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Abstract

In this paper, we propose a homogeneous model for solving monotone mixed complementarity problems over symmetric cones, by extending the results in [11] for standard form of the problems. We show that the extended model inherits the following desirable features: (a) A path exists, is bounded and has a trivial starting point without any regularity assumption concerning the existence of feasible or strictly feasible solutions. (b) Any accumulation point of the path is a solution of the homogeneous model. (c) If the original problem is solvable, then every accumulation point of the path gives us a finite solution. (d) If the original problem is strongly infeasible, then every accumulation point of the path gives us a finite certificate proving infeasibility. We also show that the homogeneous model is directly applicable to the primal-dual convex quadratic problems over symmetric cones.

Key words. Complementarity problem, nonlinear optimization, optimality condition, symmetric cone, homogeneous algorithm, interior point method, detecting infeasibility.

1 Introduction

Let $(V, \circ)$ be a Euclidian Jordan algebra with an identity element e. We denote by K the symmetric cone of V, which is a self-dual closed convex cone such that for any two elements $x \in \operatorname{int} K$ and $y \in \operatorname{int} K$, there exists an invertible map $\Gamma : V \to V$ satisfying $\Gamma(K) = K$ and $\Gamma(x) = y$. It is known that a cone in V is symmetric if and only if it is the cone of squares of V given by $K = \{x \circ x : x \in V\}$. The (nonlinear) complementarity problem (CP) over the symmetric cone K is given by

$$
\begin{aligned}
\text{(CP) Find} \quad & (x, y, z) \in K \times K \times \Re^m \\
\text{Subject to} \quad & F(x, y, z) = 0, \ x \circ y = 0
\end{aligned}
\tag{1}
$$

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where $F : K \times K \times \Re^m \to V \times \Re^m$ is continuous. The class of CPs covers a wide range of optimization problems such as primal-dual linear, quadratic, semidefinite, and second-order cone programs. Recently, an interior point map and associated trajectories have been studied in the paper [11] based on the results in [1, 4, 5, 7]. The paper has also provided a *homogeneous model* for a special class of CPs of the form

$$\text{(SCP) Find} \qquad (x, y) \in K \times K \tag{2}$$
$$\text{Subject to} \ \ y - \psi(x) = 0, \ x \circ y = 0$$

where $\psi : K \to V$ is continuous. Choose an appropriate inner product $\langle \cdot, \cdot \rangle$ on $V \times V$, and suppose that the function ϕ is monotone on K, i.e., ϕ satisfies

$$\langle \psi(x) - \psi(x'), x - x' \rangle \geq 0 \ \ \text{for all } x, x' \in K.$$

Then the homogenous model has the following remarkable features (cf. Theorems 5.4 and 5.5 in [11]):

(a) A path exists, is bounded and has a trivial starting point without any regularity assumption concerning the existence of feasible or strictly feasible solutions.

(b) Any accumulation point of the path is a solution of the homogeneous model.

(c) If the original problem is solvable, then every accumulation point of the path gives us a finite certificate proving infeasibility.

(d) If the original problem is strongly infeasible, then every accumulation point of the path gives us a finite certificate proving infeasibility.

In this paper, we extend the above results for a wider class of CPs, which are so called *mixed* (cf. [3]) and given by

$$\text{(MiCP) Find} \qquad (x, y, z) \in K \times K \times \Re^m$$
$$\text{Subject to} \quad F(x, y, z) := \begin{pmatrix} y - \psi_1(x, z) \\ \psi_2(x, z) \end{pmatrix}, \ x \circ y = 0 \tag{3}$$

where $\psi := (\psi_1, \psi_2) : K \times \Re^m \to V \times \Re^m$.

The paper is organized as follows.

In Section 2, we summarize some basic properties of the MiCP, most of which have appeared in [11].

In Section 3, we propose a homogeneous model for the MiCP and show that the proposed model maintain the properties (a)–(d) described above under certain assumptions on F.

The assumptions used in Section 3 are slightly theoretical. In order to make them more tangible, we consider the following assumption in Section 4:

Assumption 1.1 (i) *ψ is affine. i.e., there exist linear operators A and B, and $d \in V \times \Re^m$, such that $\psi(x, z) = Ax + Bz + d$.*

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(ii) ψ *is monotone on* $K \times \Re^m$, *i.e., for all* $(x, z), (x', z') \in K \times \Re^m$,

$$\langle \psi_1(x, z) - \psi_1(x', z'), x - x' \rangle + [\psi_2(x, z) - \psi_2(x', z')]^T (z - z') \geq 0.$$

(iii) *The rank of the linear operator* B *is* m.

We show that if the function ψ satisfies the above assumption then the properties (a)-(d) hold. It should be noted that our results are new even for the (classical) mixed CPs where K is the n-dimensional nonnegative orthant.

Section 5 is devoted to discussions on some applications of our results. We conclude that our homogeneous model is *directly* applicable to the optimal condition of the following quadratic conic optimization problem:

$$\text{(QO) Minimize} \quad \tfrac{1}{2} z^T Q z + c^T z$$
$$\text{Subject to} \quad Az - b \in -K \tag{4}$$

where $A : \Re^m \rightarrow V$ is a linear operator of rank m, $Q \in \Re^{m \times m}$ is a symmetric positive semidefinite matrix, $b \in V$, and $c \in \Re^m$. We also refer to the results obtained when we apply our homogeneous model to the linear conic programming.

Note that the functions appearing in the paper are not necessarily defined on the boundary of the set $K \times K \times \Re^m$. By this reason, we introduce the following *asymptotic* definitions:

Definition 1.2 *The CP is* asymptotically feasible *if and only if there exists a bounded sequence*
$\{x^{(k)}, y^{(k)}, z^{(k)}\} \subseteq \text{int}K \times \text{int}K \times \Re^m$ *such that* $\lim_{k \to \infty} F(x^{(k)}, y^{(k)}, z^{(k)}) = 0$.

The CP is asymptotically solvable *if and only if there exists a bounded sequence* $\{x^{(k)}, y^{(k)}, z^{(k)}\} \subseteq \text{int}K \times \text{int}K \times \Re^m$ *such that* $\lim_{k \to \infty} F(x^{(k)}, y^{(k)}, z^{(k)}) = 0$ *and* $\lim_{k \to \infty} x^{(k)} \circ y^{(k)} = 0$.

2 Preliminaries

Let $(V, \circ)$ be a Euclidean Jordan algebra with the identity element e, where $(x, y) \mapsto x \circ y : V \times V \rightarrow V$ is a bilinear map satisfying

(i) $x \circ y = y \circ x$,

(ii) $x \circ (y \circ x^2) = (x \circ y) \circ x^2$ where $x^2 = x \circ x$,

(iii) $x \circ e = e \circ x = x$,

for all $x, y, z \in V$. Since $(x, y) \mapsto x \circ y$ is a bilinear map, for each $x \in V$, there exists a matrix $L(x)$ such that $L(x)y = x \circ y$ holds for all $y \in V$. For $x \in V$, the *degree* of x is the smallest integer d such that the set $\{e, x, x^2, \dots, x^d\}$ is linearly independent. The *rank* r of V is the maximum of the degree of x over all $x \in V$. For any element x in V of rank r, we can define the *characteristic polynomial* of x of the form

$$p_x(\lambda) := \lambda^r - a_1(x)\lambda^{r-1} + \cdots + (-1)^r a_r(x)$$

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(cf. Section 2 of [9]). We call the roots $\lambda_1, \cdots, \lambda_r$ of $p_x(\lambda)$ the *eigenvalues* of x and define

$$\operatorname{tr}(x) := \sum_{i=1}^{r} \lambda_i = a_1(x), \quad \det(x) := \Pi_{i=1}^{r} \lambda_i = a_r(x). \tag{5}$$

It is known that $\operatorname{tr}(x \circ y)$ gives an inner product on V. Throughout the paper, we define the scalar product of $x, y \in V$ and the norm of $x \in V$ as follows:

$$\langle x, y \rangle := \operatorname{tr}(x \circ y), \quad \|x\| := \sqrt{\operatorname{tr}(x \circ x)}. \tag{6}$$

Note that $\|e\| = \sqrt{r}$.

The set of squares $K := \{x^2 : \ x \in V\}$ is the symmetric cone of V, which is self-dual (i.e., $K = K^* := \{y : \ \langle x, y \rangle \geq 0 \ \text{for all } x \in K\}$). In the next proposition, we give some properties of the symmetric cone K for further discussion. For the proofs of the results, see Theorem III.1.2 and Corollary I.1.6 of [4], Lemma 2.6 and Proposition 2.7 of [11], etc.

Proposition 2.1 *Let K be the symmetric cone of V.*

(i) *If $y \in \operatorname{int} K$ and $\eta > 0$, then the set $\{x \in K : \langle x, y \rangle \leq \eta\}$ is compact.*

(ii) *If $x \in V$, then there exist real numbers $\lambda_1, \ldots, \lambda_r$ and a Jordan frame $c_1, \ldots, c_r$ such that $x = \sum_{j=1}^{r} \lambda_j c_j$. Here the numbers λ_j (with their multiplicities) are uniquely determined by x and λ_j's are the eigenvalues (multiplicities included) of x.*

(iii) *If $x \in K$ and $y \in K$, then $\langle x, y \rangle = 0$ if and only if $x \circ y = 0$.*

(iv) *$\operatorname{int} K \times \{\alpha e : \ \alpha \in \Re_{++}\} \subseteq \mathcal{U}, \ \{\alpha e : \ \alpha \in \Re_{++}\} \times \operatorname{int} K \subseteq \mathcal{U},$*
$K \times \{\alpha e : \ \alpha \in \Re_+\} \subseteq \operatorname{cl}(\mathcal{U}), \ \{\alpha e : \ \alpha \in \Re_+\} \times K \subseteq \operatorname{cl}(\mathcal{U}),$
where $\Re_+ := \{\alpha \in \Re : \ \alpha \geq 0\}$ and $\Re_{++} := \{\alpha \in \Re : \ \alpha > 0\}$.

(v) *There exists $\omega_1 > 0$ and $\omega_2 > 0$ for which $0 < \omega_1 \leq \|c\| \leq \omega_2$ holds for any nonzero idempotent c of V.*

Here we introduce the so-called *interior point map* $H : \operatorname{int} K \times \operatorname{int} K \times \Re^m \to V \times V \times \Re^m$ of the form

$$H := \begin{pmatrix} x \circ y \\ F(x, y, z) \end{pmatrix}. \tag{7}$$

Consider the following assumption on F:

Assumption 2.2 (i) *F is (x, y)-everywhere-monotone on its domain, i.e., there exist continuous functions ϕ from the domain of F to the set $V \times \Re^m$ and $c : (V \times \Re^m) \times (V \times \Re^m) \to \Re$ such that for any $r \in V \times \Re^m$ and for any (x, y, z) and (x', y', z') in the domain of F, we have*

$$c(r, r) = 0$$

and

$$[F(x, y, z) = r \ \text{and} \ F(x', y', z') = r']$$
$$\Rightarrow \langle x - x', y - y' \rangle \geq \langle r - r', \phi(x, y, z) - \phi(x', y', z') \rangle_{V \times \Re^m} + c(r, r').$$

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Here we define

$$\langle(a, b), (a', b')\rangle_{V \times \Re^m} = \langle a, a'\rangle + b^T b'$$

for any $(a, b), (a', b') \in V \times \Re^m$.

(ii) *F is z-bounded on its domain, i.e., for any sequence $\{(x^{(k)}, y^{(k)}, z^{(k)})\}$ in the domain of F, if $\{(x^{(k)}, y^{(k)})\}$ and $\{F(x^{(k)}, y^{(k)}, z^{(k)})\}$ are bounded then the sequence $\{z^{(k)}\}$ is also bounded.*

(iii) *$F(x, y, z)$ is z-injective on its domain, i.e., for any (x, y, z) and (x, y, z') lie in the domain of F, if $F(x, y, z) = F(x, y, z')$ then $z = z'$ holds.*

The following theorem has been proposed in [11] as Theorems 3.10 and 3.12. The theorem shows that the map H is a *homeomorphism* under Assumption 2.2:

Theorem 2.3 *Suppose that a continuous map $F : \mathrm{int}K \times \mathrm{int}K \times \Re^m \to V \times \Re^m$ satisfies Assumption 2.2. Define the set*

$$\mathcal{U} := \{(x, y) \in \mathrm{int}K \times \mathrm{int}K : \ x \circ y \in \mathrm{int}K\}.$$

(i) *H maps $\mathcal{U} \times \Re^m$ homeomorphically onto $\mathrm{int}K \times F(\mathcal{U} \times \Re^m)$, i.e., H is bijective from $\mathcal{U} \times \Re^m$ onto $\mathrm{int}K \times F(\mathcal{U} \times \Re^m)$, and H and H^{-1} are continuous.*

(ii) *The set $F(\mathcal{U} \times \Re^m)$ is an open convex set.*

3 A homogeneous model for the MiCP

We define the homogeneous model (HMiCP) for the MiCP, which is a natural extension of the model in [2] for the CP with $K = \Re^n_+$, and of the one in [11] for the SCP (2):

$$\text{(HMiCP) Find} \qquad (x, \tau, y, \kappa, z) \in (K \times \Re_{++}) \times (K \times \Re_+) \times \Re^m \tag{8}$$
$$\text{Subject to} \quad F_{\mathrm{H}}(x, \tau, y, \kappa, z) = 0, (x, \tau) \circ_{\mathrm{H}} (y, \kappa) = 0$$

where F_{H} and $(x, \tau) \circ_{\mathrm{H}} (y, \kappa)$ are given by

$$F_{\mathrm{H}}(x, \tau, y, \kappa, z) := \begin{pmatrix} y - \tau\psi_1(x/\tau, z/\tau) \\ \kappa + \langle\psi_1(x/\tau, z/\tau), x\rangle + \psi_2(x/\tau, z/\tau)^T z \\ \tau\psi_2(x/\tau, z/\tau) \end{pmatrix} \tag{9}$$

and

$$(x, \tau) \circ (y, \kappa) := \begin{pmatrix} x \circ y \\ \tau\kappa \end{pmatrix}. \tag{10}$$

We also introduce the scalar product $\langle(x, \tau), (y, \kappa)\rangle_{\mathrm{H}}$ associated to the product above by

$$\langle(x, \tau), (y, \kappa)\rangle_{\mathrm{H}} := \langle x, y\rangle + \tau\kappa. \tag{11}$$

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For ease of notation, we use the following symbols

$$V_{\rm H} := V \times \Re, \ \ K_{\rm H} := K \times \Re_+, \ \ x_{\rm H} := (x, \tau) \in V_{\rm H}, \ \ y_{\rm H} := (y, \kappa) \in V_{\rm H} \tag{12}$$

and define the mapping $\psi_{\rm H} := (\psi_{\rm H_1}, \psi_{\rm H_2})$ by

$$\psi_{\rm H_1}(x_{\rm H}, z) = \psi_{\rm H_1}(x, \tau, z) := \begin{pmatrix} \tau \psi_1(x/\tau, z/\tau) \\ -\langle \psi_1(x/\tau, z/\tau), x \rangle - \psi_2(x/\tau, z/\tau)^T z \end{pmatrix} \tag{13}$$

$$\psi_{\rm H_2}(x_{\rm H}, z) = \psi_{\rm H_2}(x, \tau, z) := \tau \psi_2(x/\tau, z/\tau)$$

for every $(x_{\rm H}, z) = (x, \tau, z) \in (K \times \Re_{++}) \times \Re^m$. We can easily see that $\mathrm{int} K_{\rm H} = \mathrm{int} K \times \Re_{++}$ and

$$F_{\rm H}(x_{\rm H}, y_{\rm H}, z) = \begin{pmatrix} y_{\rm H} - \psi_{\rm H_1}(x_{\rm H}, z) \\ \psi_{\rm H_2}(x_{\rm H}, z) \end{pmatrix}. \tag{14}$$

In addition,

$$K_{\rm H} = \{x_{\rm H}^2 = (x^2, \tau^2) : x_{\rm H} \in V_{\rm H}\}$$

holds, hence the closed convex cone $K_{\rm H}$ is the symmetric cone of $V_{\rm H}$.

Note that the function $F_{\rm H}$ is defined on the set $\mathrm{int} K_{\rm H} \times \mathrm{int} K_{\rm H} \times \Re^m$ (but not necessarily on its boundary). In what follows, we set the domain of $F_{\rm H}$ to be $\mathrm{int} K_{\rm H} \times \mathrm{int} K_{\rm H} \times \Re^m$.

Let us consider the map

$$H_{\rm H} := \begin{pmatrix} x_{\rm H} \circ_{\rm H} y_{\rm H} \\ F_{\rm H}(x_{\rm H}, y_{\rm H}, z) \end{pmatrix} \tag{15}$$

and choose an initial point $(x_{\rm H}^{(0)}, y_{\rm H}^{(0)}, z^{(0)})$ such that

$$(x_{\rm H}^{(0)}, y_{\rm H}^{(0)}, z^{(0)}) \in \mathrm{int} K_{\rm H} \times \mathrm{int} K_{\rm H} \times \Re^m \text{ and } x_{\rm H} \circ_{\rm H} y_{\rm H} \in \mathrm{int} K_{\rm H}.$$

For simplicity, we set

$$(x_{\rm H}^{(0)}, y_{\rm H}^{(0)}, z^{(0)}) = (x^{(0)}, \tau_0, y^{(0)}, \kappa_0, z^{(0)}) = (e, 1, e, 1, 0) \in \mathrm{int} K_{\rm H} \times \mathrm{int} K_{\rm H} \times \Re^m. \tag{16}$$

Define

$$h_{\rm H}^{(0)} := \begin{pmatrix} p_{\rm H}^{(0)} \\ f_{\rm H}^{(0)} \end{pmatrix} := \begin{pmatrix} (x_{\rm H}^{(0)}, y_{\rm H}^{(0)}) \\ F_{\rm H}(x_{\rm H}^{(0)}, y_{\rm H}^{(0)}, z^{(0)}) \end{pmatrix} = \begin{pmatrix} e_{\rm H} \\ \begin{pmatrix} y_{\rm H}^{(0)} - \psi_{\rm H_1}(x_{\rm H}^{(0)}, z^{(0)}) \\ \psi_{\rm H_2}(x_{\rm H}^{(0)}, z^{(0)}) \end{pmatrix} \end{pmatrix} \tag{17}$$

where $e_{\rm H} = (e, 1)$ is the identity element in $V_{\rm H}$ satisfying

$$\mathrm{Tr}(e_{\rm H}) = \mathrm{rank}(V_{\rm H}) = r + 1. \tag{18}$$

The next two theorems follow from the results described in Section 2. We give the proofs in the appendix. The proofs are analogous to those of Theorems 5.4 and 5.5 in [11]. Theorem 3.1 below shows that we can find whether the MiCP is solvable, infeasible or in other cases, by observing any accumulation point of a bounded path, whose existence is guaranteed by Theorem 3.2.

Theorem 3.1 *Suppose that $F : K \times K \times \Re^m \to \Re^m$ satisfies Assumption 2.2.*

(i) *For every $(x_H, z) \in \mathrm{int}K_H \times \Re^m$,*

$$\langle x_H, \psi_{H_1}(x_H, z) \rangle_H + z^T \psi_{H_2}(x_H, z) = 0.$$

(ii) *Every asymptotically feasible solution $(\hat{x}_H, \hat{y}_H, z)$ of (HCP) is an asymptotically complementary solution.*

(iii) *The HCP is asymptotically feasible.*

(iv) *The CP has a solution if and only if the HCP has an asymptotical solution $(x_H^*, y_H^*, z^*) = (x^*, \tau^*, y^*, \kappa^*, z^*)$ with $\tau^* > 0$. In this case, $(x^*/\tau^*, y^*/\tau^*, z^*/\tau^*)$ is a solution of (CP).*

(v) *The CP is strongly infeasible if and only if the HCP has an asymptotical solution $(x^*, \tau^*, y^*, \kappa^*, z^*)$ with $\kappa^* > 0$.*

Here, the asymptotic feasibility and solvability of the problem are given in Definition 1.2.

Theorem 3.2 *Suppose that $F_H : \mathrm{int}K_H \times \mathrm{int}K_H \times \Re^m \to V_H$ defined by (14) satisfies Assumption 2.2.*

(i) *For any $t \in (0, 1]$, there exists a point $(x_H(t), y_H(t), z(t)) \in \mathrm{int}K_H \times \mathrm{int}K_H \times \Re^m$ such that*

$$H_H(x_H(t), y_H(t), z(t)) = t h_H^{(0)}.$$

(ii) *The set*

$$P := \{(x_H(t), y_H(t), z(t)) : H_H(x_H(t), y_H(t), z(t)) = t h_H^{(0)}, t \in (0, 1]\} \tag{19}$$

forms a bounded path $\in \mathrm{int}K_H \times \mathrm{int}K_H \times \Re^m$. Any accumulation point $(x_H(t), y_H(t), z(t))$ is an asymptotically complementary solution to the HCP.

(iii) *If the HCP has an asymptotically complementarity solution $(x_H^*, y_H^*, z^*) = (x^*, \tau^*, y^*, \kappa^*, z^*)$ with $\tau^* > 0$ $(\kappa^* > 0, respectively)$, then any accumulation point*

$$(x_H(0), y_H(0), z(0)) = (x(0), \tau(0), y(0), \kappa(0), z(0))$$

of the bounded path P satisfies $\tau(0) > 0$ $(\kappa(0) > 0, respectively)$.

Note that we assume that Assumption 2.2 holds for F in Theorem 3.1 and for F_H in Theorem 3.2, respectively. In the next section, we will show that Assumption 1.1 is sufficient.

4 A sufficient condition on the function ψ

Monteiro and Pang [7] showed several sufficient conditions to ensure that the function F satisfies Assumption 2.2, when K is the cone of positive semidefinite matrices. The issue is more complicated in our analysis: Not only the function F but also the homogeneous function F_H should satisfy Assumption 2.2 (see Theorems 3.1 and 3.2). In this section, we show that these requirements are satisfied under Assumption 2.2.

Proposition 4.1 *Suppose that the function $\psi = (\psi_1, \psi_2) : K \times \Re^m \to V \times \Re^m$ satisfies Assumption 1.1. Then the function $F : K \times K \times \Re^m \to V \times \Re^m$ satisfies Assumption 2.2, i.e., F is (x, y)-everywhere-monotone, z-injective and z-bounded on $K \times K \times \Re^m$.*

Proof: Suppose that ψ satisfies Assumption 1.1.
    Define $\phi : K \times K \times \Re^m \to V \times \Re^m$ and $c : (V \times \Re^m) \times (V \times \Re^m) \to \Re$ by

$$\phi(x, y, z) := (x, -z), \ \ c := 0.$$

Let $r := (a, b) = F(x, y, z)$ and $r' := (a', b') = F(x', y', z')$ where $(x, z), (x', z') \in K \times \Re^m$. Then

$$\psi_1(x, z) - \psi_1(x', z') = (y - y') - (a - a'), \ \ \psi_2(x, z) - \psi_2(x', z') = b - b',$$

and the monontonicity of ψ implies that

$$\begin{aligned}
0 &\le \langle \psi_1(x, z) - \psi_1(x', z'), x - x' \rangle + [\psi_2(x, z) - \psi_2(x', z')]^T (z - z') \\
&= \langle (y - y') - (a - a'), x - x' \rangle + (b - b')^T (z - z') \\
&= \langle y - y', x - x' \rangle - \langle a - a', x - x' \rangle + (b - b')^T (z - z') \\
&= \langle y - y', x - x' \rangle - \langle r - r', \phi(x, y, z) - \phi(x', y', z') \rangle + c(r, r').
\end{aligned}$$

Thus, the function F is (x, y)-everywhere-monotone.
    By (i) and (iii) of Assumption 1.1, we can easily see that ψ is z-bounded and z-injective, and hence, from the definition (3), F is z-bounded and z-injective. ∎

Proposition 4.2 *Suppose that the function $\psi = (\psi_1, \psi_2) : K \times \Re^m \to V \times \Re^m$ satisfies Assumption 1.1.*

(i) *ψ_H is monotone on $\mathrm{int} K_H \times \Re^m$.*

(ii) *$F_H : \mathrm{int} K_H \times \mathrm{int} K_H \times \Re^m$ satisfies Assumption 2.2, i.e., F_H is (x_H, y_H)-everywhere-monotone, z-bounded and z-injective on $\mathrm{int} K_H \times \mathrm{int} K_H \times \Re^m$.*

Proof: (i): For every $(x_H, z), (x'_H, z') \in \mathrm{int} K_H \times \Re^m$, it follows from the definition (13) that

$$\langle \psi_{H_1}(x_H, z) - \psi_{H_1}(x'_H, z'), x_H - x'_H \rangle_H + [\psi_{H_2}(x_H, z) - \psi_{H_2}(x'_H, z')]^T (z - z')$$
$$= \langle \tau \psi_1(x/\tau, z/\tau) - \tau' \psi_1(x'/\tau', z'/\tau'), x - x' \rangle + [\tau \psi_2(x/\tau, z/\tau) - \tau' \psi_2(x'/\tau', z'/\tau')]^T (z - z')$$
$$- (\tau - \tau') \{ \langle \psi_1(x/\tau, z/\tau), x \rangle - \langle \psi_1(x'/\tau', z'/\tau'), x' \rangle \} - (\tau - \tau') \{ \psi_2(x/\tau, z/\tau)^T z - \psi_2(x'/\tau', z'/\tau')^T z' \}$$

By rearranging the right-hand side, we have

$$\begin{aligned}
\langle \psi_{H_1}(x_H, z) &- \psi_{H_1}(x'_H, z'), x_H - x'_H \rangle_H + [\psi_{H_2}(x_H, z) - \psi_{H_2}(x'_H, z')]^T (z - z') \\
&= \tau \tau' \{ \langle \psi_1(x/\tau, z/\tau) - \psi_1(x'/\tau', z'/\tau'), (x/\tau) - (x'/\tau') \rangle \\
&\quad + [\psi_2(x/\tau, z/\tau) - \psi_2(x'/\tau', z'/\tau')]^T [(z/\tau) - (z'/\tau')] \} \\
&\ge 0
\end{aligned}$$

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where the last inequality follows from the monotonicity of $\psi = (\psi_1, \psi_2)$. Thus the map $\psi_{\mathrm{H}} = (\psi_{\mathrm{H}_1}, \psi_{\mathrm{H}_2})$ is monotone on the set $\mathrm{int} K_{\mathrm{H}} \times \Re^m$.

(ii): The monotonicity of F_{H} follows from (i) above and an analogous discussion to the proof of Proposition 4.1.

We are going to show that F_{H} is z-bounded and z-injective. Note that we have already seen that ψ is z-bounded and z-injective in Proposition 4.1.

Suppose that
$$\{(x_{\mathrm{H}}^{(k)}, y_{\mathrm{H}}^{(k)})\} \subseteq \mathrm{int} K_{\mathrm{H}} \times \mathrm{int} K_{\mathrm{H}}$$
and
$$\{F_{\mathrm{H}}(x_{\mathrm{H}}^{(k)}, y_{\mathrm{H}}^{(k)}, z^{(k)})\} = \{y_{\mathrm{H}}^{(k)} - \psi_{\mathrm{H}}(x_{\mathrm{H}}^{(k)}, z^{(k)})\} \subseteq V \times \Re^m$$

are bounded. Then $\{\psi_{\mathrm{H}}(x_{\mathrm{H}}^{(k)}, z^{(k)})\}$ is also bounded. Since we assume that $\psi = (\psi_1, \psi_2)$ is affine, ψ_1 and ψ_2 are given by

$$\psi_1(x, z) = A_1 x + B_1 z + d_1, \quad \psi_2(x, z) = A_2 x + B_2 z + d_2,$$

for some linear operators A_i, B_i $(i = 1, 2)$, $d_1 \in V$ and $d_2 \in \Re^m$. Therefore, for any $\tau_k > 0$, we have

$$
\begin{aligned}
\tau_k \|\psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k)\| &= \|A_1 x^{(k)} + B_1 z^{(k)} + \tau_k d_1\| \\
&= \|A_1 x^{(k)} + B_1 z^{(k)} + d_1 - (1 - \tau_k) d_1\| \\
&\geq \|A_1 x^{(k)} + B_1 z^{(k)} + d_1\| - \|(1 - \tau_k) d_1\|,
\end{aligned}
$$

and by the definition (13) of ψ_{H_1},

$$
\begin{aligned}
\|\psi_1(x^{(k)}, z^{(k)})\| &= \|A_1 x^{(k)} + B_1 z^{(k)} + d_1\| \\
&\leq \tau_k \|\psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k)\| + \|(1 - \tau_k) d_1\| \\
&= \|\psi_{\mathrm{H}_1}(x_{\mathrm{H}}^{(k)}, z^{(k)})\| + \|(1 - \tau_k) d_1\|.
\end{aligned}
$$

By the boundedness of $\{(x_{\mathrm{H}}^{(k)}, z^{(k)}) = (x^{(k)}, \tau_k, z^{(k)})\}$ and of $\{\psi_{\mathrm{H}}(x_{\mathrm{H}}^{(k)}, z^{(k)})\}$, we know that $\{\psi_1(x^{(k)}, z^{(k)})\}$ is bounded. The boundedness of $\{\psi_2(x^{(k)}, z^{(k)})\}$ can be obtained similarly. Since we have seen that ψ is z-bounded, the above facts guarantee that $\{z^{(k)}\}$ is bounded, which implies the z-boundedness of F_{H}.

Next, we show that F_{H} is z-injective. Suppose that $(x_{\mathrm{H}}, y_{\mathrm{H}}, z), (x_{\mathrm{H}}, y_{\mathrm{H}}, z') \in \mathrm{int} K_{\mathrm{H}} \times \mathrm{int} K_{\mathrm{H}} \times \Re^m$ satisfy $F_{\mathrm{H}}(x_{\mathrm{H}}, y_{\mathrm{H}}, z) = F_{\mathrm{H}}(x_{\mathrm{H}}, y_{\mathrm{H}}, z')$. Then, by the definition (9) of F_{H}, we have

$$y - \tau \psi_1(x/\tau, z/\tau) = y - \tau \psi_1(x/\tau, z'/\tau), \quad \tau \psi_2(x/\tau, z/\tau) = \tau \psi_2(x/\tau, z/\tau).$$

Since ψ is z-injective, the equivalence $z = z'$ follows. ∎

5 Convex quadratic optimization problems over K

In this section, we discuss applications of Theorem 3.1 to the function ψ. Consider the quadratic convex optimization problem QO given by (4). The QO is a special case of the convex optimization problem CO:

$$\text{(CO) Minimize} \quad f(z)$$
$$\text{Subject to} \ \ g(z) \in -K \tag{20}$$

where $f : \Re^m \to \Re$ is continuously differentiable and convex, $g : \Re^m \to V$ is continuously differentiable and K-convex, i.e., for any $z, z' \in \Re^m$ and $\tau \in (0, 1)$,

$$\tau g(z) + (1 - \tau)g(z') - g(\tau z + (1 - \tau)z') \in K$$

holds. Rockafellar [8] discussed the optimality condition of the CO, and showed that under a suitable constraint qualification, there must exist $z \in \Re^m$ and $x \in K$ such that

$$g(z) \in -K, \ \ \nabla_z L(x, z) = 0, \ \ \langle x, g(z) \rangle = 0 \tag{21}$$

where $L : K \times \Re^m \to \Re$ is the Lagrangian function given by

$$L(x, z) = f(z) + \langle x, g(z) \rangle$$

(see also Shapiro [10] for semidefinite programming cases). Define

$$\psi(x, z) := \begin{pmatrix} \psi_1(x, z) \\ \psi_2(x, z) \end{pmatrix} := \begin{pmatrix} -g(z) \\ \nabla_z L(x, z) \end{pmatrix}. \tag{22}$$

Then, by (iii) of Proposition 2.1, the optimal condition (21) is equivalent to the MiCP with (22). The following proposition shows that the function ψ is monotone whenever the CO is convex.

Proposition 5.1 *Suppose that f is continuously differentiable and convex, and g is continuously differentiable and K-convex on $\Re^m$. Then the function ψ given by (22) is monotone on $K \times \Re^m$ in the sense of Assumption 1.1.*

Proof: Since the cone K is self-dual (i.e., $K = K^* := \{y : \langle x, y \rangle \geq 0 \ \text{ for all } x \in K\}$), it is easy to see that the K-convexity of g implies that $\langle x, g(z) \rangle$ is convex on $\Re^m$ for any fixed $x \in K$. Thus, for any $x \in K$, the function $L(x, \cdot)$ is continuously differentiable and convex on $\Re^m$. Let $x, x' \in K$. Then, we can see that

$$L(x, z') - L(x, z) - (z' - z)^T \nabla_z L(x, z) \geq 0, \ \ L(x', z) - L(x', z') - (z - z')^T \nabla_z L(x', z') \geq 0$$

hold for any $z, z' \in \Re^m$. By adding these two inequalities and by the definition (22) of ψ, we have

$$0 \leq L(x, z') - L(x, z) + L(x', z) - L(x', z') + (z - z')^T [\nabla_z L(x, z) - \nabla_z L(x', z')]$$
$$= \langle x - x', -g(x) + g(z') \rangle + (z - z')^T [\nabla_z L(x, z) - \nabla_z L(x', z')]$$
$$= \langle \psi_1(x, z) - \psi_1(x', z'), x - x' \rangle + [\psi_2(x, z) - \psi_2(x', z')]^T (z - z').$$

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Thus, ψ is monotone in the sense of Assumption 1.1.   ∎

The Lagrangian function $L : K \times \Re^m \to \Re$ for the QO is given by

$$L(x, z) = \frac{1}{2} z^T Q z + c^T z + \langle x, Az - b \rangle$$

and we see that

$$\psi(x, z) := \begin{pmatrix} \psi_1(x, z) \\ \psi_2(x, z) \end{pmatrix} := \begin{pmatrix} -Az + b \\ Qz + c + A^* x \end{pmatrix} \tag{23}$$

where $A^* : V \to \Re^m$ is the adjoint of A. Here, we assume that $\mathrm{rank}(A) = m$ and Q is positive semidefinite. Thus, it is easy to see that the function ψ satisfies (i) and (iii) of Assumption 1.1. In addition, we can see that $g(z) = Az - b$ is K-convex, and $f(z) = \frac{1}{2} z^T Q z$ is convex. Thus, the function ψ defined above is monotone by Proposition 5.1. Since the above properties are invalid if we restrict the domain of ψ to $K \times \Re^m$, we obtain the following corollary.

Corollary 5.2 *Suppose that the function ψ is given by (23). If the rank of the linear operator A is m and the symmetric matrix Q is positive semidefinite, then ψ whose domain is restricted to $K \times \Re^m$ satisfies Assumption 1.1.*

It should be noted that the QO is equivalent to

$$\text{Minimize} \quad \alpha$$
$$\text{Subject to} \quad \tfrac{1}{2} z^T Q z + c^T z \le \alpha, \quad Az - b \in -K.$$

We can represent the above problem as

$$\text{(QO1) Minimize} \quad \alpha$$
$$\text{Subject to} \quad h(z, \alpha) \in -K_0 \quad Az - b \in -K$$

where $h : \Re^{m+1} \to \Re^{r+1}$ with $r = \mathrm{rank}(Q)$ is an affine function and K_0 is a second-order cone. Since the QO1 is a linear optimization problem over a symmetric cone, the QO can be solved by using a primal-dual framework for linear conic optimization problems. However, we should add $r = \mathrm{rank}(Q)$ new variables to the dual form, and the size of primal-dual problems may become considerably larger than the size of the original problem QO. A merit of our homogeneous model is that we can deal with the QO directly without adding a large number of variables.

Finally, we consider a more special case of the QO, i.e., the linear optimization problem (LO) over the symmetric cone K, which can be obtained by setting $Q = O$ in the QO:

$$\text{(LO) Minimize} \quad c^T z$$
$$\text{Subject to} \quad Az - b \in -K$$

It is well known that the (primal) LO has the following mutually exclusive cases:

- The problem is strictly feasible, i.e., there exists $\bar{z}$ such that $Az - b \in -\mathrm{int} K$.

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Primal

		feasible		infeasible	
		strictly fesible	others	asympt. feasible	others
Dual feasible	strictly fesible	$\tau > 0$ $\kappa = 0$	$\tau >= 0$ $\kappa = 0$	$\tau = 0$ $\kappa = 0$	$\tau = 0$ $\kappa > 0$
	others	$\tau >= 0$ $\kappa = 0$	$\tau >= 0$ $\kappa = 0$	$\tau = 0$ $\kappa = 0$	$\tau = 0$ $\kappa > 0$
infeasible	asympt. feasible	$\tau = 0$ $\kappa = 0$	$\tau = 0$ $\kappa = 0$	$\tau = 0$ $\kappa = 0$	$\tau = 0$ $\kappa > 0$
	others	$\tau = 0$ $\kappa > 0$	$\tau = 0$ $\kappa > 0$	$\tau = 0$ $\kappa > 0$	$\tau = 0$ $\kappa > 0$

Table 1: Theorem 3.1 for the linear conic optimization.

- The problem is feasible, i.e., there exists $\bar{z}$ such that $Az - b \in -K$, but not strictly feasible.

- The problem is infeasible, but asymptotically feasible, i.e. there exists an (unbounded) sequence $\{z^{(k)}\}$ such that $\lim_{k \to \infty} Az^{(k)} - b \in -K$.

- The problem is infeasible and not asymptotically feasible.

Note that the terminology *asymptotically* above is used differently from the way in Definition 1.2, where the *asymptotically* converging sequence $\{\zeta^{(k)}\}$ should be bounded. It is also known that the dual problem has the corresponding four cases similarly, and all of the possible 16 cases of primal and dual pair of LOs have concrete examples.

Table 1 shows the results obtained by applying our homogeneous model to primal-dual pair of LOs (see Theorem 3.1). Each case shows the possible signs of the variables $\tau(0)$ and $\kappa(0)$ at any accumulation point of the path (19). The results shown in Table 1 are weaker than those obtained for the self-dual embedding model for the LOs proposed in [6], in terms of the discriminant ability to detect the primal infeasibility or the dual infeasibility. A merit of our homogeneous model is that, as we have seen in Theorems 3.1 and 3.2, it has applicability to optimality conditions of nonlinear optimization problems over symmetric cones, whenever the corresponding functions F and F_{H} satisfy Assumption 2.2.

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A   Proof of Theorem 3.1

(i): The equation follows from the definition of (13).
(ii): Suppose that $(\hat{x}_\mathrm{H}, \hat{y}_\mathrm{H}, \hat{z})$ is an asymptotically feasible solution (see Definition 1.2). Then there exists a bounded sequence $(x_\mathrm{H}^k, y_\mathrm{H}^k, z^{(k)}) \in \mathrm{int}K_\mathrm{H} \times \mathrm{int}K_\mathrm{H} \times \Re^m$ such that

$$\lim_{k\to 0}(y_\mathrm{H}^k - \psi_{\mathrm{H}1}(x_\mathrm{H}^k, z^{(k)})) = \text{ and } \lim_{k\to\infty}\psi_{\mathrm{H}2}(x_\mathrm{H}^k, z^{(k)})) = 0.$$

The assertion (i) implies that

$$\langle x_\mathrm{H}^k, y_\mathrm{H}^k\rangle_\mathrm{H} = \langle x_\mathrm{H}^k, y_\mathrm{H}^k\rangle_\mathrm{H} - \left\{\langle \psi_{\mathrm{H}1}(x_\mathrm{H}^k, z^{(k)}), x_\mathrm{H}^k\rangle_\mathrm{H} + \psi_{\mathrm{H}2}(x_\mathrm{H}^k, z^{(k)})^T z^{(k)}\right\}$$
$$= \langle y_\mathrm{H}^{(k)} - \psi_{\mathrm{H}1}(x_\mathrm{H}^k, z^{(k)}), x_\mathrm{H}^k\rangle_\mathrm{H} - \psi_{\mathrm{H}2}(x_\mathrm{H}^k, z^{(k)})^T z^{(k)}$$

holds for every $k \geq 0$. Thus, by the boundedness of $z^{(k)}$, we see that $\lim_{k\to\infty}\langle x_\mathrm{H}^k, y_\mathrm{H}^k\rangle_\mathrm{H} = 0$ and $(\hat{x}_\mathrm{H}, \hat{y}_\mathrm{H}, \hat{z})$ is an asymptotically complementary solution.

(iii): For every $k \geq 0$, define

$$x^{(k)} := (1/2)^k e \in \mathrm{int}K, \ \tau_k := (1/2)^k \in \Re_{++},$$
$$y^{(k)} := (1/2)^k e \in \mathrm{int}K, \ \kappa_k := (1/2)^k \in \Re_{++}, \ z^{(k)} := 0 \in \Re^m.$$

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It is easy to see that the bounded sequence $(x_{\mathrm{H}}^k, y_{\mathrm{H}}^k, z^{(k)}) = \{(x^{(k)}, \tau_k, y^{(k)}, \kappa_k, z^{(k)})\} \subseteq \mathrm{int} K_{\mathrm{H}} \times \Re^m$ satisfies

$$\lim_{k \to \infty} (y_{\mathrm{H}}^k - \psi_{\mathrm{H}_1}(x_{\mathrm{H}}^k, z^{(k)})) = 0, \quad \lim_{k \to \infty} \psi_{\mathrm{H}_2}(x_{\mathrm{H}}^{(k)}, z^{(k)}) = 0.$$

(iv): If $(x^*, \tau^*, y^*, \kappa^*, z^*) \in (K \times \Re_{++}) \times (K \times \Re_+) \times \Re^m$ is a solution of the HMiCP with $\tau^* > 0$ then

$$y^*/\tau^* - \psi_1(x^*/\tau^*, z^*/\tau^*) = 0, \quad \tau^* \psi_2(x^*/\tau^*, z^*/\tau^*) = 0, \quad x^* \circ y^* = 0$$

and $(x^*/\tau^*, y^*/\tau^*, z^*/\tau^*) \in K \times K \times \Re^m$ is a solution of the MiCP. Conversely, if $(\hat{x}, \hat{y}, \hat{z}) \in K \times K \times \Re^m$ is a solution of the MiCP, then $(\hat{x}, 1, \hat{y}, 0, \hat{z}) \in (K \times \Re_{++}) \times (K \times \Re_+) \times \Re^m$ is a solution of the MiHCP.

(v): By (ii) of Theorem 2.3, the set $F(\mathcal{U} \times \Re^m)$ is open and convex.

If the MiCP is strongly infeasible, then we must have $0 \notin \mathrm{cl}(F(\mathcal{U} \times \Re^m))$. Since the set $\mathrm{cl}(F(\mathcal{U} \times \Re^m))$ is a closed convex set, by the separating hyperplane theorem, there exists $a = (a_1, a_2) \in V \times \Re^m$ with $\|a\| = 1$ and $\xi \in \Re$ that

$$\langle a, b \rangle \geq \xi > 0 \quad \text{for all} \quad b = (b_1, b_2) \in \mathrm{cl}(F(\mathcal{U} \times \Re^m)). \tag{24}$$

Since F is continuous on the set $\mathrm{cl}(\mathcal{U} \times \Re^m) \subseteq K \times K \times \Re^m$, we can see that $F(\mathrm{cl}(\mathcal{U} \times \Re^m)) \subseteq \mathrm{cl}(F(\mathcal{U} \times \Re^m))$. Therefore (24) implies that

$$\begin{aligned}
\langle a, F(x, y, z) \rangle &= \langle a_1, y - \psi_1(x, z) \rangle + a_2^T \psi_2(x, z) \\
&= \langle a_1, y \rangle - \langle a_1, \psi_1(x, z) \rangle + a_2^T \psi_2(x, z) \\
&\geq \xi > 0
\end{aligned} \tag{25}$$

for all $(x, y, z) \in \mathrm{cl}(\mathcal{U} \times \Re^m)$. Note that (iv) of Proposition 2.1 ensures that the above relation (25) holds at $(x, y, z) = (0, \alpha \bar{y}, 0)$ for any $\bar{y} \in K$ and $\alpha > 0$. Thus, it must be true that $\langle a_1, \bar{y} \rangle \geq 0$ for all $\bar{y} \in K$. which implies that $a_1 \in K$. Similarly, since $(x, 0, z) \in \mathrm{cl}(\mathcal{U} \times \Re^m)$ for every $(x, z) \in K \times \Re^m$, it follows from (25) that

$$-\langle a_1, \psi_1(x, z) \rangle + a_2^T \psi_2(x, z) \geq \xi > 0 \tag{26}$$

for all $(x, z) \in K \times \Re^m$. Combining with the fact that $(a_1, -a_2) \in K \times \Re^m$, we see that

$$-\langle a_1, \psi_1(\beta a_1, -\beta a_2) \rangle + a_2^T \psi_2(\beta a_1, -\beta a_2) \geq \xi > 0 \text{ for all } \beta \geq 0. \tag{27}$$

From the monotonicity of the map ψ on the set $K \times \Re^m$, we see that

$$\begin{aligned}
0 &\leq \langle \beta x - x, \psi_1(\beta x, \beta z) - \psi_1(x, z) \rangle + [\psi_2(\beta x, \beta z) - \psi_2(x, z)]^T (\beta z - z) \\
&= (\beta - 1)\langle x, \psi_1(\beta x, \beta z) - \psi_1(x, z) \rangle + (\beta - 1)[\psi_2(\beta x, \beta z) - \psi_2(x, z)]^T z \\
&= (\beta - 1)\left\{ \langle x, \psi_1(\beta x, \beta z) - \psi_1(x, z) \rangle + [\psi_2(\beta x, \beta z) - \psi_2(x, z)]^T z \right\}
\end{aligned}$$

for all $(x, z) \in K \times \Re^m$ and $\beta \geq 0$. Thus, for all $\beta \geq 1$, we should have

$$\langle x, \psi_1(\beta x, \beta z) - \psi_1(x, z) \rangle + [\psi_2(\beta x, \beta z) - \psi_2(x, z)]^T z \geq 0 \tag{28}$$

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and hence,

$$\lim_{\beta \to \infty} \left\{ \langle x, \psi_1(\beta x, \beta z) \rangle + \psi_2(\beta x, \beta z)^T z \right\} / \beta \geq 0. \tag{29}$$

For each $(x, z) \in K \times \Re^m$, define the set

$$\Psi^{\infty}(x, z) := \{ \psi^{\infty}(x, z) = (\psi_1^{\infty}(x, z), \psi_2^{\infty}(x, z)) : \ \psi(\beta^k x, \beta^k z) / \beta^k \to \psi^{\infty}(x, z) \text{ for some } \beta^k \to \infty \}$$

where $\psi^{\infty}(x, z) \in \Psi^{\infty}(x, z)$ may have elements of ∞ or $-\infty$.

We claim that $\Psi_1^{\infty}(a_1, -a_2) \subseteq K$. Let $\{\beta_k\}$ be a subsequence such that $\beta_k \to +\infty$ and $\psi(\beta^k a_1, \beta^k(-a_2))/\beta^k \to \psi^{\infty}(a_1, -a_2)$, and let

$$\frac{\psi_1(\beta_k a_1, -\beta a_2)}{\beta_k} = \sum_{i=1}^{r} \lambda_i^{(k)} c_i^{(k)}, \ (k = 1, 2, \ldots) \tag{30}$$

be a decomposition given by (ii) of Proposition 2.1. We also define

$$\lambda_k := \min\{\lambda_i^{(k)}(i = 1, 2, ..., r)\}, \ j_k \in \operatorname{argmin}\{\lambda_i^{(k)}(i = 1, 2, ..., r)\}, \ c^{(k)} := c_{j_k}^{(k)}. \tag{31}$$

Note that $\{c^{(k)}\}$ is a sequence of primitive (i.e., nozero) idempostents of a Euclidean Jordan algebra $(V, \circ)$. Thus, by (v) of Proposition 2.1, there exists $\omega_1 > 0$ and $\omega_2 > 0$ such that

$$0 < \omega_1 \leq \|c^{(k)}\| \leq \omega_2 \text{ for every } k. \tag{32}$$

Suppose that $\psi_1^{\infty}(a_1, -a_2) \notin K$. Then there exists a $\delta > 0$ for which $\lambda_k \leq -\delta < 0$ for sufficiently large k's. Define $x^{(k)} := a_1 + \epsilon c^{(k)}$ for $\epsilon > 0$. We can see that

$$\begin{aligned}
&\left\{ \langle x^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2) \rangle - \psi_2(\beta_k x^{(k)}, -\beta_k a_2)^T a_2 \right\} / \beta_k \\
&= \left\{ \langle a_1 + \epsilon c^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2) \rangle - \psi_2(\beta_k x^{(k)}, -\beta_k a_2)^T a_2 \right\} \ \beta_k \\
&= \left\{ \langle a_1, \psi_1(\beta_k x^{(k)}, -\beta_k a_2) \rangle - \psi_2(\beta_k x^{(k)}, -\beta_k a_2)^T a_2 \right\} / \beta_k + \epsilon \langle c^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2) \rangle / \beta_k \\
&< \epsilon \langle c^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2) \rangle / \beta_k \quad \text{(by (26))} \\
&= \epsilon \left\{ \langle c^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2) - \psi_1(\beta_k a_1, -\beta_k a_2) \rangle + \langle c^{(k)}, \psi_1(\beta_k a_1, -\beta_k a_2) \rangle \right\} / \beta_k. \tag{33}
\end{aligned}$$

The definitions (30) and (31) and the boundedness (32) of $\{c^{(k)}\}$ ensure that

$$\langle c^{(k)}, \psi_1(\beta_k a_1, -\beta_k a_2) \rangle / \beta_k = \lambda_k \langle c^{(k)}, c^{(k)} \rangle \leq -\delta \omega_1^2 < 0 \tag{34}$$

for sufficiently large k's. In addition, since we set $x^{(k)} = a_1 + \epsilon c^{(k)}$, by the continuity of ψ_1 and the boundedness of $\{c^{(k)}\}$, we have

$$\langle c^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2) - \psi_1(\beta_k a_1, -\beta_k a_2) \rangle / \beta_k = \mathcal{O}(\epsilon) \tag{35}$$

for sufficiently small ϵ's. Thus, by (34) and (35),

$$\langle c^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2) - \psi_1(\beta_k a_1, -\beta_k a_2) \rangle / \beta_k + \langle c^{(k)}, \psi_1(\beta_k a_1, -\beta_k a_2) \rangle / \beta_k \leq -\delta \omega_1^2 / 2 < 0$$

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and by (33),

$$\left\{ \langle x^{(k)}, \psi_1(\beta_k x^{(k)}, -\beta_k a_2) \rangle - \psi_2(\beta_k x^{(k)}, -\beta_k a_2)^T a_2 \right\} / \beta_k \leq -\epsilon \delta \omega_1^2 / 2 < 0$$

holds for sufficiently large k's and sufficiently small ϵ's. Since $x^{(k)} = a + \epsilon c^{(k)} \in K$, by fixing a suitably small $\epsilon > 0$, the above inequality contradicts to (29) and we must have $\psi_1^\infty(a_1, -a_2) \in K$.

Next we claim that $\psi_2^\infty(a_1, -a_2) = 0$. Suppose that $\psi_2^\infty(a_1, -a_2) \neq 0$, then there exists a $v \in \Re^m$ with $v \neq 0$, for which $v^T \psi_2^\infty(a_1, -a_2) < -\omega$ for some $\omega > 0$. Define $z(\epsilon) = -a_2 + \epsilon v$ for $\epsilon > 0$. Since the v is a constant vector, we see that

$$v^T \psi_2(\beta_k a_1, -\beta_k a_2) / \beta_k < -\omega / 2 < 0 \tag{36}$$

for sufficiently large k's. In addition, by the continuity of ψ_2, we have

$$v^T [\psi_2(\beta_k a_1, -\beta_k z(\epsilon)) - \psi_2(\beta_k a_1, -\beta_k a_2)] / \beta_k = \mathcal{O}(\epsilon) \tag{37}$$

for sufficiently small ϵ's. Thus, by (36) and (37),

$$v^T [\psi_2(\beta_k a_1, -\beta_k z(\epsilon))] < 0 \tag{38}$$

holds for sufficiently large k's and sufficiently small ϵ's. Here we can calculate that

$$\begin{aligned}
&\left\{ \langle a_1, \psi_1(\beta_k a_1, \beta_k z(\epsilon)) \rangle + \psi_2(\beta_k a_1, \beta_k z(\epsilon))^T z(\epsilon) \right\} / \beta_k \\
&= \left\{ \langle a_1, \psi_1(\beta_k a_1, \beta_k z(\epsilon)) \rangle - \psi_2(\beta_k a_1, -\beta_k z(\epsilon))^T a_2 \right\} / \beta_k + \epsilon \delta \psi_2(\beta_k a_1, -\beta_k z(\epsilon))^T / \beta_k \\
&< \epsilon \delta \psi_2(\beta_k a_1, -\beta_k z(\epsilon))^T / \beta_k
\end{aligned}$$

by (26) and hence, from (38),

$$\left\{ \langle a_1, \psi_1(\beta_k a_1, \beta_k z(\epsilon)) \rangle + \psi_2(\beta_k a_1, \beta_k z(\epsilon))^T z(\epsilon) \right\} / \beta_k < 0$$

for sufficiently large k's and sufficiently small ϵ's,

By fixing a suitably small $\epsilon > 0$, the above inequality contradicts to (29). Thus, we must have $\psi_2^\infty(a_1, -a_2) = 0$.

We are going to show that $\psi^\infty(a_1, -a_2)$ is bounded. Since $(\beta_k a_1, -\beta_k a_2)$ and $(e, 0)$ are in $K \times \Re^m$, by the monotonicity of ψ, we have

$$\begin{aligned}
0 &\leq \langle (\beta_k a_1, -\beta_k a_2) - (e, 0), \psi(\beta_k a_1, -\beta_k a_2) - \psi(e, 0) \rangle / \beta_k \\
&= \langle a_1, \psi_1(\beta_k a_1, -\beta_k a_2) \rangle - a_2^T \psi_2(\beta_k a_1, -\beta_k a_2) \\
&\quad + a_2^T \psi_2(e, 0) + \langle e, \psi_1(x', z(\epsilon)) \rangle / \beta_k - \langle a_1, \psi_1(e, 0) \rangle - \langle e, \psi_1(\beta_k a_1, -\beta_k a_2) \rangle / \beta_k \\
&\leq a_2^T \psi_2(e, 0) + \langle e, \psi_1(x', z(\epsilon)) \rangle / \beta_k - \langle a_1, \psi_1(e, 0) \rangle - \langle e, \psi_1(\beta_k a_1, -\beta_k a_2) \rangle / \beta_k
\end{aligned}$$

where the last inequality follows from (26). Taking a limit as $k \to \infty$ from both sides, we have

$$\langle e, \psi_1^\infty(a_1, -a_2) \rangle \leq a_2^T \psi_2(e, 0) - \langle a_1, \psi_1(e, 0) \rangle,$$

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which implies that $\psi_1^\infty(a_1, -a_2) \in K$ is bounded (see (i) of Proposition 2.1). Note that from (27) and (28), we have

$$-\xi \geq \langle a_1, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle - a_2^T \psi_2(\beta_k a_1, -\beta_k a_2) \geq \langle a_1, \psi_1(a_1, -a_2)\rangle - a_2^T \psi_2(a_1, -a_2).$$

Thus the sequence $\{\langle a_1, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle - a_2^T \psi_2(\beta_k a_1, -\beta_k a_2)\}$ is also bounded. To summarize, by setting

$$x^* := a_1 \in K, \qquad \tau^* := \lim_{\beta_k \to \infty} \frac{1}{\beta_k} = 0, \qquad z^* := -a_2 \in \Re^m,$$

$$y^* := \psi_1^\infty(a_1, -a_2) \in K, \; \kappa^* := \lim_{k \to \infty} \{-\langle a_1, \psi_1(\beta_k a_1, -\beta_k a_2)\rangle + a_2^T \psi_2(\beta_k a_1, -\beta_k a_2)\} \geq \xi > 0,$$

the MiHCP has an asymptotical solution $(x^*, \tau^*, y^*, \kappa^*, z^*) \in (K \times \Re_+) \times (K \times \Re_+) \times \Re^m$ with $\kappa^* > 0$.

Conversely, suppose that exists a bounded sequence $\{(x^{(k)}, \tau_k, y^{(k)}, \kappa_k, z^{(k)})\} \subseteq (\text{int}K \times \Re_{++}) \times (\text{int}K \times \Re_{++}) \times \Re^m$ such that

$$\lim_{k \to \infty} y^{(k)} = \lim_{k \to \infty} \tau_k \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k) \in K,$$
$$\lim_{k \to \infty} \kappa_k = \lim_{k \to \infty} \{-\langle \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k), x^{(k)}\rangle - \psi_2(x^{(k)}/\tau_k, z^{(k)}/\tau_k)^T z^{(k)}\} \geq \xi > 0,$$
$$0 = \lim_{k \to \infty} \tau_k \psi_2(x^{(k)}/\tau_k, z^{(k)}/\tau_k).$$

Let us show that there is no feasible point $(x, y, z) \in K \times K \times \Re^m$ satisfying $y - \psi_1(x, z) = 0$ and $\psi_2(x, z) = 0$. Suppose that $(x, y, z) \in K \times K \times \Re^m$ satisfies $y - \psi_1(x, z) = 0$ and $\psi_2(x, z) = 0$, and define $x_H = (x, 1)$. Since ψ_H is monotone on $(K \times \Re_{++}) \times (K \times \Re_+) \times \Re^m$, by the definition (13) of ψ_H and by the assumptions $\psi_1(x, z) = y$ and $\psi_2(x, z) = 0$, we have

$$\begin{aligned}
0 &\leq \langle (x_H^{(k)}, z^{(k)}) - (x_H, z), \psi_H(x_H^{(k)}, z^{(k)}) - \psi_H(x_H, z)\rangle \\
&= \langle \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k), x^{(k)}\rangle + \psi_2(x^{(k)}/\tau_k, z^{(k)}/\tau_k)^T z^{(k)} \\
&\quad -\langle x^{(k)}, y\rangle - \langle \tau_k \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k), x\rangle - \tau_k \psi_2(x^{(k)}/\tau_k, z^{(k)}/\tau_k)^T z + \tau_k \langle x, y\rangle. \quad (39)
\end{aligned}$$

Here, we see that $\langle x^{(k)}, y\rangle \geq 0$,

$$\lim_{k \to \infty} \langle x, y^{(k)}\rangle = \langle x, \lim_{k \to \infty} \tau_k \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k)\rangle \geq 0,$$

and $\lim_{k \to \infty} \tau_k = 0$ since $\lim_{k \to \infty} \kappa_k \geq \xi > 0$. Thus the relation (39) ensures that

$$\lim_{k \to \infty} \langle \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k), x^{(k)}\rangle + \psi_2(x^{(k)}/\tau_k, z^{(k)}/\tau_k)^T z^{(k)} \geq 0$$

which contradicts to

$$\kappa_k = -\langle \psi_1(x^{(k)}/\tau_k, z^{(k)}/\tau_k), x^{(k)}\rangle - \psi_2(x^{(k)}/\tau_k, z^{(k)}/\tau_k)^T z^{(k)} \geq \xi > 0.$$

In addition, any limit of $x^{(k)}$ gives a separation hyperplane, i.e., a certificate proving infeasibility. ∎

B   Proof of Theorem 3.2

(i) : Since the map F_{H} satisfies Assumption 2.2, by (ii) of Theorem 2.3, the set $H_{\mathrm{H}}(\mathcal{U}_{\mathrm{H}} \times \Re^m)$ with

$$\mathcal{U}_{\mathrm{H}} := \{(x_{\mathrm{H}}, y_{\mathrm{H}}) \in \mathrm{int}K_{\mathrm{H}} \times \mathrm{int}K_{\mathrm{H}} : x_{\mathrm{H}} \circ_{\mathrm{H}} y_{\mathrm{H}} \in \mathrm{int}K_{\mathrm{H}}\}$$

is an open convex subset of $\mathrm{int}K_{\mathrm{H}} \times V_{\mathrm{H}} \times \Re^m$. Note that we have already seen that $0 \in \mathrm{cl}\left(H_{\mathrm{H}}(\mathcal{U}_{\mathrm{H}} \times \Re^m)\right)$ in (ii) and (iii) of Theorem 3.1, and $h_{\mathrm{H}}^{(0)} \in H_{\mathrm{H}}(\mathcal{U}_{\mathrm{H}} \times \Re^m)$ by the definition (17). Since the set $H_{\mathrm{H}}(\mathcal{U}_{\mathrm{H}} \times \Re^m)$ is convex, the fact above implies that $th_{\mathrm{H}}^{(0)} \in H_{\mathrm{H}}(\mathcal{U}_{\mathrm{H}} \times \Re^m)$ for every $t \in (0, 1]$. Combining this with the homeomorphism of the map H_{H} in (i) of Theorem 2.3, we obtain the assertion (i).

(ii): It follows from (i) of Theorem 2.3 that the map H_{H} is a homeomorphism and the set P forms a path in $\mathrm{int}K_{\mathrm{H}} \times \mathrm{int}K_{\mathrm{H}} \times \Re^m$. Therefore, it suffices to show that the path P is bounded. Let $(x_{\mathrm{H}}, y_{\mathrm{H}}, z) = (x, \tau, y, \kappa, z) \in P$. Then there exists a $t \in (0, 1]$ for which $H_{\mathrm{H}}(x_{\mathrm{H}}, y_{\mathrm{H}}, z) = th_{\mathrm{H}}^{(0)}$ i.e.,

$$x_{\mathrm{H}} \circ_{\mathrm{H}} y_{\mathrm{H}} = te_{\mathrm{H}}, \ y_{\mathrm{H}} - \psi_1(x_{\mathrm{H}}, z) = t\left(y_{\mathrm{H}}^{(0)} - \psi_{\mathrm{H}1}(x_{\mathrm{H}}^{(0)}, z^{(0)})\right), \ \psi_{\mathrm{H}2}(x_{\mathrm{H}}, z) = t\psi_{\mathrm{H}2}(x_{\mathrm{H}}^{(0)}, z^{(0)})$$

hold (see (17)). By analogous discussions as in the proof of Theorem 5.5 in [11], we can see that

$$
\begin{aligned}
\langle x_{\mathrm{H}}, y_{\mathrm{H}}^{(0)} \rangle_{\mathrm{H}} + \langle x_{\mathrm{H}}^{(0)}, y_{\mathrm{H}} \rangle_{\mathrm{H}} &\leq t\langle x_{\mathrm{H}}^{(0)}, y_{\mathrm{H}}^{(0)} \rangle_{\mathrm{H}} + \langle x_{\mathrm{H}}^{(0)}, y_{\mathrm{H}}^{(0)} \rangle_{\mathrm{H}} \\
&= (1 + t)\langle x_{\mathrm{H}}^{(0)}, y_{\mathrm{H}}^{(0)} \rangle_{\mathrm{H}} \\
&= (1 + t)(r + 1) \qquad \text{(by (17))} \\
&\leq 2(r + 1).
\end{aligned}
$$

Thus, the boundedness of the set P follows from (i) of Proposition 2.1 and z-boundedness of F_{H}.

(iii): Let $(x_{\mathrm{H}}^*, y_{\mathrm{H}}^*, z^*) = (x^*, \tau^*, y^*, \kappa^*, z^*)$ be an asymptotical solution to the HCP. Then there exists a bounded sequence

$$\{(x_{\mathrm{H}}^{(k)}, y_{\mathrm{H}}^{(k)}, z^{(k)})\} = \{(x^{(k)}, \tau_k, y^{(k)}, \kappa_k, z^{(k)})\} \subseteq \mathrm{int}K_{\mathrm{H}} \times \mathrm{int}K_{\mathrm{H}} \times \Re^m$$

such that

$$\lim_{k \to \infty}(x_{\mathrm{H}}^{(k)}, y_{\mathrm{H}}^{(k)}, z^{(k)}) = (x_{\mathrm{H}}^*, y_{\mathrm{H}}^*, z^*), \lim_{k \to \infty} y_{\mathrm{H}}^{(k)} - \psi_{\mathrm{H}1}(x_{\mathrm{H}}^{(k)}, z^{(k)}) = 0,$$

$$\lim_{k \to \infty} \psi_{\mathrm{H}2}(x_{\mathrm{H}}^{(k)}, z^{(k)}) = 0, \lim_{k \to \infty} x_{\mathrm{H}}^{(k)} \circ_{\mathrm{H}} y_{\mathrm{H}}^{(k)} = 0.$$

Let $(x_{\mathrm{H}}(t), y_{\mathrm{H}}(t), z(t)) = (x(t), \tau(t), y(t), \kappa(t), z(t))$ be any point on the path P. Then,

$$
\begin{aligned}
x_{\mathrm{H}}(t) \circ_{\mathrm{H}} y_{\mathrm{H}}(t) &= te_{\mathrm{H}}, \\
y_{\mathrm{H}}(t) - \psi_{\mathrm{H}1}(x_{\mathrm{H}}(t), z(t)) &= t[y_{\mathrm{H}}^{(0)} - \psi_{\mathrm{H}1}(x_{\mathrm{H}}^{(0)}, z^{(0)})], \qquad\qquad (40) \\
\psi_{\mathrm{H}2}(x_{\mathrm{H}}(t), z(t)) &= t\psi_{\mathrm{H}2}(x_{\mathrm{H}}^{(0)}, z^{(0)}).
\end{aligned}
$$

By the boundedness of the set P as we have see in (ii) above, there exists an $\epsilon \in (0, 1]$ such that

$$\|x_{\mathrm{H}}(t)\| \leq 1/\epsilon, \ \|y_{\mathrm{H}}(t)\| \leq 1/\epsilon, \ \|z(t)\| \leq 1/\epsilon \qquad\qquad (41)$$

holds for every $t \in (0, 1]$. In addition, for each $t \in (0, 1]$, there exists an index $k(t)$ such that for every $k \geq k(t)$, we have

$$\begin{aligned}
&\|x_\mathrm{H}^{(k)} - x_\mathrm{H}^*\| \leq \epsilon, \quad \|y_\mathrm{H}^{(k)} - y_\mathrm{H}^*\| \leq \epsilon, \quad \|z^{(k)} - z^*\| \leq \epsilon, \\
&\|y_\mathrm{H}^{(k)} - \psi_{\mathrm{H}_1}(x_\mathrm{H}^{(k)}, z^{(k)})\| \leq t\epsilon, \quad \|\psi_{\mathrm{H}_2}(x_\mathrm{H}^{(k)}, z^{(k)})\| \leq t\epsilon.
\end{aligned} \tag{42}$$

Since ψ_H is monotone and (i) of Theorem 3.1 holds, by analogous calculations as in the proof of Theorem 5.5 in [11], we can see that

$$\begin{aligned}
&\langle x_\mathrm{H}(t), y_\mathrm{H}^{(k)} \rangle_\mathrm{H} + \langle y_\mathrm{H}(t), x_\mathrm{H}^{(k)} \rangle_\mathrm{H} \\
&\leq \langle x_\mathrm{H}(t), y_\mathrm{H}^{(k)} - \psi_{\mathrm{H}_1}(x_\mathrm{H}^{(k)}, z^{(k)}) \rangle_\mathrm{H} + \langle x_\mathrm{H}^{(k)}, y_\mathrm{H}(t) - \psi_{\mathrm{H}_1}(x_\mathrm{H}(t), z(t)) \rangle_\mathrm{H} \\
&\quad - \psi_{\mathrm{H}_2}(x_\mathrm{H}(t), z(t))^T z^{(k)} - \psi_{\mathrm{H}_2}(x_\mathrm{H}^{(k)}, z^{(k)})^T z(t)
\end{aligned}$$

for every $t \in (0, 1]$ and every $\kappa \geq \kappa(t)$. Therefore, it follows from (40), (41) and (42) that

$$\begin{aligned}
&\langle x_\mathrm{H}(t), y_\mathrm{H}^{(k)} \rangle_\mathrm{H} + \langle y_\mathrm{H}(t), x_\mathrm{H}^{(k)} \rangle_\mathrm{H} \\
&\leq \langle x_\mathrm{H}(t), y_\mathrm{H}^{(k)} - \psi_{\mathrm{H}_1}(x_\mathrm{H}^{(k)}, z^{(k)}) \rangle_\mathrm{H} + \langle x_\mathrm{H}^{(k)}, y_\mathrm{H}(t) - \psi_{\mathrm{H}_1}(x_\mathrm{H}(t), z(t)) \rangle_\mathrm{H} \\
&\quad - \psi_{\mathrm{H}_2}(x_\mathrm{H}(t), z(t))^T z^{(k)} - \psi_{\mathrm{H}_2}(x_\mathrm{H}^{(k)}, z^{(k)})^T z(t) \\
&= \langle x_\mathrm{H}(t), y_\mathrm{H}^{(k)} - \psi_{\mathrm{H}_1}(x_\mathrm{H}^{(k)}, z^{(k)}) \rangle_\mathrm{H} + \langle x_\mathrm{H}^{(k)}, t[y_\mathrm{H}^{(0)} - \psi_{\mathrm{H}_1}(x_\mathrm{H}^{(0)}, z^{(0)})] \rangle_\mathrm{H} \\
&\quad - [t\psi_{\mathrm{H}_2}(x_\mathrm{H}^{(0)}, z^{(0)})]^T z^{(k)} - \psi_{\mathrm{H}_2}(x_\mathrm{H}^{(k)}, z^{(k)})^T z(t) \\
&\leq \|x_\mathrm{H}(t)\| \|y_\mathrm{H}^{(k)} - \psi_{\mathrm{H}_1}(x_\mathrm{H}^{(k)}, z^{(k)})\| + t\|x_\mathrm{H}^{(k)}\| \|y_\mathrm{H}^{(0)} - \psi_{\mathrm{H}_1}(x_\mathrm{H}^{(0)}, z^{(0)})\| \\
&\quad + t\|\psi_{\mathrm{H}_2}(x_\mathrm{H}^{(0)}, z^{(0)})\| \|z^{(k)}\| + \|\psi_{\mathrm{H}_2}(x_\mathrm{H}^{(k)}, z^{(k)})\| \|z(t)\| \\
&\leq (1/\epsilon)(t\epsilon) + t(\|x_\mathrm{H}^*\| + \epsilon)\|y_\mathrm{H}^{(0)} - \psi_{\mathrm{H}_1}(x_\mathrm{H}^{(0)}, z^{(0)})\| \\
&\quad + t\|\psi_{\mathrm{H}_2}(x_\mathrm{H}^{(0)}, z^{(0)})\|(\|z^*\| + \epsilon) + (t\epsilon)(1/\epsilon) \\
&\leq t\delta
\end{aligned}$$

where $\delta := 2 + \|h_\mathrm{H}^{(0)}\|(\|x_\mathrm{H}^*\| + \|z^*\| + 2)$. Note that (40) implies

$$x_\mathrm{H}(t) = ty_\mathrm{H}(t)^{-1}, \quad y_\mathrm{H}(t) = tx_\mathrm{H}(t)^{-1}.$$

Combining the relations above, it must hold that for every $t \in (0, 1]$ and $k \geq k(t)$

$$\begin{aligned}
t\delta &\geq \langle x_\mathrm{H}(t), y_\mathrm{H}^{(k)} \rangle_\mathrm{H} + \langle y_\mathrm{H}(t), x_\mathrm{H}^{(k)} \rangle_\mathrm{H} \\
&= \langle ty_\mathrm{H}(t)^{-1}, y_\mathrm{H}^{(k)} \rangle_\mathrm{H} + \langle tx_\mathrm{H}(t)^{-1}, x_\mathrm{H}^{(k)} \rangle_\mathrm{H} \\
&= t\left\{ \langle y(t)^{-1}, y^{(k)} \rangle + \frac{\kappa_k}{\kappa(t)} + \langle x(t)^{-1}, x^{(k)} \rangle + \frac{\tau_k}{\tau(t)} \right\}
\end{aligned}$$

Since $\langle y(t)^{-1}, y^{(k)} \rangle > 0$ and $\langle x(t)^{-1}, x^{(k)} \rangle > 0$, we finally obtain that

$$\frac{\kappa_k}{\kappa(t)} < \delta, \quad \frac{\tau_k}{\tau(t)} < \delta$$

for every $t \in (0, 1]$ and $k \geq k(t)$. The assertion (iii) follows from the facts $\kappa_k \to \kappa^*$, $\tau_k \to \tau^*$ and $\delta > 0$. ∎