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IMPOSSIBILITY AND POSSIBILITY THEOREMS OF SOCIAL CHOICE FUNCTION WITH RESTRICTED ALTERNATIVE SET

by

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IMPOSSIBILITY AND POSSIBILITY THEOREMS OF SOCIAL CHOICE FUNCTION WITH RESTRICTED ALTERNATIVE SET

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Abstract. We study the existence and properties of social choice function when individual's alternative set is restricted: each individual expresses his/her preference on his/her alternative set that is a subset of the whole set of alternatives. We define the strategy-proofness and show that if at least one individual's alternative set contains the range of social choice function, the strategy-proofness is not coexistent with non-dictatorship. In addition, we propose to weaken the strategy-proofness as well as the dictator, and show the existence of a social choice function with eligible properties.

1. Introduction

We often encounter the problem of aggregating preferences of individuals in a society. Arrow [2] introduced the social choice theory for this problem and established the monumental impossibility theorem, and then Gibbard [4] and Satterthwaite [6] showed the impossibility theorem concerning strategy-proofness. From then onward, difficulty of the problem has been well recognized, and a variety of impossibility theorems in Arrow's or Gibbard-Satterthwaite's framework has been developed. See Sen [7], Tanaka [8] and survey papers in [3].

In this paper we study the existence and properties of social choice function when individual's alternative set is restricted: one expresses one's preference on one's alternative set that is a subset of the whole set of alternatives. Accordingly, strategy-proofness and generalized strong positive association are defined. We show that if there exists an individual whose alternative set contains the range of social choice function, the social choice function satisfying strategy-proofness is dictatorial and one of such individuals is a dictator. After showing the equivalence of generalized strong positive association and strategy-proofness, we study the situation of mutual evaluation as a special case, and show the non-existence of social choice function satisfying strategy-proofness. We then weaken the condition of strategy-proofness as well as dictator and show the existence of a non-dictatorial social choice function with eligible properties in a constructive manner.

The organization of this paper is as follows. In Section 2, we introduce the framework and notation. In Section 3, we give an impossibility theorem when individual's alternative set is restricted. In Section 4, we focus on generalized strong positive association and show the equivalence to strategy-proofness. In Section 5, we discuss the mutual evaluation situation, redefine the strategy-proofness as well as the dictator, and present a possibility theorem. Section 6 summarizes the results.

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2. Framework and Notation

Let us consider a society consisting of a finite number, say n, of individuals. Each individual in the society has his/her own preference on the set X of a finite number of alternatives. The problem faced by the society is to aggregate the individual preferences into a society's choice. The rule of choice is called the *social choice function.* In the framework of Gibbard [4] and Satterthwaite [6], each individual in the society is interested in the whole set X of alternatives, and his/her preference is defined as a preference ordering on X. Preference ordering, denoted by $\succsim$, is a binary relation on X satisfying

(i) completeness : $x \succsim y$, $y \succsim x$ or both hold for any pair of alternatives $x, y \in X$, and
(ii) transitivity : if $x \succsim y$ and $y \succsim z$, then $x \succsim z$ holds for any alternatives $x, y, z \in X$.

We say that x is *weakly preferred* to y when $x \succsim y$. We write $x \sim y$ when both $x \succsim y$ and $y \succsim x$ hold and say that x is *indifferent* to y. When $x \succsim y$ and $y \not\succsim x$ we write $x \succ y$, reading that x is *strictly preferred* to y. For a subset $Y \subseteq X$, we denote by $\succsim\!|Y$ the restriction of binary relation $\succsim$ to Y, i.e., $\succsim\!|Y$ is defined on $Y \times Y$ and $x \succsim\!|Y y$ if and only if $x \succsim y$ and $x, y \in Y$.

Let $\mathcal{W}$ denote the set of all preference orderings on X and $\mathcal{W}^n$ denote its n-time Cartesian product $\underbrace{\mathcal{W} \times \cdots \times \mathcal{W}}_{n}$. We call an element, denoted by $\succsim^p$ or simply p, of $\mathcal{W}^n$ a *profile*, which is a combination of preference orderings $\succsim_i^p$ of individual $i \in N$. Then the social choice function is a function that assigns an alternative of X to each profile. Throughout this paper we denote the set of individuals by $N = \{1, 2, \ldots, n\}$ and assume that $n \geq 2$ except Section 5, where $n \geq 3$ is assumed.

Gibbard and Satterthwaite require strategy-proofness as one of the property that the social choice function should acquire. The strategy-proofness and dictator are defined by Gibbard and Satterthwaite as follows.

Definition 2.1 (Strategy-proofness in Gibbard-Satterthwaite's Framework)**.** We say that the social choice function $C\colon \mathcal{W}^n \to X$ is *strategically manipulable* by individual $i \in N$ at profile $p \in \mathcal{W}^n$ if there is a preference ordering $\succsim \in \mathcal{W}$ such that

$$C(p/_{-i} \succsim) \succ_i^p C(p) \tag{2.1}$$

holds, where

$$p/_{-i} \succsim = (\succsim_1^p, \ldots, \succsim_{i-1}^p, \succsim, \succsim_{i+1}^p, \ldots, \succsim_n^p).$$

When C is not strategically manipulable, it is said to satisfy *strategy-proofness.*

The social choice function is required to avoid being strategically manipulable because otherwise individual i can profit from misrepresentation of his/her preference ordering.

Definition 2.2 (Dictator)**.** An individual $k \in N$ is said to be a *dictator* if

$$C(p) \in \{\, x \in R(C) \mid x \succsim_k^p y \text{ for all alternatives } y \in R(C) \,\}$$

holds for each profile $p \in \mathcal{W}^n$, where $R(C)$ is the range of C defined by

$$R(C) = \{\, x \in X \mid x = C(p) \text{ for some } p \in \mathcal{W}^n \,\}.$$

Namely, the society always chooses an alternative out of those that the dictator prefers best. A social choice function that admits a dictator is said to be *dictatorial.*

Gibbard [4] and Satterthwaite [6] independently proved that the strategy-proofness and non-dictatorship are not coexistent.

Theorem 2.3 (Gibbard-Satterthwaite's Theorem). *If the social choice function* $C\colon \mathcal{W}^n \to X$ *satisfies strategy-proofness in Definition 2.1 and* $|R(C)| \geq 3$*, then it is dictatorial in the sense of Definition 2.2.*

There might be some individuals who are not allowed to express his/her preference on the whole set of alternatives. To formulate such a situation we introduce the alternative set on which individual i expresses his/her preference. For $i \in N$ let $X_i \subseteq X$ denote individual i's alternative set. We assume that $|X_i| \geq 2$ for all $i \in N$ and each alternative $x \in X$ belongs to some X_i, i.e., $X = \bigcup_{i \in N} X_i$. For each $i \in N$ let $\mathcal{W}_i$ denote the set of all preference orderings on X_i. We write $\mathcal{P} = \mathcal{W}_1 \times \mathcal{W}_2 \times \cdots \times \mathcal{W}_n$. Then the *social choice function* for this situation, which will be denoted by D henceforth, assigns an alternative of X to a *profile* $p \in \mathcal{P}$, i.e.,

$$D\colon \mathcal{P} \to X.$$

Now we introduce several definitions on the social choice function D.

Definition 2.4 (Strategy-proofness (SP)). We say that the social choice function $D\colon \mathcal{P} \to X$ is *strategically manipulable* by individual $i \in N$ at profile $p \in \mathcal{P}$ if there is a preference ordering $\succsim \in \mathcal{W}_i$ such that

$$\begin{aligned} &\text{either } D(p/_{-i} \succsim) \succ_i^p D(p) \\ &\text{or } \{D(p), D(p/_{-i} \succsim)\} \not\subseteq X_i \text{ and } D(p) \neq D(p/_{-i} \succsim). \end{aligned}$$

holds, where

$$p/_{-i} \succsim = (\succsim_1^p, \ldots, \succsim_{i-1}^p, \succsim, \succsim_{i+1}^p, \ldots, \succsim_n^p).$$

When D is not manipulable, it is said to satisfy *strategy-proofness*.

Individual i is not allowed to express his/her preference on the alternatives outside of X_i. Therefore we have no big picture of his/her preference, and are forced to assume that $D(p/_{-i} \succsim) \succ_i^p D(p)$ when $\{D(p), D(p/_{-i} \succsim)\} \not\subseteq X_i$. This is the reason why we adopt the above definition of strategy-proofness.

Definition 2.5. For a subset Y of X, $N(Y)$ denotes the set of all individuals whose alternative set contains Y, i.e.,

$$N(Y) = \{\, i \in N \mid Y \subseteq X_i \,\}.$$

Definition 2.6 (Dictator). An individual $k \in N$ is called a *dictator* if

$$D(p) \in \{\, x \in R(D) \mid x \succsim_k^p y \text{ for all alternatives } y \in R(D) \,\}$$

holds for each profile $p \in \mathcal{P}$. A social choice function that admits a dictator is said to be *dictatorial*.

This definition is indeed the same as Definition 2.2, but it requires the dictator's alternative set be large enough to contain the range of the social choice function.

Definition 2.7 (Generalized Strong Positive Association (GSPA)). The social choice function $D\colon P \to X$ is said to satisfy *generalized strong positive association* if it satisfies the following property for any pair of distinct profiles p and $q \in \mathcal{P}$.

If either $\succsim_i^p = \succsim_i^q$ for all $i \in N(\{D(p)\})$
or there exists a nonempty subset of individuals $M \subseteq N(\{D(p)\})$ such that
for all $i \in M, D(p) \succsim_i^p x$ implies $D(p) \succ_i^q x$ for all $x \in X_i \setminus \{D(p)\}$ and
for all $j \in N(\{D(p)\}) \setminus M$, $\succsim_j^p = \succsim_j^q$,
then $D(q) = D(p)$.

This means that if the alternative chosen by the society at profile p receives a not worse evaluation from all individuals at profile q, then it is chosen also at profile q.

3. Impossibility Theorem

The main theorem of this section below states that if at least one individual's alternative set contains the range of social choice function, the strategy-proofness is not coexistent with non-dictatorship.

Theorem 3.1. *Suppose that the social choice function $D\colon \mathcal{P} \to X$ satisfies the strategy-proofness (SP) in Definition 2.4, $|R(D)| \geq 3$ and $N(R(D)) \neq \emptyset$. Then it is dictatorial, and the dictator is in $N(R(D))$.*

We define the restriction $r(p)$ of profile $p = (\succsim_1^p, \succsim_2^p, \cdots, \succsim_n^p) \in \mathcal{W}^n$ as

$$r(p) = (\succsim_1^p|X_1,\ \succsim_2^p|X_2,\ \ldots,\ \succsim_n^p|X_n), \tag{3.1}$$

and introduce a function $C\colon \mathcal{W}^n \to X$ derived from the social choice function $D\colon \mathcal{P} \to X$ as

$$C(p) = D\left(r(p)\right) \text{ for each } p \in \mathcal{W}^n. \tag{3.2}$$

Lemma 3.2. *Suppose $D\colon \mathcal{P} \to X$ satisfies the strategy-proofness (SP) in Definition 2.4. Then the function $C\colon \mathcal{W}^n \to X$ defined by* (3.2) *satisfies the strategy-proofness in Gibbard-Satterthwaite's sense, Definition 2.1.*

Proof. We start the proof by assuming that C does not satisfy the strategy-proofness in Gibbard-Satterthwaite's sense, i.e., there exist an individual $i \in N$, profile $p \in \mathcal{W}^n$ and preference $\succsim \in \mathcal{W}$ such that

$$C(p/_{-i} \succsim) \succ_i^p C(p).$$

We will show that this assumption leads to a contradiction.

First note that

$$r(p/_{-i} \succsim) = r(p)/_{-i}(\succsim|X_i) \tag{3.3}$$

by the definitions of profile $p/_{-i} \succsim$ and restriction r. When $\{C(p), C(p/_{-i} \succsim)\} \subseteq X_i$, we have $D(r(p/_{-i} \succsim)) \succ_i^{r(p)} D(r(p))$. Therefore by (3.3) we see $D(r(p)/_{-i}(\succsim|X_i)) \succ_i^{r(p)} D(r(p))$, which contradicts the strategy-proofness (SP) of D in Definition 2.4.

When $\{C(p), C(p/_{-i} \succsim)\} \not\subseteq X_i$, we have $\{D(r(p)), D(r(p/_{-i} \succsim)\} \not\subseteq X_i$. We see $D(r(p)) \neq D(r(p/_{-i} \succsim))$ since $C(p) \neq C(p/_{-i} \succsim)$. By (3.3), these facts again contradict (SP) of D in Definition 2.4. □

Proof of Theorem 3.1.
Since

$$\{\, r(p) \mid p \in \mathcal{W}^n \,\} = \mathcal{P}, \tag{3.4}$$

we see

$$R(C) = R(D)$$

and $|R(C)| = |R(D)| \geq 3$. This and Lemma 3.2 show that C defined by (3.2) is dictatorial by Gibbard-Satterthwaite Theorem, Theorem 2.3. Namely, there is a dictator $k \in N$ of C, which satisfies

$$C(p) \in \{\, x \in R(C) \mid x \succsim_k^p y \text{ for all } y \in R(C) \,\}$$

for any profile $p \in \mathcal{W}^n$. We will show that this individual k belongs to $N(R(D))$ and is a dictator of D.

Assume that $R(C) \not\subseteq X_k$, choose arbitrarily an alternative $x \in R(C) \setminus X_k$, and consider a profile $p_1 \in \mathcal{W}^n$ such that

$$x \succ_k^{p_1} y \text{ for any } y \in R(C) \setminus \{x\}.$$

Since individual k is a dictator of C, we see that $C(p_1) = x$. Consider another profile $p_2 \in \mathcal{W}^n$ such that

$$\begin{aligned}
&y \succ_k^{p_2} x \text{ for some } y \in R(C) \setminus \{x\},\\
&\succsim_k^{p_2} |(X \setminus \{x\}) = \succsim_k^{p_1} |(X \setminus \{x\}) \text{ and}\\
&\succsim_j^{p_2} = \succsim_j^{p_1} \quad \text{for all } j \in N \setminus \{k\}.
\end{aligned}$$

Again by the dictatorialness of individual k, we see $C(p_2) \neq x$. Then $D(r(p_1)) \neq D(r(p_2))$.

On the other hand, we see

$$\succsim_j^{r(p_1)} = \succsim_j^{r(p_2)} \quad \text{for all } j \in N \setminus \{k\}$$

since $\succsim_j^{p_1} = \succsim_j^{p_2}$ for all $j \in N \setminus \{k\}$, and also

$$\succsim_k^{r(p_1)} = \succsim_k^{r(p_2)}$$

by the construction of profile p_1 and p_2 and the fact that $x \notin X_k$. Therefore $r(p_1) = r(p_2)$, implying $D(r(p_1)) = D(r(p_2))$. This is a contradiction, and hence we conclude that $R(C) \subseteq X_k$, i.e., $k \in N(R(D))$.

Therefore we see that

$$D(r(p)) \in \{\, x \in R(D) \mid x \succsim_k^{r(p)} y \text{ for all } y \in R(D) \,\}$$

for each profile $p \in \mathcal{W}^n$. By (3.4), we conclude that for each profile $q \in \mathcal{P}$

$$D(q) \in \{\, x \in R(D) \mid x \succsim_k^q y \text{ for all } y \in R(D) \,\},$$

i.e., individual k is a dictator of D. □

4. Equivalence of Generalized Strong Positive Association and Strategy-proofness

We will prove in this section the equivalence of generalized strong positive association (GSPA) and strategy-proofness (SP).

Theorem 4.1. *The social choice function $D \colon \mathcal{P} \to X$ satisfies strategy-proofness (SP) in Definition 2.4 if and only if it satisfies generalized strong positive association (GSPA) in Definition 2.7.*

Proof of the Sufficiency. Suppose that there exists a social choice function D that satisfies (GSPA) but not (SP). Then there is an individual $i \in N$, profile $p \in \mathcal{P}$ and preference $\succsim \in \mathcal{W}_i$ such that

$$\text{either } D(p/_{-i} \succsim) \succ_i^p D(p)$$
$$\text{or } \{D(p), D(p/_{-i} \succsim)\} \not\subseteq X_i \text{ and } D(p) \neq D(p/_{-i} \succsim).$$

Let us denote the profile $p/_{-i} \succsim$ by q for the sake of simplicity.

First we consider the case where $D(q) \succ_i^p D(p)$. Partition X_i into two exhaustive and disjoint subsets

$$U = \{\, x \in X_i \mid x \succ_i^p D(p) \,\} \quad \text{and} \quad V = \{\, x \in X_i \mid D(p) \succsim_i^p x \,\}.$$

Note that $D(p) \in V$ and $D(q) \in U$. Then choose a preference $\succsim' \in \mathcal{W}_i$ such that

$$D(q) \succ' x \text{ for all } x \in X_i \setminus \{D(q)\} \quad \text{and} \quad D(p) \succ' x \text{ for all } x \in V \setminus \{D(p)\}.$$

Let us denote $p/_{-i} \succsim'$ by q', and consider the social choice $D(q')$ at profile q'. Since $D(p) \succsim_i^p x$ implies $D(p) \succ' x$ for all $x \in X_i \setminus \{D(p)\}$ and $\succsim_j^p = \succsim_j^{q'}$ for all $j \in N(\{D(p)\}) \setminus \{i\}$, we see

$$D(q') = D(p) \tag{4.1}$$

by (GSPA) in Definition 2.7. Observe that $D(q) \succsim_i^q x$ implies $D(q) \succ' x$ for all $x \in X_i \setminus \{D(q)\}$ and $\succsim_j^q = \succsim_j^{q'}$ for all $j \in N(\{D(q)\}) \setminus \{i\}$. These imply by (GSPA) in Definition 2.7 that

$$D(q') = D(q),$$

which by (4.1) yields $D(p) = D(q)$, a contradiction.

Next, we consider the case where $\{D(p), D(q)\} \not\subseteq X_i$ and $D(p) \neq D(q)$. When $D(p) \notin X_i$, we have $i \notin N(\{D(p)\})$. Note that

$$\succsim_j^q = \succsim_j^p \ \text{ for all } j \in N(\{D(p)\})$$

since $\succsim_j^p = \succsim_j^q$ for all $j \in N \setminus \{i\}$. Applying (GSPA) in Definition 2.7, we see that $D(q) = D(p)$, which is a contradiction. When $D(q) \notin X_i$, we have $i \notin N(\{D(q)\})$, implying that

$$\succsim_j^p = \succsim_j^q \ \text{ for all } j \in N(\{D(q)\}).$$

Therefore $D(p) = D(q)$ by (GSPA) in Definition 2.7, which again contradicts $D(p) \neq D(q)$. □

Proof of the Necessity. Suppose that there exists a social choice function D that satisfies (SP) but not (GSPA).

First, we consider the case where there are two distinct profiles $p, q \in \mathcal{P}$ and a nonempty subset of individuals $M \subseteq N(\{D(p)\})$ such that

$$D(p) \succsim_i^p x \text{ implies } D(p) \succ_i^q x \text{ for all } x \in X_i \setminus \{D(p)\} \text{ and for all } i \in M,$$
$$\succsim_j^p = \succsim_j^q \ \text{ for all } j \in N(\{D(p)\}) \setminus M, \text{ and}$$
$$D(q) \neq D(p).$$

Let $M = \{1, \ldots, m\}$ and $N \setminus M = \{m+1, \ldots, n\}$ ($n \geq m \geq 1$) by renumbering if necessary. Consider the sequence of $n+1$ profiles

$$
\begin{aligned}
&(\succsim_1^p, \succsim_2^p, \ldots, \succsim_m^p, \succsim_{m+1}^p, \ldots, \succsim_n^p) = r_0, \\
&(\succsim_1^q, \succsim_2^p, \ldots, \succsim_m^p, \succsim_{m+1}^p, \ldots, \succsim_n^p) = r_1, \\
&\qquad \vdots \\
&(\succsim_1^q, \ldots, \succsim_{j-1}^q, \succsim_j^p, \succsim_{j+1}^p, \ldots, \succsim_m^p, \succsim_{m+1}^p, \cdots \succsim_n^p) = r_{j-1}, \\
&(\succsim_1^q, \ldots, \succsim_{j-1}^q, \succsim_j^q, \succsim_{j+1}^p, \ldots, \succsim_m^p, \succsim_{m+1}^p, \cdots \succsim_n^p) = r_j, \\
&\qquad \vdots \\
&(\succsim_1^q, \ldots, \succsim_{m+1}^q, \ldots, \succsim_{n-1}^q, \succsim_n^p) = r_{n-1}, \\
&(\succsim_1^q, \ldots, \succsim_{m+1}^q, \ldots, \succsim_{n-1}^q, \succsim_n^q) = r_n.
\end{aligned}
$$

Since $D(r_0) = D(p)$ and $D(r_n) = D(q)$, there exists an individual, say $k \in N$ such that $D(r_{k-1}) = D(p)$ and $D(r_k) \neq D(p)$. Let $D(r_k) = w$ and note that w might be equal to $D(q)$. Concerning k, the following three cases A, B and C are possible.

Case A: $k \in M$.

Four possibilities should be considered for the preference of individual k at profile r_{k-1} between $D(p)$ and w: $D(p) \succ_k^{r_{k-1}} w$, $D(p) \sim_k^{r_{k-1}} w$, $w \succ_k^{r_{k-1}} D(p)$ and $\{D(p), w\} \not\subseteq X_k$. We will show that each of four possibilities leads to a contradiction.

(1) If either $D(p) \succ_k^{r_{k-1}} w$ or $D(p) \sim_k^{r_{k-1}} w$, then we see $D(p) \succ_k^{r_k} w$ since $D(p) \succsim_k^p x$ implies $D(p) \succ_k^q x$ for all $x \in X_k \setminus \{D(p)\}$. This means that $D(r_k/_{-k} \succsim_k^p) = D(r_{k-1}) \succ_k^{r_k} D(r_k)$, which is contrary to (SP), Definition 2.4.

(2) If $w \succ_k^{r_{k-1}} D(p)$, then this means that $D(r_{k-1}/_{-k} \succsim_k^q) = D(r_k) \succ_k^{r_{k-1}} D(r_{k-1})$, which is also contrary to (SP), Definition 2.4.

(3) If $\{D(p), w\} \not\subseteq X_k$, then this together with $D(r_{k-1}) = D(p) \neq w = D(r_k)$ contradicts (SP), Definition 2.4 since $\succsim_j^{r_{k-1}} = \succsim_j^{r_k}$ for all $j \in N \setminus \{k\}$.

Case B: $k \in N(\{D(p)\}) \setminus M$.

We see that $r_{k-1} = r_k$ since $\succsim_i^p = \succ_i^q$ for all $i \in N(\{D(p)\}) \setminus M$. This implies that $D(r_{k-1}) = D(r_k)$, which is a contradiction.

Case C: $k \in N \setminus N(\{D(p)\})$.

Since $D(r_{k-1}) = D(p)$, we see that $D(r_{k-1}) \notin X_k$, meaning that $\{D(r_{k-1}), D(r_k)\} \not\subseteq X_k$. This together with $D(r_{k-1}) \neq D(r_k)$ contradicts (SP), Definition 2.4 since $\succsim_j^{r_{k-1}} = \succsim_j^{r_k}$ for all $j \in N \setminus \{k\}$.

Next, we consider the case where there exist distinct profiles $p, q \in \mathcal{P}$ such that $\succsim_i^p = \succsim_i^q$ for all $i \in N(\{D(p)\})$ and $D(p) \neq D(q)$. Let $N \setminus N(\{D(p)\}) = \{1, .., m\}$ and $N(\{D(p)\}) = \{m+1, ..., n\}$ ($n \geq m \geq 1$) by renumbering if necessary and consider the sequence of $m+1$

profiles.

$$
\begin{aligned}
&(\succsim_1^p, \succsim_2^p, \dots, \succsim_m^p, \succsim_{m+1}^p, \dots, \succsim_n^p) = r_0,\\
&(\succsim_1^q, \succsim_2^p, \dots, \succsim_m^p, \succsim_{m+1}^p, \dots, \succsim_n^p) = r_1,\\
&\qquad\vdots\\
&(\succsim_1^q, \dots, \succsim_{k-1}^q, \succsim_k^p, \succsim_{k+1}^p, \dots, \succsim_m^p, \succsim_{m+1}^p, \dots, \succsim_n^p) = r_{k-1},\\
&(\succsim_1^q, \dots, \succsim_{k-1}^q, \succsim_k^q, \succsim_{k+1}^p, \dots, \succsim_m^p, \succsim_{m+1}^p, \dots, \succsim_n^p) = r_k,\\
&\qquad\vdots\\
&(\succsim_1^q, \dots, \succsim_{m-1}^q, \succsim_m^p, \succsim_{m+1}^p, \dots, \succsim_n^p) = r_{m-1},\\
&(\succsim_1^q, \dots, \succsim_{m-1}^q, \succsim_m^q, \succsim_{m+1}^p, \dots, \succsim_n^p)) = r_m.
\end{aligned}
$$

Since $\succsim_i^p = \succsim_i^q$ for all $i \in N(\{D(p)\}) = \{m+1, ..., n\}$, we see that $r_m = q$ hence $D(r_m) = D(q)$. Then there exists $k \in N \setminus N(\{D(p)\})$ such that $D(r_{k-1}) = D(p)$ and $D(r_k) \neq D(p)$. Let $D(r_k) = w$. Since $k \notin N(\{D(p)\})$, we have $\{D(p), w\} \not\subseteq X_k$. This together with $D(r_{k-1}) = D(p) \neq w = D(r_k)$ contradicts (SP) in Definition 2.4 since $\succsim_i^{r_{k-1}} = \succsim_i^{r_k}$ for all $i \in N \setminus \{k\}$. □

By the equivalence of two properties we have the following corollary from Theorem 3.1.

Corollary 4.2. *Suppose that the social choice function $D\colon \mathcal{P} \to X$ satisfies generalized strong positive association (GSPA) in Definition 2.7, $|R(D)| \geq 3$ and $N(R(D)) \neq \emptyset$. Then it is dictatorial, and the dictator is in $N(R(D))$.*

5. Social Choice Function for Mutual Evaluation

We consider in this section the social choice function for mutual evaluation. We assume that there are at least three individuals, i.e., $n \geq 3$, and each individual evaluates all individuals in the society but him/herself. Namely, the set of alternatives coincides with the set of all individuals in the society, i.e., $X = N$, and individual i's alternative set X_i is given by

$$X_i = N \setminus \{i\}.$$

We use $\mathcal{W}_i$ and $\mathcal{P}$ in the same manner as in the preceding sections. To avoid confusion we denote the social choice function for mutual evaluation by D_m.

Firstly we will show that if the range of the social choice function is so large that any individual can be chosen by the society, the function is strategically manipulable.

Theorem 5.1. *There does not exist a social choice function $D_m\colon \mathcal{P} \to X$ that satisfies both (SP) in Definition 2.4 and $R(D_m) = N$.*

The key to the proof of Theorem 5.1 is two-fold: the fact that strategy-proofness (SPP) implies the following weak Pareto principle and the "cyclic" profile to be introduced in (5.3).

Definition 5.2 (Weak Pareto Principle (WPP))**.** The social choice function $D_m\colon \mathcal{P} \to X$ is said to satisfy the *weak Pareto principle* if it holds that

$$i \succ_k^p j \text{ for all } k \in N \setminus \{i, j\} \text{ implies } D_m(p) \neq j$$

for any pair of distinct individuals $i, j \in N$ and for any profile $p \in \mathcal{P}$.

Lemma 5.3. *If the social choice function $D_m : \mathcal{P} \to X$ satisfies (SP) in Definition 2.4 and $R(D_m) = N$, then it satisfies (WPP) in Definition 5.2.*

Proof. We suppose that there exists a social choice function D_m that satisfies (SP) and $R(D_m) = N$ but not (WPP). Then there exist a profile $p \in \mathcal{P}$ and distinct individuals $i, j \in N$ such that

$$i \succ_k^p j \text{ for all } k \in N \setminus \{i, j\} \text{ and } D_m(p) = j.$$

Consider a profile $q \in \mathcal{P}$ such that

$$\begin{aligned} &\succsim_i^q = \succsim_i^p, \\ &i \succ_j^q k && \text{for all } k \in N \setminus \{i, j\}, \text{ and} \\ &i \succ_k^q j \succ_k^q l && \text{for all } l \in N \setminus \{i, j, k\} \text{ and for all } k \in N \setminus \{i, j\}. \end{aligned}$$

Observe that $\succsim_i^p = \succsim_i^q$ and $j \succsim_k^p l$ implies $j \succ_k^q l$ for all $l \in N \setminus \{j, k\}$ and for all $k \in N \setminus \{i, j\}$. Note that (SP) is equivalent to (GSPA) and that $j \notin N(\{j\})$. Then we obtain

$$D_m(q) = D_m(p) \tag{5.1}$$

by applying (GSPA) to profiles p and q. Since $i \in N = R(D_m)$, there exists a profile $s \in \mathcal{P}$ such that $D_m(s) = i$. Between s and q we observe that $i \succsim_k^s l$ implies $i \succ_k^q l$ for all $l \in N \setminus \{i, k\}$ and for all $k \in N \setminus \{i\}$. Note that $i \notin N(\{i\})$. Therefore

$$D_m(q) = D_m(s) \tag{5.2}$$

by applying (GSPA) to profile s and q. From (5.1) and (5.2) we have

$$j = i,$$

which is a contradiction. □

Proof of Theorem 5.1.

We take the "cyclic" profile c defined by

$$\begin{aligned} &2 \succ_1^c 3 \succ_1^c \cdots \succ_1^c n, \\ &i+1 \succ_i^c i+2 \succ_i^c \cdots \succ_i^c n-1 \succ_i^c n \succ_i^c 1 \succ_i^c \cdots \succ_i^c i-1 \quad \text{for } i = 2, \cdots, n-1, \\ &1 \succ_n^c 2 \succ_n^c \cdots \succ_n^c n-1. \end{aligned} \tag{5.3}$$

and consider what alternative D_m should choose at this profile. Take an individual i out of $N \setminus \{n\}$, then $i \succ_j^c i+1$ for all $j \in N \setminus \{i, i+1\}$, which means

$$D_m(c) \neq i+1 \text{ for } i = 1, 2, \ldots, n-1$$

by Lemma 5.3. Concerning the individual 1 we see that $n \succ_j^c 1$ for all $j \in N \setminus \{1, n\}$, meaning

$$D_m(c) \neq 1.$$

Therefore $D_m(c)$ cannot be any alternative, and we conclude that the social choice function D_m satisfying both (SP) in Definition 2.4 and $R(D_m) = N$ is impossible. □

It is pointed out in [1] that the social welfare function for mutual evaluation degenerates or does not exist due to the presence of cyclic profile in $\mathcal{P}$. To exclude the controversial cyclic profile Ohbo et al. [5] introduced individuals who are entitled to evaluate all individuals in the society and proved the existence of a dictator among the introduced individuals. Concerning the social choice function, we readily see, by applying Theorem 3.1, that every social choice function satisfying (SP) in Definition 2.4 is dictatorial and one of the introduced individuals is a dictator.

We have assumed thus far that each individual is interested in all the individuals but is allowed to express his/her preference on $N \setminus \{i\}$. We suppose hereafter that every individual is interested in only $N \setminus \{i\}$. For this situation we redefine strategy proofness (SP) in Definition 2.4 and dictator in Definition 2.6 as follows.

Definition 5.4 (Weak Strategy-proofness (WSP))**.** We say that the social choice function $D_m : \mathcal{P} \to N$ is *strongly strategically manipulable* by individual $i \in N$ at profile $p \in \mathcal{P}$ if there is a preference ordering $\succsim \in \mathcal{W}_i$ such that

$$D_m(p/_{-i} \succsim) \succ_i^p D_m(p)$$

holds, where

$$p/_{-i} \succsim = (\succsim_1^p, \ldots, \succsim_{i-1}^p, \succsim, \succsim_{i+1}^p, \ldots, \succsim_n^p).$$

When D_m is not strongly strategically manipulable, it is said to satisfy *weak strategy-proofness.*

Note that we do not consider that D_m is manipulable when either $D_m(p/_{-i} \succsim)$ or $D_m(p)$ is outside of $N \setminus \{i\}$.

We weaken the definition of dictator.

Definition 5.5. An individual $i \in N$ is called a *dictator for mutual evaluation* if

$$D_m(p) \in \{ j \in R(D_m) \mid \text{there does not exist } k \in R(D_m) \text{ such that } k \succ_i^p j \}$$

for any profile $p \in \mathcal{P}$. The social choice function D_m that admits a dictator is said to be *dictatorial.*

This definition, which does not require the dictator's alternative set contain the range of the social choice function, forms a contrast to Definition 2.6. The following theorem claims the existence of a social choice function D_m of eligible property.

Theorem 5.6. *There exists a non-dictatorial social choice function $D_m : \mathcal{P} \to N$ for mutual evaluation satisfying both weak strategy-proofness (WSP) in Definition 5.4 and $R(D_m) = N$.*

Proof. We will prove this theorem by induction over n, the number of individuals in the society. Table 1 in Appendix demonstrates an example of non-dictatorial social choice function D_m satisfying (WSP) and $R(D_m) = N$ when $N = \{1, 2, 3\}$.[1] That is, the theorem has been proved when $n = 3$.

For $h \geq 3$ let $N^h = \{1, \ldots, h\}$ and $\mathcal{W}_i^h$ be the set of all preference orderings defined on $N^h \setminus \{i\}$ for $i \in N^h$, and $\mathcal{P}^h = \mathcal{W}_1^h \times \cdots \times \mathcal{W}_h^h$.

Assuming the existence of a social choice function $D_m^h : \mathcal{P}^h \to N^h$ which is non-dictatorial and satisfies (WSP) and $R(D_m^h) = N^h$, we will show that there exists a non-dictatorial social choice function $D_m^{h+1} : \mathcal{P}^{h+1} \to N^{h+1}$ satisfying (WSP) and $R(D_m^{h+1}) = N^{h+1}$.

For each profile $p \in \mathcal{P}^{h+1}$, let

$$r(p) = (\succsim_1^p|(N^h \setminus \{1\}), \succsim_2^p|(N^h \setminus \{2\}), \ldots, \succsim_h^p|(N^h \setminus \{h\})).$$

That is to say, $r(p)$ is the restriction of profile $p \in \mathcal{P}^{h+1}$ to $\mathcal{P}^h$. We define $D_m^{h+1} : \mathcal{P}^{h+1} \to N^{h+1}$ as

$$D_m^{h+1}(p) = \begin{cases} D_m^h(r(p)) & \text{when } \succsim_{h+1}^p = (1 \succ 2 \succ \cdots \succ h), \\ h+1 & \text{otherwise} \end{cases} \tag{5.4}$$

[1] We carried out exhaustive enumeration for the case of $N = \{1, 2, 3\}$ and found more than 1.6×10^9 different social choice functions satisfying the conditions in Theorem 5.6 .

for each profile $p \in \mathcal{P}^{h+1}$.

Firstly, to show that D_m^{h+1} defined by (5.4) satisfies (WSP) we suppose the contrary, i.e., there exists an individual $i \in N^{h+1}$, profile $p \in \mathcal{P}^{h+1}$ and preference $\succsim \in \mathcal{W}_i^{h+1}$ such that

$$D_m^{h+1}(p/_{-i} \succsim) \succ_i^p D_m^{h+1}(p). \tag{5.5}$$

Letting $q = p/_{-i} \succsim$ for the sake of notational simplicity, we consider the following two possible cases.

Case A: $i \in N^h$.

If $\succsim_{h+1}^p \neq (1 \succ 2 \succ \cdots \succ h)$, then $D_m^{h+1}(p) = D_m^{h+1}(p/_{-i} \succsim) = h+1$ by the definition (5.4) of D_m^{h+1}. Since this fact contradicts (5.5), we have $\succsim_{h+1}^p = (1 \succ 2 \succ \cdots \succ h)$, implying that $D_m^{h+1}(p) = D_m^h(r(p))$ and $D_m^{h+1}(q) = D_m^h(r(q))$. Then

$$D_m^h(r(q)) \succ_i^{r(p)} D_m^h(r(p)).$$

However, since $\succsim_j^p = \succsim_j^q$ for all $j \in N^{h+1} \setminus \{i\}$,

$$\succsim_j^{r(p)} = \succsim_j^{r(q)} \text{ for all } j \in N^h \setminus \{i\}.$$

These contradict (WSP) of D_m^h.

Case B: $i = h+1$.

Since $\succsim_{h+1}^p$ is a preference ordering on $N^{h+1} \setminus \{h+1\} = N^h$, we see that both $D_m^{h+1}(p)$ and $D_m^{h+1}(q)$ are an element of N^h from (5.5). Therefore we have by the construction (5.4) of D_m^{h+1} that $\succsim_{h+1}^p = \succsim = (1 \succ 2 \succ \cdots \succ h)$. This implies that $q = p/_{-i} \succsim = p$, which contradicts (5.5).

Secondly, we will show $R(D_m^{h+1}) = N^{h+1}$. We have known from (5.4) that there exists a profile $p \in \mathcal{P}^{h+1}$ such that $D_m^{h+1}(p) = h+1$. Since $\{\, r(p) \mid p \in \mathcal{P}^{h+1} \text{ such that } \succsim_{h+1}^p = (1 \succ 2 \succ \cdots \succ h) \,\} = \mathcal{P}^h$ and $R(D_m^h) = N^h$ from the induction assumption, we have $N^h \subseteq R(D_m^{h+1})$. Therefore we see that $R(D_m^{h+1}) = N^{h+1}$.

Lastly, we will show that D_m^{h+1} is non-dictatorial. Let j be an arbitrary individual in N^h. Since D_m^h is non-dictatorial, we see that there exist a profile $p_1 \in \mathcal{P}^h$ and $k, l \in N^h \setminus \{j\}$ such that

$$k \succ_j^{p_1} l \text{ and } D_m^h(p_1) = l.$$

Take a profile $q_1 \in \mathcal{P}^{h+1}$ such that

$$r(q_1) = p_1 \text{ and } \succsim_{h+1}^{q_1} = (1 \succ 2 \succ \cdots \succ h).$$

Then $D_m^{h+1}(q_1) = D_m^h(r(q_1)) = D_m^h(p_1) = l$ and $k \succ_j^{q_1} l$ by the construction of profile q_1, implying that individual $j \in N^h$ is not a dictator of D_m^{h+1}.

Let p_2 be a profile of $\mathcal{P}^h$ such that $D_m^h(p_2) = 2$.[2] Consider a profile $q_2 \in \mathcal{P}^{h+1}$ such that

$$r(q_2) = p_2 \text{ and } \succsim_{h+1}^{q_2} = (1 \succ 2 \succ \cdots \succ h).$$

We see that $D_m^{h+1}(q_2) = D_m^h(r(q_2)) = D_m^h(p_2) = 2$ and $1 \succ_{h+1}^{q_2} 2$. These facts imply that individual $h+1$ is not a dictator of D_m^{h+1}, either. Therefore we conclude that D_m^{h+1} is non-dictatorial. □

[2] $D_m^h(p_2)$ can be any natural number between 2 and h.

6. Conclusion

We considered a society where each individual's alternative set is restricted to a subset of the whole set of alternatives. We have shown an impossibility theorem: the social choice function satisfying strategy-proofness is dictatorial whenever at least one individual's alternative set contains the range of social choice function. We also have shown the equivalence of generalized positive association and strategy-proofness. We studied the social choice function for mutual evaluation as a special case. Then we weakened the condition of strategy-proofness as well as dictator, and showed the existence of a non-dictatorial social choice function satisfying the weak strategy-proofness in a constructive manner.

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Appendix

individual	preference
1	$2 \succ_1^{p_1} 3$
2	$1 \succ_2^{p_1} 3$
3	$1 \succ_3^{p_1} 2$

$D_m(p_1) = 1$

individual	preference
1	$2 \succ_1^{p_2} 3$
2	$1 \succ_2^{p_2} 3$
3	$2 \succ_3^{p_2} 1$

$D_m(p_2) = 1$

individual	preference
1	$2 \succ_1^{p_3} 3$
2	$1 \succ_2^{p_3} 3$
3	$1 \sim_3^{p_3} 2$

$D_m(p_3) = 1$

individual	preference
1	$2 \succ_1^{p_4} 3$
2	$3 \succ_2^{p_4} 1$
3	$1 \succ_3^{p_4} 2$

$D_m(p_4) = 1$

individual	preference
1	$2 \succ_1^{p_5} 3$
2	$3 \succ_2^{p_5} 1$
3	$2 \succ_3^{p_5} 1$

$D_m(p_5) = 1$

individual	preference
1	$2 \succ_1^{p_6} 3$
2	$3 \succ_2^{p_6} 1$
3	$1 \sim_3^{p_6} 2$

$D_m(p_6) = 1$

individual	preference
1	$2 \succ_1^{p_7} 3$
2	$1 \sim_2^{p_7} 3$
3	$1 \succ_3^{p_7} 2$

$D_m(p_7) = 1$

individual	preference
1	$2 \succ_1^{p_8} 3$
2	$1 \sim_2^{p_8} 3$
3	$2 \succ_3^{p_8} 1$

$D_m(p_8) = 2$

individual	preference
1	$2 \succ_1^{p_9} 3$
2	$1 \sim_2^{p_9} 3$
3	$1 \sim_3^{p_9} 2$

$D_m(p_9) = 2$

individual	preference
1	$3 \succ_1^{p_{10}} 2$
2	$1 \succ_2^{p_{10}} 3$
3	$1 \succ_3^{p_{10}} 2$

$D_m(p_{10}) = 1$

individual	preference
1	$3 \succ_1^{p_{11}} 2$
2	$1 \succ_2^{p_{11}} 3$
3	$2 \succ_3^{p_{11}} 1$

$D_m(p_{11}) = 1$

individual	preference
1	$3 \succ_1^{p_{12}} 2$
2	$1 \succ_2^{p_{12}} 3$
3	$1 \sim_3^{p_{12}} 2$

$D_m(p_{12}) = 1$

individual	preference
1	$3 \succ_1^{p_{13}} 2$
2	$3 \succ_2^{p_{13}} 1$
3	$1 \succ_3^{p_{13}} 2$

$D_m(p_{13}) = 2$

individual	preference
1	$3 \succ_1^{p_{14}} 2$
2	$3 \succ_2^{p_{14}} 1$
3	$2 \succ_3^{p_{14}} 1$

$D_m(p_{14}) = 3$

individual	preference
1	$3 \succ_1^{p_{15}} 2$
2	$3 \succ_2^{p_{15}} 1$
3	$1 \sim_3^{p_{15}} 2$

$D_m(p_{15}) = 3$

individual	preference
1	$3 \succ_1^{p_{16}} 2$
2	$1 \sim_2^{p_{16}} 3$
3	$1 \succ_3^{p_{16}} 2$

$D_m(p_{16}) = 2$

individual	preference
1	$3 \succ_1^{p_{17}} 2$
2	$1 \sim_2^{p_{17}} 3$
3	$2 \succ_3^{p_{17}} 1$

$D_m(p_{17}) = 2$

individual	preference
1	$3 \succ_1^{p_{18}} 2$
2	$1 \sim_2^{p_{18}} 3$
3	$1 \sim_3^{p_{18}} 2$

$D_m(p_{18}) = 2$

individual	preference
1	$2 \sim_1^{p_{19}} 3$
2	$1 \succ_2^{p_{19}} 3$
3	$1 \succ_3^{p_{19}} 2$

$D_m(p_{19}) = 1$

individual	preference
1	$2 \sim_1^{p_{20}} 3$
2	$1 \succ_2^{p_{20}} 3$
3	$2 \succ_3^{p_{20}} 1$

$D_m(p_{20}) = 3$

individual	preference
1	$2 \sim_1^{p_{21}} 3$
2	$1 \succ_2^{p_{21}} 3$
3	$1 \sim_3^{p_{21}} 2$

$D_m(p_{21}) = 1$

individual	preference
1	$2 \sim_1^{p_{22}} 3$
2	$3 \succ_2^{p_{22}} 1$
3	$1 \succ_3^{p_{22}} 2$

$D_m(p_{22}) = 1$

individual	preference
1	$2 \sim_1^{p_{23}} 3$
2	$3 \succ_2^{p_{23}} 1$
3	$2 \succ_3^{p_{23}} 1$

$D_m(p_{23}) = 3$

individual	preference
1	$2 \sim_1^{p_{24}} 3$
2	$3 \succ_2^{p_{24}} 1$
3	$1 \sim_3^{p_{24}} 2$

$D_m(p_{24}) = 2$

individual	preference
1	$2 \sim_1^{p_{25}} 3$
2	$1 \sim_2^{p_{25}} 3$
3	$1 \succ_3^{p_{25}} 2$

$D_m(p_{25}) = 2$

individual	preference
1	$2 \sim_1^{p_{26}} 3$
2	$1 \sim_2^{p_{26}} 3$
3	$2 \succ_3^{p_{26}} 1$

$D_m(p_{26}) = 2$

individual	preference
1	$2 \sim_1^{p_{27}} 3$
2	$1 \sim_2^{p_{27}} 3$
3	$1 \sim_3^{p_{27}} 2$

$D_m(p_{27}) = 2$

Table 1. Example of social choice function for mutual evaluation when $N = \{1, 2, 3\}$

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