

Using Algorithm $B.1$, we are now in a position to develop computational procedures for evaluating the prices of zero-coupon discount bonds and the prices of European options on the zero-coupon discount bonds for the Vasicek model, where both the reversion function $\alpha(t)$ and the volatility function $\sigma(t)$ are constant. While the analytical results are available in a closed form for this case, the algorithms described in this section provide a computational vehicle for dealing with the Hull-White model with stepwise reversion and volatility functions which is analytically intractable. The numerical results of the algorithms are compared with the analytical results as well as those of the Hull-White trinomial tree approach, demonstrating excellent accuracy of the Ehrenfest approximation involved in the algorithms.

In order to evaluate the prices of the zero-coupon discount bonds, a discrete time backward recursive formula is employed together with the shift function $\theta(t)$ of (1.3). Let Δt be the length of each time step with the maturity time $T = K\Delta t$. Let $D(k,m|K)$ $(0 \le k \le K)$ be the discount bond price at time $\tau = k\Delta t$ at a state corresponding to state m of the underlying Ehrenfest process. For simplicity, this state is called m from now on. One then sees that, for $0 \le k < K$,

$$D(k,m|K) = e^{-r(k,m)\Delta t} \sum_{n \in \mathcal{N}_V} \tilde{p}_{2V:mn}(\Delta t) D(k+1,n|K), \tag{2.1}$$

where $D(K,m|K) = d_m$ for $m \in \mathcal{N}_V$, and

$$r(k,m) = \tilde{x}_V(m) + \theta(k\Delta t) \tag{2.2}$$

with $\tilde{x}_V(m)$ as in (1.14). Normally, one has $\underline{d} = [d_0, \cdots, d_{2V}]^\top = \underline{1}$. In order to prepare for patching different O-U processes over different time intervals to be discussed later, the procedure in (2.1) is defined for an arbitrary positive vector $\underline{d}$. Since $\underline{\underline{\widetilde{\mathrm{P}}}}_{2V}(\Delta t) = [\tilde{p}_{2V:mn}(\Delta t)]$

can be computed using Algorithm $B.1$, the discount bond price $D(0, m|K)$ of interest can be readily evaluated based on (2.1).

We next turn our attention to the prices of European options on the zero-coupon discount bond. Since the prices of put options can be derived from those of call options as shown later, we focus on call options only. For discrete economies, it is well known that the price of any security with known payoffs at time T can be viewed as a portfolio of Arrow-Debreu securities and can be priced as the payoff-weighted sum over all states of the prices of the Arrow-Debreu securities (see e.g. Pelsser [7]). Accordingly, we first evaluate the price of an Arrow-Debreu security. Let $Q_0(k, m)$ be the present value at time $t = 0$ of the Arrow-Debreu security with maturity price of 1 at time $\tau = k\Delta t$ $(k \geq 0)$ and state $m \in \mathcal{N}_V$. Then, given an initial state $m_0 \in \mathcal{N}_V$, one has

$$Q_0(k+1, m) = \sum_{n \in \mathcal{N}_V} e^{-r(k,n)\Delta t} \tilde{p}_{2V:nm}(\Delta t) Q_0(k, n), \tag{2.3}$$

starting with $Q_0(0, m) = \delta_{m_0 m}$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. We note from (1.3) and (2.2) that $\theta(0) = \widehat{X}_{\mathrm{OU}}(0) = r(0, m_0)$ and hence $m_0 = V$ because $\tilde{x}_V(V) = 0$ and $\tilde{x}_V(m) \neq 0$ for $m \neq V$. When different O-U process are patched over different time intervals, Equation (2.3) has to be modified since the transition probability matrix $\underline{\underline{\widetilde{\mathrm{P}}}}_{2V}(\Delta t)$ would change from one time interval to another. In order to accommodate this modification, we define $Q_0(k, m|k_0)$ for $0 < k_0 \leq k$ by

$$Q_0(k+1, m|k_0) = \sum_{n \in \mathcal{N}_V} e^{-r(k,n)\Delta t} \tilde{p}_{2V:nm}(\Delta t) Q_0(k, n|, k_0), \tag{2.4}$$

starting with $[Q_0(k_0, m|k_0)]_{m \in \mathcal{N}_V} = \underline{s} = [s_0, \cdots, s_{2V}]^\top$.

Remark 2.1 It may be worthwhile to note that the discount bond price $D(0, V|K)$ at time $t = 0$ and state $m = V$ can be expressed in terms of $Q_0(K, m)$ as

$$D(0, V|K) = \sum_{m \in \mathcal{N}_V} Q_0(K, m). \tag{2.5}$$

However, one needs to employ (2.1) for computing $D(0, m|K)$ for $m \neq V$.

Let $\pi_c(0|M)$ be the present value at time $t = 0$ of a European call option with maturity at time $\tau_M = M\Delta t$ and the strike price K_S on the zero-coupon discount bond maturing at time $T = K\Delta t$ $(K > M)$. The corresponding payoff function $h_c(D)$ at time $M\Delta t$ and state $m \in \mathcal{N}_V$ can be written as

$$h_c(D(M, m|K)) = [D(M, m|K) - K_S]^+, \tag{2.6}$$

where $[x]^+ \stackrel{\text{def}}{=} \max\{x, 0\}$. It then follows that

$$\pi_c(0|M) = \sum_{m \in \mathcal{N}_V} h_c(D(M, m|K)) Q_0(M, m). \tag{2.7}$$

Using (2.1) and Algorithm $B.1$, $D(M, m|K)$ can be readily computed. $Q_0(M, m)$ can also be obtained from (2.3). Consequently, one can evaluate the price of interest, $\pi_c(0|M)$, from (2.6) and (2.7). These procedures are summarized as Algorithms $B.2$ through $B.4$ in Appendix B, where the backward formula for $D(k, m|K)$ in (2.1) and the forward formula

for $Q_0(k+1, m|k_0)$ in (2.4) can be computed starting from an arbitrary terminal value vector $\underline{d}$ and an arbitrary beginning value vector $\underline{s}$ in Algorithms $B.2$ and $B.3$ respectively.

For the associated European put option, the payoff function $h_p(D)$ is given by

$$h_p(D(M, m|K)) = [K_S - D(M, m|K)]^+. \tag{2.8}$$

Its present value $\pi_p(0|M)$ can be obtained from (2.7) by substituting $h_p(D(M, m|K))$ in place of $h_c(D(M, m|K))$. Alternatively, $\pi_c(0|M)$ and $\pi_p(0|M)$ are related to each other by

$$\pi_c(0|M) - \pi_p(0|M) = D(0, V|K) - K_S D(0, V|M). \tag{2.9}$$

This relationship is known as "put-call parity".

Hull and White [1] show that, with $\sigma(t) = \sigma$ and $\alpha(t) = \alpha$, the price at time t of a zero-coupon discount bond maturing at time T is given by

$$D(t|T) = H_1(t, T)e^{-H_2(t,T)R(t)}, \ 0 \le t < T, \tag{2.10}$$

where

$$\begin{cases} H_2(t, T) & = & \frac{1}{\alpha}(1 - e^{-\alpha(T-t)}), \\ \log H_1(t, T) & = & \frac{1}{2}\int_t^T \sigma^2 H_2^2(u, T)du - \int_t^T \phi(u)H_2(u, T)du. \end{cases}$$

The explicit formula for the price of the associated European call option can then be expressed as

$$\pi_c(0|\tau) = D(0|T)\Phi(d) - KD(0|\tau)\Phi(d - \sigma_F), \quad 0 \le \tau < T, \tag{2.11}$$

where

$$d = \frac{\ln D(0|T) - \ln(KD(0|\tau))}{\sigma_F} + \frac{\sigma_F}{2},$$

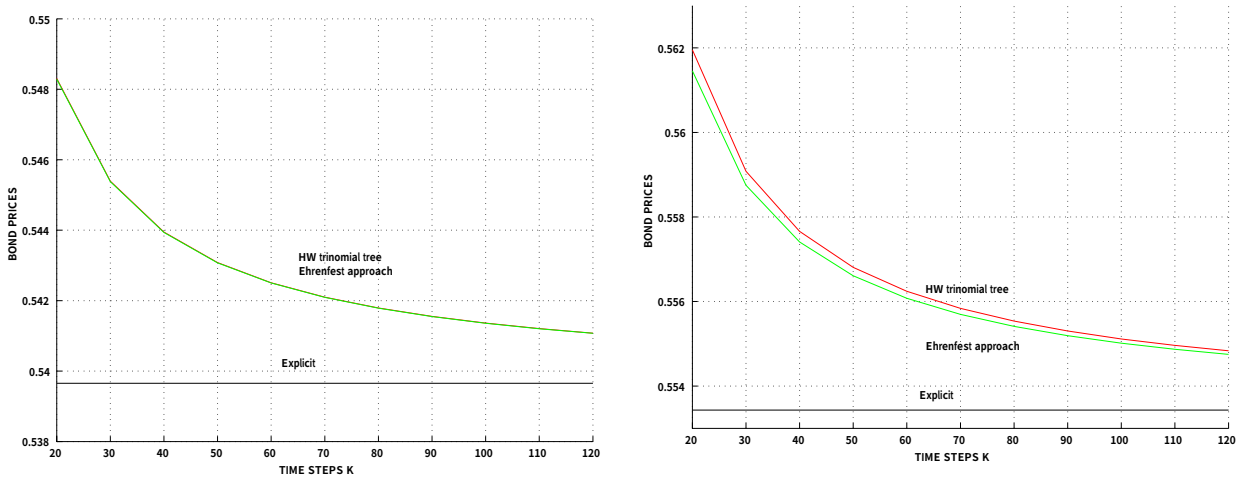
$$\sigma_F^2 = \frac{\sigma^2}{2\alpha}\left(\frac{e^{-\alpha T} - e^{-\alpha\tau}}{\alpha}\right)^2(e^{2\alpha\tau} - 1),$$

and

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}dt, \quad x \in \mathbb{R}.$$

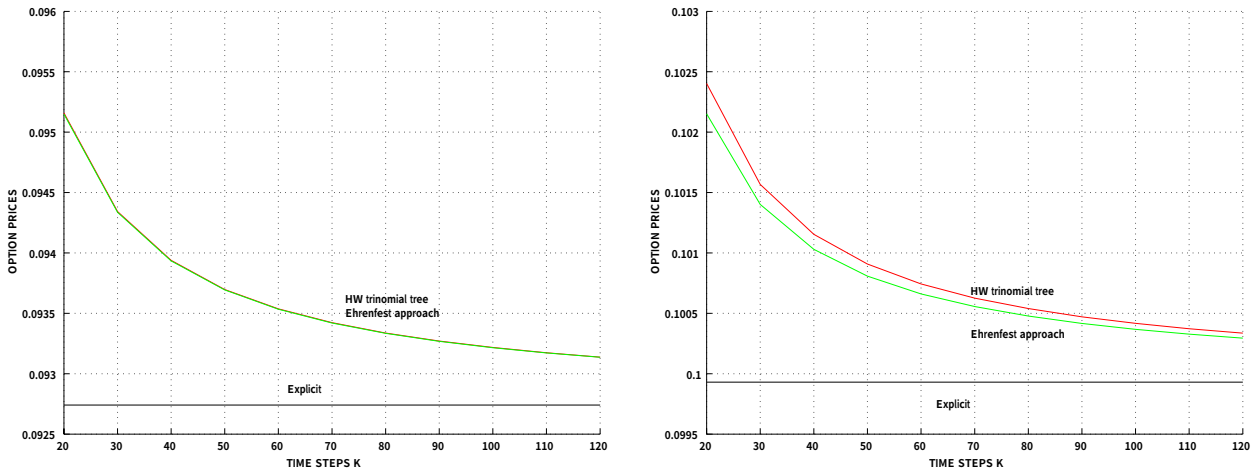
Accuracy of the computational procedures developed in this section can be tested by comparing numerical results with the exact known solutions of (2.10) and (2.11) where $\phi(t) = \phi$. Numerical results based on the Hull-White trinomial tree approach are also obtained, since it will be modified later so as to cope with the case of stepwise reversion and volatility functions for comparison purposes, see Appendix A.

Numerical experiments are implemented for a parameter range of $\alpha = 0.05$ to 0.5 and $\sigma = 0.005$ to 0.05 with $\phi = 0.05$ and $R(0) = 0.05$. Figures 2.1(a) and (b) demonstrate the convergence of zero-coupon discount bond prices as a function of K where $K = T/\Delta t$, with maturity $T = 5$, $\alpha = 0.2$, and $\sigma = 0.01$ for the former and $\sigma = 0.05$ for the latter respectively. In each figure, "Explicit" indicates that the prices are calculated using the explicit formula (2.10), "HW trinomial tree" based on the Hull-White trinomial procedure, and "Ehrenfest approach" employing the new proposed approach. Similarly, Figures 2.2(a) and (b) show



(a) $\phi = 0.05, a = 0.2, \sigma = 0.01$ (b) $\phi = 0.05, a = 0.2, \sigma = 0.05$

Figure 2.1: Convergence Behavior of Discount Bond Prices



(a) $\phi = 0.05, a = 0.2, \sigma = 0.01$ (b) $\phi = 0.05, a = 0.2, \sigma = 0.05$

Figure 2.2: Convergence Behavior of European Call Option Prices

the convergence of the prices of the associated European call options with strike price of $K_S = 0.7$ and maturity at time $\tau_M = M\Delta t = 4$.

From these figures, one sees that the computational results by both the Ehrenfest approach and the Hull-White trinomial tree approach converge to the analytical solution monotonically. The accuracy of the absolute relative error for the Ehrenfest approach is 0.263% for Figure 2.1(a) and 0.238% for Figure 2.1(b) at $K = 120$, while these numbers for the Hull-White trinomial tree approach are 0.264% and 0.253% respectively. While the former performs slightly but consistently better than the latter, and this performance difference is enlarged when α and σ become large, the performance difference is minimal in that the absolute relative errors for the discount bond prices are contained within 2.664% for the Ehrenfest approach and 2.665% for the Hull-White trinomial tree approach throughout the

parameter range of $K = 20$ to 120, $\alpha = 0.05$ to 0.5 and $\sigma = 0.005$ to 0.05 with $\phi = 0.05$ and $R(0) = 0.05$. In case of the prices of the associated European call options, the best accuracy of the Ehrenfest approach is attained at $K = 120$ with accuracy of the absolute relative error of 0.426% for Figure 2.2(a) and 0.364% for Figure 2.2(b). The absolute relative errors within the parameter range mentioned above are bounded by 6.629%. These figures for the Hull-White trinomial tree approach are 0.428%, 0.406% and 6.631% respectively.

3 Pricing Discount Bonds and Associated European Options with Stepwise Reversion and Volatility Functions

When the reversion and volatility functions are dependent on time t, it is necessary to find the short rate $R(t)$ for computing the price at time t of a zero-coupon discount bond based on (2.10). However, $R(t)$ is not known except at the initial time $t = 0$ and the explicit formula in (2.10) cannot be used. It is possible to modify the Hull-White trinomial tree approach so as to deal with stepwise reversion and volatility functions, which we call the modified trinomial tree approach. A succinct summary of how to construct the modified version is given in Appendix A. Because of the reasons stated in Section 0, however, the modified trinomial tree approach is rather limited. The purpose of this section is to establish a new computational approach for computing the prices of discount bonds and associated European options with stepwise reversion and volatility functions based on the Ehrenfest approach described in Sections 1 and 2. For descriptive simplicity, we illustrate the new approach using the example given in Figure 0.1.

The stochastic differential equation corresponding to the example in Figure 0.1 can be written as

$$\mathrm{d}R(t) = (\phi - \alpha(t)R(t))\mathrm{d}t + \sigma(t)\mathrm{d}W(t), \tag{3.1}$$

where

$$\alpha(t) = \begin{cases} \alpha_1, & 0 \le t < T_2, \\ \alpha_2, & T_2 \le t < T_3, \\ \alpha_3, & T_3 \le t \le T_4; \end{cases} \quad \sigma(t) = \begin{cases} \sigma_1, & 0 \le t < T_1, \\ \sigma_2, & T_1 \le t < T_3, \\ \sigma_3, & T_3 \le t \le T_4, \end{cases} \tag{3.2}$$

for $0 < T_1 < T_2 < T_3 < T_4$. As discussed in Section 0, the stochastic process $R(t)$ describing the short rate can then be constructed by patching O-U processes with constant reversion and volatility functions defined on individual time intervals separately. More specifically, one has

$$R(t) = \begin{cases} \widehat{X}^1_{\mathrm{OU}}(t): & \mathrm{d}\widehat{X}^1_{\mathrm{OU}}(t) = (\phi - \alpha_1\widehat{X}^1_{\mathrm{OU}}(t))\mathrm{d}t + \sigma_1\mathrm{d}W(t), & 0 \le t < T_1, \\ \widehat{X}^2_{\mathrm{OU}}(t): & \mathrm{d}\widehat{X}^2_{\mathrm{OU}}(t) = (\phi - \alpha_1 d\widehat{X}^2_{\mathrm{OU}}(t))\mathrm{d}t + \sigma_2\mathrm{d}W(t), & T_1 \le t < T_2, \\ \widehat{X}^3_{\mathrm{OU}}(t): & \mathrm{d}\widehat{X}^3_{\mathrm{OU}}(t) = (\phi - \alpha_2 d\widehat{X}^3_{\mathrm{OU}}(t))\mathrm{d}t + \sigma_2\mathrm{d}W(t), & T_2 \le t < T_3, \\ \widehat{X}^4_{\mathrm{OU}}(t): & \mathrm{d}\widehat{X}^4_{\mathrm{OU}}(t) = (\phi - \alpha_3 d\widehat{X}^4_{\mathrm{OU}}(t))\mathrm{d}t + \sigma_3\mathrm{d}W(t), & T_3 \le t \le T_4. \end{cases} \tag{3.3}$$

Based on the relationship between $\widehat{X}_{\mathrm{OU}}(t)$ and $\widetilde{X}_{\mathrm{OU}}(t)$ given in (1.2) combined with Theorem 1.1 assuring the convergence of $\widetilde{X}_V(t)$ to $\widetilde{X}_{\mathrm{OU}}(t)$ as $V \to \infty$, $\widehat{X}^i_{\mathrm{OU}}(t)$ in each time interval can be approximated by $\widehat{X}^i_{\mathrm{OU}}(t) \approx \widetilde{X}^i_V(t) + \theta^i(t)$, $i = 1, 2, 3, 4$. For the processes

$\{\widetilde{X}_V^i(t) : t \geq 0\}$ $(i = 1, 2, 3, 4)$, the size of the state space is fixed as $2V + 1$ which is independent of the underlying parameters α_i and σ_i $(i = 1, 2, 3)$. Accordingly, the patching process can be repeated without any modification. Because of this, it is possible to automate the entire procedures with speed and accuracy. This is one advantage of the Ehrenfest approach to the modified trinomial tree approach.

The numerical algorithm for pricing a zero-coupon discount bond with stepwise reversion and volatility functions is now summarized in Algorithm 3.1 below, using Algorithm $B.2$ given in Appendix B as a subroutine. The algorithm is described for a general case of I intervals. (The example discussed in (3.3) has $I = 4$ as illustrated in Figure 3.1 where the discrete points K_1, K_2, K_3 and K_4 correspond to T_1, T_2, T_3 and T_4 respectively.) We suppose that the prices at different states at time τ_M of a zero-coupon discount bond maturing at time T are needed. The stepwise reversion function is given by $\alpha(t) = \alpha_l,\ t \in [T_{l-1}^\alpha, T_l^\alpha],\ 1 \leq l \leq L$ with $T_0^\alpha = 0 < T_1^\alpha < \cdots < T_L^\alpha = T$, while the stepwise volatility function by $\sigma(t) = \sigma_j,\ t \in [T_{j-1}^\sigma, T_j^\sigma],\ 1 \leq j \leq J$ with $T_0^\sigma = 0 < T_1^\sigma < \cdots < T_J^\sigma = T$. We suppose that T_l^α's and T_j^σ's are sorted together in ascending order to form a combined and renumbered sequence $T_0 = 0 < T_1 < \cdots < T_I = T$ for some $I,\ 0 \leq I \leq L+J$. For notational convenience, we write this operation as

$$I \leftarrow JOIN[L, J]. \tag{3.4}$$

Throughout this section, we assume that Δt is chosen in such a way that, for some positive integers $K_i\,,\ 0 \leq i \leq I$ and M, one has

$$T_i = K_i \Delta t,\ 0 \leq i \leq I; \quad \tau_M = M\Delta t. \tag{3.5}$$

Let i_0 be such that

$$M \in [K_{i_0-1},\ K_{i_0}]. \tag{3.6}$$

The two stepwise functions $\alpha(t)$ and $\sigma(t)$ can be rewritten as

$$\alpha(k\Delta t) = \alpha_{l_i} \ \text{ and } \ \sigma(k\Delta t) = \sigma_{j_i} \ \text{ for } \ k \in [K_{i-1},\ K_i],\ 1 \leq i \leq I. \tag{3.7}$$

Of interest is $D_{SW}(M, m|K_I)$, the discount bond price at time $\tau_M = M\Delta t$ with maturity at time $T = K_I \Delta t$, when $\alpha(t)$ and $\sigma(t)$ are stepwise functions as in (3.7). The index sets $\mathcal{L}$, $\mathcal{J}$ and $\mathcal{I}$ are defined as $\mathcal{L} = \{1, \cdots, L\}$, $\mathcal{J} = \{1, \cdots, J\}$ and $\mathcal{I} = \{1, \cdots, I\}$ respectively.

Algorithm 3.1 (Discount Bond Prices of $[D_{SW}(M, m|K_I)]_{m \in \mathcal{N}_V}$)

Input :

- ▷ V : size of the state space of $\{\widetilde{N}_{2V}(t) : t \geq 0\}$
- ▷ T : maturity time of the discount bond
- ▷ τ_M : time at which the discount bound prices are desired
- ▷ $[\alpha_l, T_l^\alpha]_{l \in \mathcal{L}}$: stepwise reversion function where $\alpha(t) = \alpha_l$ for $t \in [T_{l-1}^\alpha,\ T_l^\alpha]$, $1 \leq l \leq L$, with $T_L^\alpha = T$
- ▷ $[\sigma_j, T_j^\sigma]_{j \in \mathcal{J}}$: stepwise volatility function where $\sigma(t) = \sigma_j$ for $t \in [T_{j-1}^\sigma,\ T_j^\sigma]$, $1 \leq j \leq J$, with $T_J^\sigma = T$
- ▷ $[\theta_l(t)]_{l \in \mathcal{L}}$: shift functions with $t \in [0, T]$

Output:

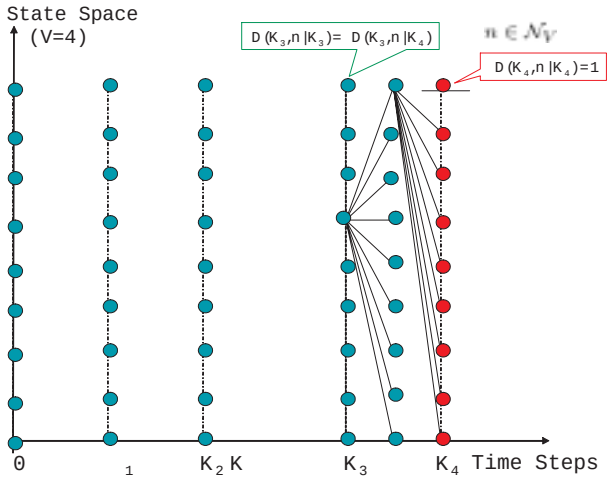


Figure 3.1: Backward Scheme of Algorithm 3.1

$\triangleleft$ $[D_{SW}(M, m|K_I)]_{m\in\mathcal{N}_V}$: Discount Bond Prices

Procedure:

0) Perform $I \leftarrow JOIN[L, J]$, yielding $[T_i]_{i\in\mathcal{I}}$ and set $\underline{d} \leftarrow \underline{1}$.

1) Choose $\Delta t > 0$ so that (3.5) is satisfied, yielding $[K_i]_{i\in\mathcal{I}}$ and M.

2) If $i_0 = I$ with i_0 as in (3.6), obtain

$$[D_{SW}(M, m|K_I)]_{m\in\mathcal{N}_V} \leftarrow \text{ALGO}B.2(V, \tau_M, T, \Delta t, \alpha_{l_I}, \sigma_{j_I}, \theta_{l_I}(t), \underline{d})$$

and stop.

3) Set $i \leftarrow I$.

4) LOOP: Set $\underline{d} \leftarrow \text{ALGO}B.2(V, T_{i-1}, T_i, \Delta t, \alpha_{l_i}, \sigma_{j_i}, \theta_{l_i}(t), \underline{d})$

5) If $i > i_0 + 1$, set $i \leftarrow i - 1$ and go to LOOP.

6) Obtain $[D_{SW}(M, m|K_I)]_{m\in\mathcal{N}_V} \leftarrow \text{ALGO}B.2(V, \tau_M, T_{i_0}, \Delta t, \alpha_{l_{i_0}}, \sigma_{j_{i_0}}, \theta_{l_{i_0}}(t), \underline{d})$ and stop.

We next turn our attention to computation of Arrow-Debreu prices. Let $Q_{SW:0}(M, m)$ be the present value of an Arrow-Debreu price at node (M, m), when $\alpha(t)$ and $\sigma(t)$ are stepwise functions as in (3.7). The next algorithm computes $Q_{SW:0}(M, m)$, using Algorithm $B.3$ given in Appendix B as a subroutine. Figure 3.2 illustrates how the algorithm works for the example given in Figure 0.1 with $I = 4$.

Algorithm 3.2 (Arrow-Debreu Prices of $[Q_{SW:0}(M, m)]_{m\in\mathcal{N}_V}$)

Input :

$\triangleright$ V : size of the state space of $\{\widetilde{N}_{2V}(t) : t \geq 0\}$

$\triangleright$ T : maturity time of the discount bond

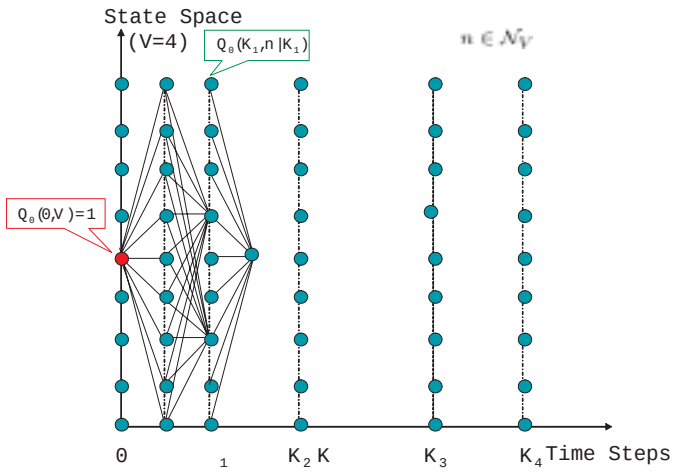


Figure 3.2: Forward Scheme of Algorithm 3.2

▷ τ_M : maturity time of the Arrow-Debreu security whose present values are desired

▷ $[\alpha_l, T_l^\alpha]_{l\in\mathcal{L}}$: stepwise reversion function where $\alpha(t) = \alpha_l$ for $t \in [T_{l-1}^\alpha,\ T_l^\alpha]$, $1 \le l \le L$, with $T_L^\alpha = T$

▷ $[\sigma_j, T_j^\sigma]_{j\in\mathcal{J}}$: stepwise volatility function where $\sigma(t) = \sigma_j$ for $t \in [T_{j-1}^\sigma,\ T_j^\sigma]$, $1 \le j \le J$, with $T_J^\sigma = T$

▷ $[\theta_l(t)]_{l\in\mathcal{L}}$: shift functions with $t \in [0, T]$

Output:

◁ $[Q_{SW:0}(M, m)]_{m\in\mathcal{N}_V}$: Arrow-Debreu Prices

Procedure:

0) Perform $I \leftarrow JOIN[L, J]$, yielding $[T_i]_{i\in\mathcal{I}}$ and set $\underline{s} \leftarrow \underline{u}_V$.

1) Choose Δt so that (3.5) is satisfied, yielding $[K_i]_{i\in\mathcal{I}}$ and M.

2) If $i_0 = 1$ with i_0 as in (3.6), obtain

$$[Q_{SW:0}(M, m)]_{m\in\mathcal{N}_V} \leftarrow \text{ALGO}B.3(V, 0, \tau_M, \Delta t, \alpha_{l_1}, \sigma_{j_1}, \theta_{l_1}(t), \underline{s})$$

and stop.

3) Set $i \leftarrow 1$.

4) LOOP: Set $\underline{s} \leftarrow \text{ALGO}B.3(V, T_{i-1}, T_i, \Delta t, \alpha_{l_i}, \sigma_{j_i}, \theta_{l_i}(t), \underline{s})$.

5) If $i < i_0 - 1$, set $i \leftarrow i + 1$ and go to LOOP.

6) Obtain $[Q_{SW:0}(M, m)]_{m\in\mathcal{N}_V} \leftarrow \text{ALGO}B.3(V, T_{i_0-1}, \tau_M, \Delta t, \alpha_{l_{i_0}}, \sigma_{j_{i_0}}, \theta_{l_{i_0}}(t), \underline{s})$ and stop.

Assuming that both $\alpha(t)$ and $\sigma(t)$ are step functions as in (3.7), let $\pi_{SW:c}(0|M)$ be the price of a European call option associated with the discount bond of maturity at time $T = K_I\Delta t$, having the maturity of the option at time $\tau_M = M\Delta t$ with strike price of K_S. We are now in a position to summarize a new algorithm to compute $\pi_{SW:c}(0|M)$ via the Ehrenfest approach using Algorithms 3.1 and 3.2.

Algorithm 3.3 (European Call Option Price of $\pi_{SW:c}(0|M)$)

Input :

- ▷ V : size of the state space of $\{\widetilde{N}_{2V}(t) : t \geq 0\}$
- ▷ T : maturity time of the discount bond
- ▷ τ_M : maturity time of the option
- ▷ $[\alpha_l, T_l^\alpha]_{l\in\mathcal{L}}$: stepwise reversion function where $\alpha(t) = \alpha_l$ for $t \in [T_{l-1}^\alpha,\ T_l^\alpha]$, $1 \leq l \leq L$, with $T_L^\alpha = T$
- ▷ $[\sigma_j, T_j^\sigma]_{j\in\mathcal{J}}$: stepwise volatility function where $\sigma(t) = \sigma_j$ for $t \in [T_{j-1}^\sigma,\ T_j^\sigma]$, $1 \leq j \leq J$, with $T_J^\sigma = T$
- ▷ $[\theta_l(t)]_{l\in\mathcal{L}}$: shift functions with $t \in [0, T]$

Output:

- ◁ $\pi_{SW:c}(0|M)$: European Call Option Price

Procedure:

1) Obtain $[D_{SW}(M, m|K_I)]_{m\in\mathcal{N}_V}$ via Algorithm 3.1.
2) Compute

$$h_c(D_{SW}(M, m|K_I)) = [D_{SW}(M, m|K_I) - K_S]^+$$

for all $m \in \mathcal{N}_V$.

3) Find $[Q_{SW:0}(M, m)]_{m\in\mathcal{N}_V}$ via Algorithm 3.2.
4) Compute

$$\pi_{SW:c}(0|M) = \sum_{m\in\mathcal{N}_V} h_c(D_{SW}(M, m|K_I))Q_{SW:0}(M, m)$$

and stop.

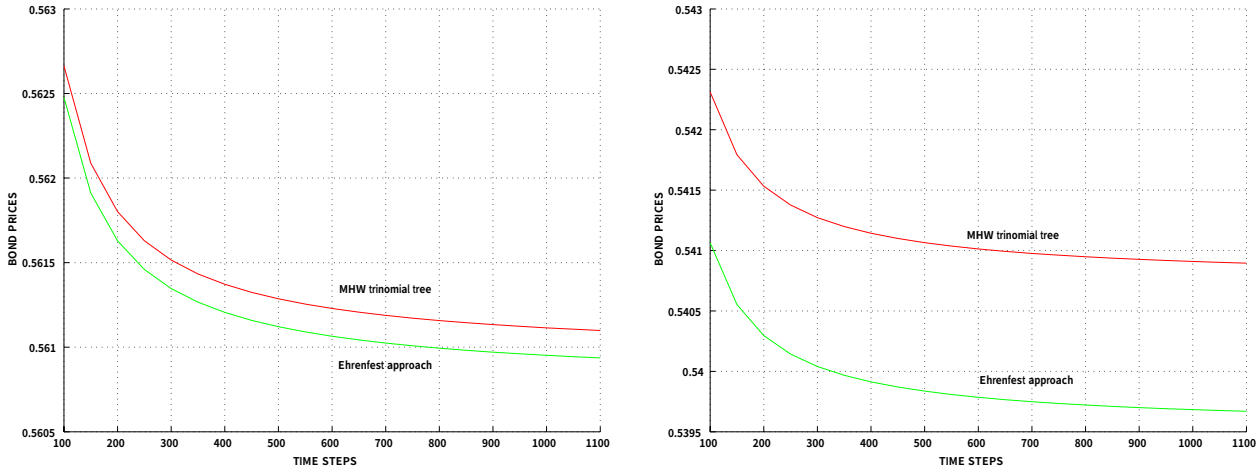
Using Algorithm 3.1 and the modified trinomial tree approach given in Appendix A, Figures 3.3(a) and (b) exhibit the zero-coupon discount bond prices as a function of K_I for the following two different sets of stepwise reversion and volatility functions:

$$\alpha(t) = \begin{cases} 0.2, & 0 \leq t < 2, \\ 0.3, & 2 \leq t < 4, \\ 0.1, & 4 \leq t \leq 5; \end{cases} \quad \sigma(t) = \begin{cases} 0.02, & 0 \leq t < 1, \\ 0.03, & 1 \leq t < 4, \\ 0.01, & 4 \leq t \leq 5, \end{cases} \tag{3.8}$$

and

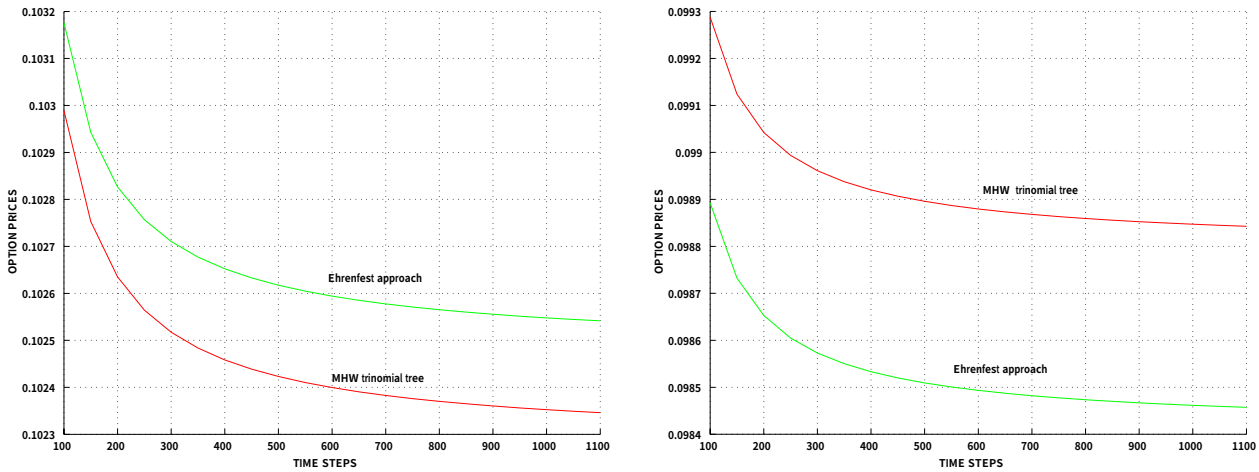
$$\alpha(t) = \begin{cases} 0.1, & 0 \leq t < 2, \\ 0.3, & 2 \leq t < 4, \\ 0.2, & 4 \leq t \leq 5; \end{cases} \quad \sigma(t) = \begin{cases} 0.01, & 0 \leq t < 1, \\ 0.03, & 1 \leq t < 4, \\ 0.02, & 4 \leq t \leq 5. \end{cases} \tag{3.9}$$

For both cases, we set $\phi = 0.05$, $R(0) = 0.05$ and $T = 5$. In each figure, "MHW trinomial tree" indicates that the calculations are based on the modified trinomial tree approach while



(a) $\phi = 0.05, \alpha = [0.2,\ 0.3,\ 0.1], \sigma = [0.02,\ 0.03,\ 0.01]$ (b) $\phi = 0.05, \alpha = [0.1,\ 0.3,\ 0.2], \sigma = [0.01,\ 0.03,\ 0.02]$

Figure 3.3: Convergence Behavior of Discount Bond Prices with Stepwise Functions



(a) $\phi = 0.05, \alpha = [0.2,\ 0.3,\ 0.1], \sigma = [0.02,\ 0.03,\ 0.01]$ (b) $\phi = 0.05, \alpha = [0.1,\ 0.3,\ 0.2], \sigma = [0.01,\ 0.03,\ 0.02]$

Figure 3.4: Convergence Behavior of European Call Option Prices with Stepwise Functions

"Ehrenfest approach" indicates the proposed new approach. Since the analytical results are not available, the first difference $\Delta D_{SW}(K_I) \stackrel{\text{def.}}{=} |D_{SW}(0,0|K_I) - D_{SW}(0,0|K_I - 50)|$ and the second difference $\Delta^2 D_{SW}(K_I) \stackrel{\text{def.}}{=} |\Delta D_{SW}(K_I) - \Delta D_{SW}(K_I - 50)|$ are summarized in Tables 3.1(a) and (b). One sees that these differences for both the Ehrenfest approach and the modified trinomial tree approach decrease as K_I increases. Moreover, the Ehrenfest approach achieves similar first and second differences to the modified trinomial tree approach. The absolute relative differences between the two approaches are contained within 0.231% for Figures 3.3(a) and (b), demonstrating the robust accuracy of the two approaches.

Figures 3.4(a) and (b) show the corresponding European call option prices on the zero-coupon discount bond, with $K_S = 0.7$ and $\tau_M = 4$. The first two differences $\Delta\pi_{SW:c}(M) \stackrel{\text{def.}}{=} |\pi_{SW:c}(0|M) - \pi_{SW:c}(0|M - 40)|$ and $\Delta\pi^2_{SW:c}(M) \stackrel{\text{def.}}{=} |\Delta\pi_{SW:c}(M) - \Delta\pi_{SW:c}(M - 40)|$ are summarized in Tables 3.2(a) and (b), where $M = K_I \cdot \tau_M / T$. Again, the first and second differences for both the Ehrenfest approach and the modified trinomial tree approach decrease

Table 3.1: First and Second Differences in Discount Bound Price Computation

(a) $\alpha = [0.2,\ 0.3,\ 0.1],\ \sigma = [0.02,\ 0.03,\ 0.01]$

K_I	Δt	First Difference		Second Difference	
		Ehrenfest	MHW Tree	Ehrenfest	MHW Tree
800	0.006250	0.00001408	0.00001430	0.00000201	0.00000204
850	0.005882	0.00001243	0.00001262	0.00000166	0.00000168
900	0.005556	0.00001105	0.00001122	0.00000138	0.00000140
950	0.005263	0.00000988	0.00001003	0.00000116	0.00000118
1000	0.005000	0.00000889	0.00000903	0.00000099	0.00000100
1050	0.004762	0.00000805	0.00000817	0.00000085	0.00000086
1100	0.004545	0.00000732	0.00000743	0.00000073	0.00000074

(b) $\alpha = [0.1,\ 0.3,\ 0.2],\ \sigma = [0.01,\ 0.03,\ 0.02]$

K_I	Δt	First Difference		Second Difference	
		Ehrenfest	MHW Tree	Ehrenfest	MHW Tree
800	0.006250	0.00001276	0.00001296	0.00000182	0.00000185
850	0.005882	0.00001126	0.00001144	0.00000150	0.00000152
900	0.005556	0.00001001	0.00001016	0.00000125	0.00000127
950	0.005263	0.00000896	0.00000909	0.00000105	0.00000107
1000	0.005000	0.00000806	0.00000818	0.00000090	0.00000091
1050	0.004762	0.00000729	0.00000740	0.00000077	0.00000078
1100	0.004545	0.00000663	0.00000673	0.00000066	0.00000067

Table 3.2: First and Second Differences in European Call Option Price Computation

(a) $\alpha = [0.2,\ 0.3,\ 0.1],\ \sigma = [0.02,\ 0.03,\ 0.01]$

K_I	Δt	First Difference		Second Difference	
		Ehrenfest	MHW Tree	Ehrenfest	MHW Tree
800	0.006250	0.00000580	0.00000588	0.00000083	0.00000084
850	0.005882	0.00000512	0.00000519	0.00000068	0.00000069
900	0.005556	0.00000455	0.00000461	0.00000057	0.00000058
950	0.005263	0.00000407	0.00000413	0.00000048	0.00000049
1000	0.005000	0.00000366	0.00000371	0.00000041	0.00000041
1050	0.004762	0.00000331	0.00000336	0.00000035	0.00000035
1100	0.004545	0.00000301	0.00000305	0.00000030	0.00000031

(b) $\alpha = [0.1,\ 0.3,\ 0.2],\ \sigma = [0.01,\ 0.03,\ 0.02]$

K_I	Δt	First Difference		Second Difference	
		Ehrenfest	MHW Tree	Ehrenfest	MHW Tree
800	0.006250	0.00000398	0.00000407	0.00000057	0.00000058
850	0.005882	0.00000351	0.00000359	0.00000047	0.00000048
900	0.005556	0.00000312	0.00000319	0.00000039	0.00000040
950	0.005263	0.00000279	0.00000285	0.00000033	0.00000034
1000	0.005000	0.00000251	0.00000257	0.00000028	0.00000029
1050	0.004762	0.00000227	0.00000232	0.00000024	0.00000024
1100	0.004545	0.00000207	0.00000211	0.00000021	0.00000021

as K_I increases and the results of the two approaches are close to each other. The absolute relative differences between the two approaches are contained within 0.399% for Figures 3.4(a) and (b), assuring the trustworthiness of the two approaches.

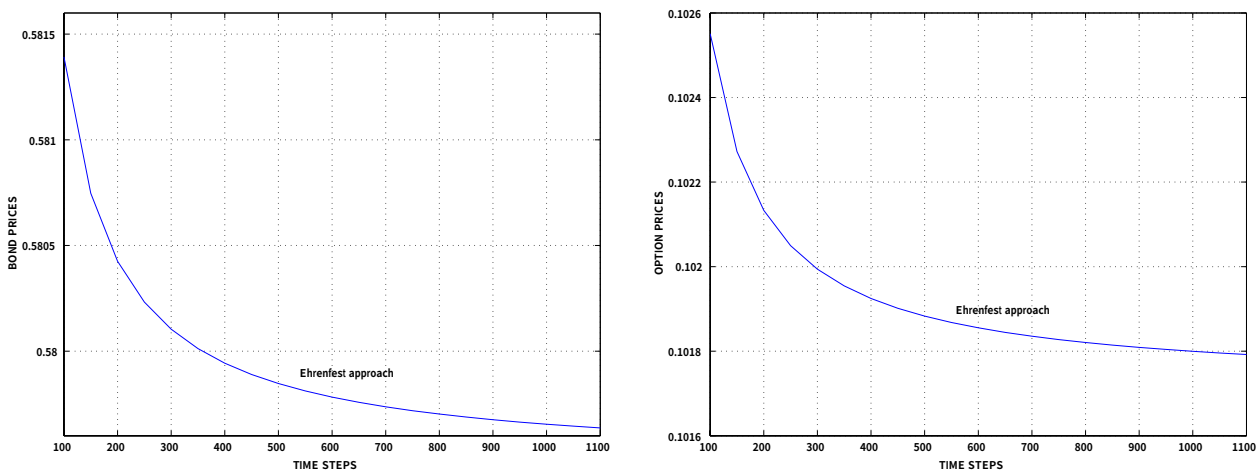
Although the performances of the Ehrenfest approach and the modified trinomial tree approach seem to be more or less comparable, the former approach is guaranteed to work for any stepwise reversion functions while the latter approach may not, as discussed in Section 0. In order to illustrate this point, the following stepwise reversion and volatility functions are considered:

$$\alpha(t) = \begin{cases} 0.05, & 0 \le t < 2, \\ 0.20, & 2 \le t < 4, \\ 0.40, & 4 \le t \le 5; \end{cases} \quad \sigma(t) = \begin{cases} 0.005, & 0 \le t < 1, \\ 0.020, & 1 \le t < 4, \\ 0.040, & 4 \le t \le 5. \end{cases} \tag{3.10}$$

Let m_k^i be the positive integer such that $m_k^i \Delta\tilde{x}$ represents the highest value at time $k\Delta t$ on the modified trinomial tree in the i-th time interval, where $\Delta\tilde{x}$ is the magnitude of each jump on the tree. With $\Delta t = 0.05$ for $i = 1, 2, 3, 4$, the range of m_k^i, denoted by $RG(i)$, should satisfy (see Appendix A for details)

$$\begin{cases} RG(1) = RG(2) = [74,\ 326], & \text{for } 0 \le t < 2, \\ RG(3) = [19,\ 81], & \text{for } 2 \le t < 4, \\ RG(4) = [10,\ 40], & \text{for } 4 \le t \le 5. \end{cases} \tag{3.11}$$

Clearly the intersection of $RG(i)$, $i = 1, 2, 3, 4$ in (3.11) is empty so that the modified trinomial tree approach is not applicable. The Ehrenfest approach can deal with this case without any difficulty as demonstrated in Figures 3.5(a), (b) and Table 3.3.



(a) Discount Bond Prices

(b) European Call Option Prices

Figure 3.5: Convergence Behavior for $\phi = 0.05, \alpha = [0.4,\ 0.2,\ 0.05], \sigma = [0.04,\ 0.02,\ 0.005]$

4 Concluding Remarks

In this paper, a novel approach is proposed for computing the prices of zero-coupon discount bonds and associated European options for the Hull-White model with stepwise reversion and volatility functions. The market fitting function is set to be constant which does not affect our proposed approach in any essential way. We first focus on the Vasicek model which

Table 3.3: First and Second Differences for $\alpha = [0.4,\ 0.2,\ 0.05]$, $\sigma = [0.04,\ 0.02,\ 0.005]$

K_I	Δt	Discount Bond		European Call	
		First Difference	Second Difference	First Difference	Second Difference
800	0.006250	0.00001604	0.00000229	0.00000694	0.00000099
850	0.005882	0.00001415	0.00000189	0.00000612	0.00000082
900	0.005556	0.00001258	0.00000157	0.00000544	0.00000068
950	0.005263	0.00001126	0.00000132	0.00000487	0.00000057
1000	0.005000	0.00001013	0.00000113	0.00000438	0.00000049
1050	0.004762	0.00000916	0.00000097	0.00000396	0.00000042
1100	0.004545	0.00000833	0.00000083	0.00000360	0.00000036

is a special case of the Hull-White model with constant reversion and volatility functions. The underlying O-U process $\widehat{X}_{\mathrm{OU}}(t)$ is expressed as a sum of another O-U process $\widetilde{X}_{\mathrm{OU}}(t)$ and a shift function $\theta(t)$. Based on SGJ [8], it is shown that a sequence of Ehrenfest processes with appropriate shifting and scaling converges in law to $\widetilde{X}_{\mathrm{OU}}(t)$ as the state space size goes to infinity. Using the uniformization procedure of Keilson [5], a numerical algorithm is then developed to compute the time dependent transition probability matrix of the Ehrenfest process, which in turn enables one to approximate the conditional transition probability density of $\widehat{X}_{\mathrm{OU}}(t)$ systematically. Consequently, the prices of zero-coupon discount bonds and associated European options for the Vasicek model can be readily computed.

For the Hull-White model with stepwise reversion and volatility functions, computational algorithms are developed by patching a sequence of $\widehat{X}^i_{\mathrm{OU}}(t)$'s, each defined in a time interval in which both the reversion and the volatility functions are constant. For comparison purposes, a modified trinomial tree approach is also developed to cope with stepwise volatility and reversion functions based on the Hull-White trinomial tree approach. Numerical experiments show that the two approaches are comparable in their performances. However, while the Ehrenfest approach can deal with any stepwise reversion and volatility functions systematically and therefore can be automated easily, it may not be always possible to apply the modified trinomial tree approach. Because of this, the modified trinomial tree approach is limited in its capacity for automated software development. Consequently, the Ehrenfest approach is more attractive than the modified trinomial tree approach for practical use.

The Ehrenfest approach relies on dynamic analysis of the underlying Markov chain in continuous time. Accordingly, one can deal with a variety of boundaries with little alteration as well as evaluation of withdrawal option values. This implies that the Ehrenfest approach opens a new path to develop algorithmic procedures for computing the prices of various barrier options and American options. Such a study is in progress and will be reported elsewhere.

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A Hull-White Trinomial Tree Approach

A.1 The Original Hull-White Trinomial Tree Approach

When $\alpha(t)$ and $\sigma(t)$ in (0.1) are constant, Hull and White [2, 3] have constructed a trinomial tree to represent the short rate stochastic process $\widehat{X}_{\mathrm{OU}}(t)$ specified in (0.5) by discretizing both the state space and the time axis with the length of each time step Δt. The transition probabilities from a state to the three adjacent states are constructed in a risk-neutral manner so that the discretized tree process converges in law to $\widehat{X}_{\mathrm{OU}}(t)$ as $\Delta t \to 0$. The construction of the trinomial tree involves two separate stages as described below.

First Stage The first stage is to build a tree representing $\{\widetilde{X}_{\mathrm{OU}}(t) : t \geq 0\}$ specified in (1.1) starting with $\widetilde{X}_{\mathrm{OU}}(0) = 0$. The magnitude of each jump on the tree, if any, is set to be $\Delta\tilde{x} = \sigma\sqrt{3\Delta t}$ for consideration of error minimization (see Hull [4]). At time $t = k\Delta t$ and state $\tilde{x} = m\Delta\tilde{x}$, the corresponding node in the tree is denoted by (k, m), where k is a positive integer and m is an integer. A state transition in the tree takes one of the three types as shown in Figure A.1. The corresponding upper, middle and lower transition probabilities at node (k, m) are denoted by $p_{m:u}(\Delta t)$, $p_{m:m}(\Delta t)$ and $p_{m:d}(\Delta t)$ respectively. The probabilities are chosen so as to match the first two moments of the trinomial tree process with those of

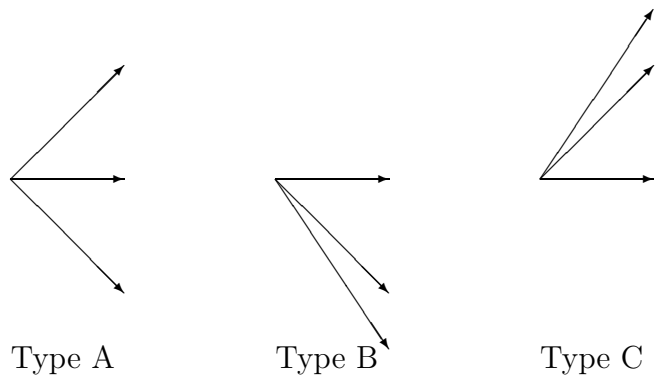

Figure A.1: Types of Tree Branching

the change of $\widetilde{X}_{\mathrm{OU}}(t)$ in the second order of Δt. More specifically, for type A, the following three equations should be satisfied.

$$\text{(A.1)} \qquad \begin{cases} p_{m:u}(\Delta t)\Delta\tilde{x} - p_{m:d}(\Delta t)\Delta\tilde{x} & = -\alpha m\Delta\tilde{x}\Delta t, \\ p_{m:u}(\Delta t)\Delta\tilde{x}^2 + p_{m:d}(\Delta t)\Delta\tilde{x}^2 & = \sigma^2\Delta t + \alpha^2 m^2\Delta\tilde{x}^2\Delta t^2, \\ p_{m:u}(\Delta t) + p_{m:m}(\Delta t) + p_{m:d}(\Delta t) & = 1. \end{cases}$$

Substituting $\Delta\tilde{x} = \sigma\sqrt{3\Delta t}$ into (A.1), solving the three equations yields the transition probabilities given by

$$\text{(A.2)} \qquad \begin{cases} p_{m:u}(\Delta t) & = \frac{1}{6} + \frac{\alpha^2 m^2\Delta t^2 - \alpha m\Delta t}{2}, \\ p_{m:m}(\Delta t) & = \frac{2}{3} - \alpha^2 m^2\Delta t^2, \\ p_{m:d}(\Delta t) & = \frac{1}{6} + \frac{\alpha^2 m^2\Delta t^2 + \alpha m\Delta t}{2}, \end{cases}$$

with $-\frac{\sqrt{6}}{3\alpha\Delta t} < m < \frac{\sqrt{6}}{3\alpha\Delta t}$ to make the three probabilities positive. Similarly, for type B, the three transition probabilities are found as

$$\text{(A.3)} \qquad \begin{cases} p_{m:u}(\Delta t) & = \frac{7}{6} + \frac{\alpha^2 m^2\Delta t^2 - 3\alpha m\Delta t}{2}, \\ p_{m:m}(\Delta t) & = -\frac{1}{3} - \alpha^2 m^2\Delta t^2 + 2\alpha m\Delta t, \\ p_{m:d}(\Delta t) & = \frac{1}{6} + \frac{\alpha^2 m^2\Delta t^2 - \alpha m\Delta t}{2}, \end{cases}$$

with $\frac{3-\sqrt{6}}{3\alpha\Delta t} < m < \frac{3+\sqrt{6}}{3\alpha\Delta t}$. The transition probabilities for type C are symmetric to those for type B given by

$$\text{(A.4)} \qquad \begin{cases} p_{m:u}(\Delta t) & = \frac{1}{6} + \frac{\alpha^2 m^2\Delta t^2 - \alpha|m|\Delta t}{2}, \\ p_{m:m}(\Delta t) & = -\frac{1}{3} - \alpha^2 m^2\Delta t^2 + 2\alpha|m|\Delta t, \\ p_{m:d}(\Delta t) & = \frac{7}{6} + \frac{\alpha^2 m^2\Delta t^2 - 3\alpha|m|\Delta t}{2}, \end{cases}$$

with $-\frac{3+\sqrt{6}}{3\alpha\Delta t} < m < -\frac{3-\sqrt{6}}{3\alpha\Delta t}$.

Let m_k be the maximum positive integer of m at time $k\Delta t$. To keep the three transition probabilities positive for all of the three types, m_k should satisfy the following inequality:

$$\text{(A.5)} \qquad \left\lceil \frac{3-\sqrt{6}}{3\alpha\Delta t} \right\rceil \le m_k \le \left\lfloor \frac{\sqrt{6}}{3\alpha\Delta t} \right\rfloor,$$

where $\lceil a \rceil$ denotes the minimum integer which is greater than or equal to a, and $\lfloor a \rfloor$ the maximum integer which is smaller than or equal to a. Let $\mathcal{N}_k \stackrel{\text{def.}}{=} \{m : -m_k \leq m \leq m_k\}$. The values of $\tilde{x}$ considered at time $k\Delta t$ consist of $m\Delta\tilde{x}$, $m \in \mathcal{N}_k$. As long as (A.5) is satisfied at time $(k+1)\Delta t$, state transitions at time $k\Delta t$ are of type A. If (A.5) is violated at time $(k+1)\Delta t$, a Type B transition takes place at node (k, m_k) and a Type C transition at $(k, -m_k)$ with all other transitions of Type A. More specifically, let

$$k_{\max} = \max\{k : m_k \text{ satisfies } (A.5)\}. \tag{A.6}$$

Then, for $0 \leq k < k_{\max}$, the state transition probabilities are given by (A.2) with $m_k = k$. For $k \geq k_{\max}$, m_k remains to be $k_{\max}$ with the state transition probabilities given by (A.3) at (k, m_k), (A.4) at $(k, -m_k)$ and (A.2) at (k, m) with $m \neq m_k$, $m \neq -m_k$.

Second Stage In the second stage, the trinomial tree constructed in Stage 1 for $\widetilde{X}_{\text{OU}}(t)$ is shifted to represent $\widehat{X}_{\text{OU}}(t)$, using the shift function $\theta(t)$ as given in (1.2). The computational algorithms for evaluating the prices of zero-coupon discount bonds, Arrow-Debreu securities and associated European options are in parallel with those of the Ehrenfest approach in (2.1), (2.4) and (2.7) respectively, where the transition probabilities obtained from the Ehrenfest process are replaced by those obtained in Stage 1. We note that Remark 1.2 is also applicable to the Hull-White trinomial tree approach and an arbitrary market fitting function $\phi(t)$ can be incorporated.

A.2 The Modified Trinomial Tree Approach

As for the Ehrenfest approach, the Hull-White trinomial tree approach can also be modified so as to cope with $R(t)$ in (0.1) having stepwise reversion and volatility functions. The idea is again to use the patching procedure described in Section 0. However, one important difference can be found in that the size of the state space for the Hull-White trinomial tree approach depends on the reversion factor α to satisfy (A.5), while that for the Ehrenfest approach is independent of all the parameters characterizing $R(t)$.

For implementing the patching procedure, it is necessary to keep the state space size the same throughout different time intervals since the state probabilities at the end of one time interval becomes the initial state probabilities for the next time interval. Because of this, the modified trinomial tree approach has to satisfy (A.5) simultaneously throughout I different time intervals, where I is given in (3.4). Let $RG(i)$ be defined by

$$RG(i) = \left[\left\lceil \frac{3-\sqrt{6}}{3\alpha_i \Delta t} \right\rceil, \ \left\lfloor \frac{\sqrt{6}}{3\alpha_i \Delta t} \right\rfloor \right]. \tag{A.7}$$

It is then necessary to have

$$\bigcap_{i=1}^{I} RG(i) \neq \emptyset, \tag{A.8}$$

for the modified trinomial tree approach to be applicable for patching. This condition, of course, is not guaranteed to hold all the time, thereby limiting the potential usefulness of the modified trinomial tree approach significantly.

B Summary of Basic Computational Algorithms

Algorithm B.1 (Transition Probability Matrix of $\{\widetilde{N}_{2V}(t) : t \geq 0\}$)

Input :

- ▷ V : parameter for the size of state space of $\{\widetilde{N}_{2V}(t) : t \geq 0\}$
- ▷ τ : future time as the argument of the transition probability matrix
- ▷ α : time scale factor where $\widetilde{N}_{2V}(\tau) = N_{2V}(\alpha\tau)$
- ▷ ε : parameter for stopping criteria for the series expansion of (1.28)

Output:

- ◁ $\underline{\underline{\widetilde{\mathrm{P}}}}_{2V}(\tau)$: Transition Probability Matrix

Procedure:

1) Set $K = \max\left\{k : \mathrm{e}^{b(V,k,\alpha\tau)} < \varepsilon, k > \alpha V\tau\right\}$, $k_0 = \min\left\{k : \mathrm{e}^{b(V,k,\alpha\tau)} > \varepsilon, k < \alpha V\tau\right\}$ and m=0.
2) LOOP1: Set $\underline{\tilde{p}}^{\top}_{2V:m}(\tau) \leftarrow \underline{0}^{\top}$, $\underline{x}^{\top} \leftarrow \underline{u}^{\top}_{m}$ and $k \leftarrow 0$.
3) LOOP2: $\underline{x}^{\top} \leftarrow \underline{x}^{\top}\underline{\underline{a}}_{V}$.
4) If $k < k_0$, set $k \leftarrow k+1$ and go to LOOP2.
5) LOOP3: $\underline{\tilde{p}}^{\top}_{2V:m}(\tau) \leftarrow \underline{\tilde{p}}^{\top}_{2V:m}(\tau) + \mathrm{e}^{b(V,k,\alpha\tau)}\underline{x}^{\top}$.
6) If $k < K$, set $\underline{x}^{\top} \leftarrow \underline{x}^{\top}\underline{\underline{a}}_{V}$, $k \leftarrow k+1$, and go to LOOP3.
7) If $m < 2V$, $m \leftarrow m+1$, and go to LOOP1.
8) Stop.

Based on the above algorithm, the transition probability matrix of $\{\widetilde{N}_{2V}(t) : t \geq 0\}$ can be evaluated, which we denote by $\underline{\underline{\widetilde{\mathrm{P}}}}_{2V}(\tau) \leftarrow \mathrm{ALGO}B.1[V, \tau, \alpha]$ for notational convenience. From (1.20), this in turn enables one to approximate the transition probabilities of $\{\widehat{X}_{\mathrm{OU}}(t) : t \geq 0\}$.

Algorithm B.2 (Discount Bond Prices of $[D(M, m|K)]_{m\in\mathcal{N}_V}$)

Input :

- ▷ V : size of the state space of $\{\widetilde{N}_{2V}(t) : t \geq 0\}$
- ▷ T : maturity time
- ▷ τ_M : time at which the discount bond prices are desired
- ▷ Δt : discretized time interval satisfying $T = K\Delta t$ and $\tau_M = M\Delta t$ for some positive integers K and M
- ▷ α : reversion factor
- ▷ σ : volatility factor
- ▷ $\theta(t)$: shift function
- ▷ $\underline{d} = [d_0, \cdots, d_{2V}]$: terminal value vector

Output:

$\triangleleft$ $[D(M,m|K)]_{m\in\mathcal{N}_V}$: Discount Bond Prices

Procedure:

1) Set $D(K,m|K)=d_m$ for all $m\in\mathcal{N}_V$, $k=K-1$, and obtain $\underline{\underline{\widetilde{\mathrm{P}}}}_{2V}(\Delta t)\leftarrow \mathrm{ALGO}B.1(V,\Delta t,\alpha)$.

2) LOOP: For all $m\in\mathcal{N}_V$, compute $r(k,m)=\tilde{x}_V(m)+\theta(k\Delta t)$ with $\tilde{x}_V(m)$ as in (1.14), and generate $D(k,m|K)$ using (2.1).

3) If $k>M$, set $k\leftarrow k-1$ and go to LOOP.

4) Stop.

For notational convenience, we write $[D(M,m|K)]_{m\in\mathcal{N}_V}\leftarrow \mathrm{ALGO}B.2\ (V,\tau_M,T,\Delta t,\alpha,\sigma,\theta(t),\underline{d})$.

Algorithm B.3 (Arrow-Debreu Prices of $[Q_0(M,m|k_0)]_{m\in\mathcal{N}_V}$)

Input :

$\triangleright$ V : size of the state space of $\{\widetilde{N}_{2V}(t):t\geq 0\}$

$\triangleright$ τ_0 : maturity time of the Arrow-Debreu security whose present values are known

$\triangleright$ τ_M : maturity time of the Arrow-Debreu security whose present values are desired

$\triangleright$ Δt : discretized time interval satisfying $\tau_0=k_0\Delta t$ and $\tau_M=M\Delta t$ for some positive integers k_0 and M

$\triangleright$ α : reversion factor

$\triangleright$ σ : volatility factor

$\triangleright$ $\theta(t)$: shift function

$\triangleright$ $\underline{s}=[s_0,\cdots,s_{2V}]$: starting value vector

Output:

$\triangleleft$ $[Q(M,m|k_0)]_{m\in\mathcal{N}_V}$: Arrow-Debreu Prices

Procedure:

1) Set $Q_0(k_0,m|k_0)=s_m$ for all $m\in\mathcal{N}_V$, $k=k_0+1$, and obtain $\underline{\underline{\widetilde{\mathrm{P}}}}_{2V}(\Delta t)\leftarrow \mathrm{ALGO}B.1(V,\Delta t,\alpha)$.

2) LOOP: For all $m\in\mathcal{N}_V$, compute $r(k,m)=\tilde{x}_V(m)+\theta(k\Delta t)$ with $\tilde{x}_V(m)$ as in (1.14), and generate $Q_0(k,m|k_0)$ using (2.4).

3) If $k<M$, set $k\leftarrow k+1$ and go to LOOP.

4) Stop.

As before, we write $[Q_0(M,m|k_0)]_{m\in\mathcal{N}_V}\leftarrow \mathrm{ALGO}B.3\ (V,\tau_0,\tau_M,\Delta t,\alpha,\sigma,\theta(t),\underline{s})$ for notational convenience.

Algorithm B.4 (European Call Option Price of $\pi_c(0|M)$)

Input :

$\triangleright$ V : size of the state space of $\{\widetilde{N}_{2V}(t):t\geq 0\}$

$\triangleright$ T : maturity time of the discount bond

$\triangleright$ τ_M : maturity time of the option

▷ Δt : discretized time interval satisfying $T = K\Delta t$ and $\tau_M = M\Delta t$ for some positive integers K and M

▷ K_S : strike price of the option

▷ α : reversion factor

▷ σ : volatility factor

▷ $\theta(t)$: shift function

Output:

◁ $\pi_c(0|M)$: Price of European Call Option on Discount Bond

Procedure:

1) Set $\underline{d} \leftarrow \underline{1}$, $\underline{s} \leftarrow \underline{u}_V$ and $\tau_0 = 0$.

2) Obtain $[D(M, m|K)]_{m \in \mathcal{N}_V} \leftarrow \text{ALGO}B.2(V, \tau_M, T, \Delta t, \alpha, \sigma, \theta(t), \underline{d})$.

3) Compute $[h_c(D(M, m|K))]_{m \in \mathcal{N}_V}$ by using (2.6).

4) Obtain $[Q_0(M, m)]_{m \in \mathcal{N}_V} \leftarrow \text{ALGO}B.3(V, \tau_0, \tau_M, \Delta t, \alpha, \sigma, \theta(t), \underline{s})$.

5) Compute $\pi_c(0|M)$ by using (2.7) and stop.