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On Evaluation of Dynamic Behavior of Modified Ornstein-Uhlenbeck Processes with Various Boundaries

by

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On Evaluation of Dynamic Behavior of Modified Ornstein-Uhlenbeck Processes with Various Boundaries

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Abstract

The dynamic behavior of O-U processes modified by various boundaries becomes numerically intractable despite the underlying simplicity of the Gaussian transition structure. The purpose of this paper is to develop computational algorithms to capture this dynamics via the Ehrenfest approximation established in Sumita, Gotoh and Jin[3]. The range of a modified O-U process with one or two boundaries is first represented by $2V+1$ discrete points. On this discrete state space, by shifting and scaling the Ehrenfest process with similar boundaries in an appropriate manner, the resulting stochastic process converges in law to the modified O-U process as $V \to \infty$. Using the uniformization procedure of Keilson [2], numerical results reveal that absorbing, replacement and reflection boundaries can be treated with speed and accuracy.

Keywords : Modified Ornstein-Uhlenbeck (O-U) process, Absorbing boundaries, Replacement boundaries, Reflection boundaries, Dynamic Behavior, Uniformization procedure

0 Introduction

The Ornstein-Uhlenbeck (O-U) process $\{X_{\rm OU}(t) : t \geq 0\}$ on $\mathbb{R}$ is a Markov diffusion process whose probability density function $f(x,t) := \frac{\rm d}{{\rm d}x}{\rm P}\{X_{\rm OU}(t) \leq x\}$ is governed by the forward diffusion equation

$$\frac{\partial}{\partial t} f(x,t) = \frac{\partial^2}{\partial x^2} f(x,t) + \frac{\partial}{\partial x}\left[x\, f(x,t)\right]. \tag{0.1}$$

Since this process is of practical importance, it has been widely studied and applied to modeling many real dynamics. Recently the usefulness of the O-U process has been reinforced in the area of financial engineering, where spot interest rates are represented by O-U processes, see e.g. Vasicek [4].

Despite the underlying simplicity associated with the Gaussian transition structure, the dynamic behavior of the O-U process becomes analytically intractable when it is modified by various types of boundaries. Typical boundaries include absorbing boundaries, replacement boundaries, and reflection boundaries which are special cases of replacement boundaries. The reader is referred to Feller [1] for further details. Figure 0.1(a) depicts the modified O-U process with one absorbing boundary. The modified O-U process with two absorbing boundaries is illustrated in Figure 0.1(b). When the upper and lower boundaries are symmetric about 0, this process expresses the first passage time of $|X_{\rm OU}(t)|$. Additional cases for

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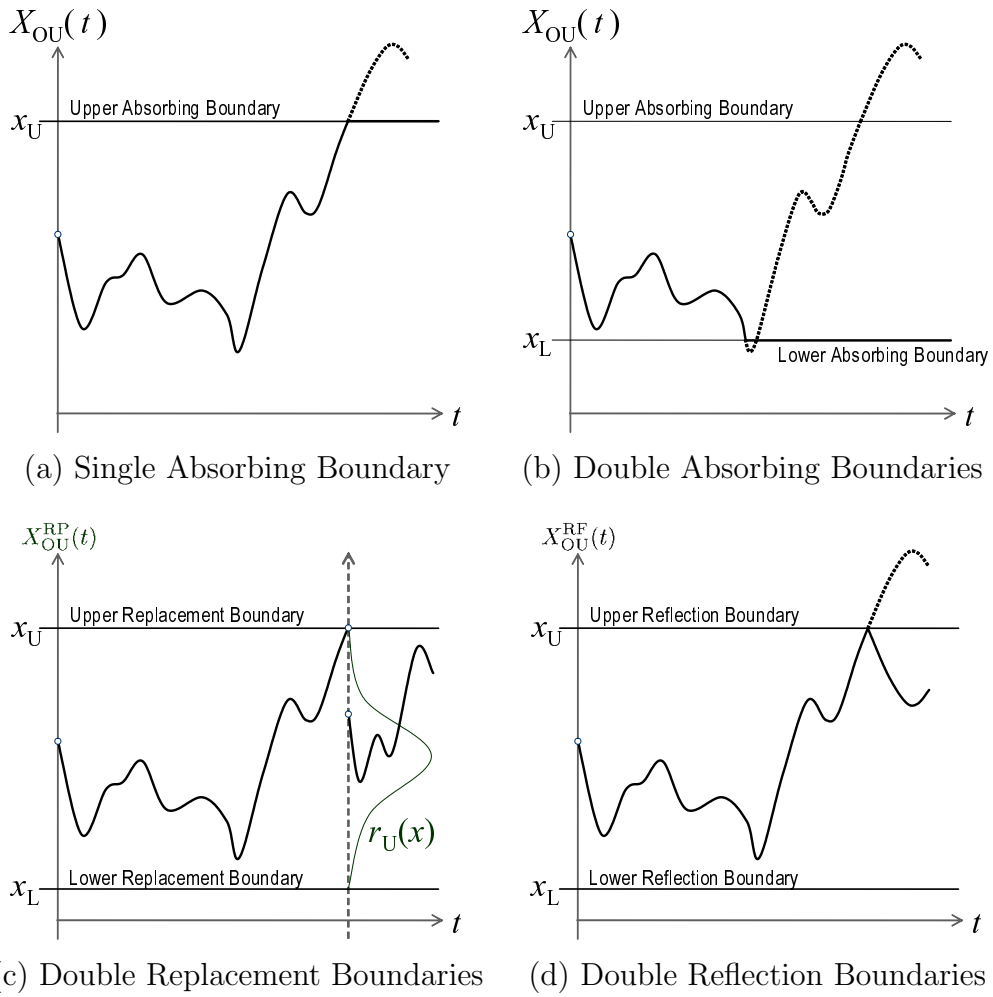


(a) Single Absorbing Boundary (b) Double Absorbing Boundaries

(c) Double Replacement Boundaries (d) Double Reflection Boundaries

Figure 0.1: Modified O-U Processes with Various Boundaries

replacement and reflection boundaries are shown in Figures 0.1(c) and 0.1(d), respectively. These boundaries play an important role in dealing with a variety of financial derivatives. The purpose of this paper is to develop computational algorithms for evaluating dynamic behavior of the modified O-U processes with such different boundaries.

In the previous paper by the authors [3], it is shown, through the spectral analysis of a birth-death process, that a sequence of Ehrenfest processes with appropriate scaling and shifting converges in law to the O-U process $\{X_{\mathrm{OU}}(t) : t \geq 0\}$. The corresponding first passage times and the historical maximum also converge in law to those of $\{X_{\mathrm{OU}}(t) : t \geq 0\}$. It is worth noting that this approach approximates the O-U process by discretizing only the state space, not the time axis. More specifically, a finite range of $\{X_{\mathrm{OU}}(t) : t \geq 0\}$ is represented by $2V + 1$ discrete states where V is a positive integer. Then the O-U process $\{X_{\mathrm{OU}}(t) : t \geq 0\}$ is approximated by $\{X_V(t) : t \geq 0\}$ which is constructed from the underlying Ehrenfest process defined on $\mathcal{N}_V = \{0, 1, ..., 2V\}$ with appropriate scaling and shifting. The zero points of the orthogonal polynomials associated with the spectral representation of

the Ehrenfest process are then computed, enabling one to evaluate the distributions of the first passage times and the historical maximum. Additional numerical experiments following the previous paper [3] have revealed that some zero points tend to cluster near the ends of $\mathcal{N}_V$ with diminishing distances among themselves. Consequently, those clustering zeros cannot be computed with accuracy for $V > 200$. For example, with $V = 200$, only two digit accuracy is assured for the survival functions of the first passage times.

In order to overcome this numerical difficulty, we propose an alternative approach based on the uniformization procedure of Keilson [2]. As we will see, the uniformization procedure is numerically stable with speed and accuracy, enabling one to cope with $V = 20,000$ or more where the computational burden increases only as a linear function of V. Based on this approach, the modified Ehrenfest processes with different boundaries are evaluated, which in turn captures the dynamic behavior of the modified O-U processes with corresponding boundaries.

The structure of this paper is as follows. In Section 1, the key results of [3] relevant to this paper are reviewed succinctly. The uniformization procedure of Keilson [2] for temporally homogeneous Markov chains in continuous time is summarized in Section 2, together with algorithms for evaluating the distributions of associated first passage times and the historical maximum. Sections 3, 4, and 5 deal with the modified O-U process with one absorbing boundary, two absorbing boundaries, and replacement and reflection boundaries, respectively. Numerical results are also presented, demonstrating the convergence of the modified Ehrenfest process as $V \to \infty$ with speed and accuracy.

For notational convenience, throughout the paper, we denote a vector by attaching single underline as $\underline{x}$, and a matrix by attaching double underlines as $\underline{\underline{a}}$. Moreover, $\underline{1}$ and $\underline{0}$ mean vectors whose all elements are 1 and 0, respectively. The vector $\underline{u}_m$ means that its element corresponding to state m is 1 and all other elements are 0. For an $N \times N$ matrix $\underline{\underline{a}}$, a submatrix on $G \subset \{1, ..., N\}$ for rows and on $B \subset \{1, ..., N\}$ for columns is denoted by $\underline{\underline{a}}_{GB} = [a_{ij}]_{i \in G, j \in B}$.

1 Convergence of Ehrenfest Process to O-U Process and Corresponding State Conversion

We consider a birth-death process $N_{2V}(t)$ on $\mathcal{N}_V = \{ 0, 1, ..., 2V \}$ governed by upward and downward transition rates given respectively by

$$\lambda_m = V - \frac{m}{2} \quad \text{and} \quad \mu_m = \frac{m}{2}, \quad m \in \mathcal{N}_V. \tag{1.1}$$

This Markov chain is called an Ehrenfest process in continuous time. From (1.1), one sees that the local growth rate of the variance is given by

$$\nu_m := \lambda_m + \mu_m = V, \quad m \in \mathcal{N}_V, \tag{1.2}$$

which is independent of m, and the local velocity is given by

$$\lambda_m - \mu_m = V - m. \tag{1.3}$$

For the associated stationary chain $\{N_{VS}(t) : t \geq 0\}$, one has

$$\mathrm{cov}\,[\,N_{VS}(t),\, N_{VS}(t+\tau)\,] = \frac{V}{4}\,\mathrm{e}^{-\tau}, \tag{1.4}$$

and asymptotic normality. The O-U process is characterized by its Markov property, normal distribution, and exponential covariance function. Because of the properties of the Ehrenfest process specified in (1.1) through (1.4) together with its asymptotic normality, one expects that a sequence of processes $\{X_V(t) : t \geq 0\}, V = 1, 2, 3, ...,$ defined by

$$X_V(t) = \sqrt{\frac{2}{V}}\,N_{2V}(t) - \sqrt{2V} \tag{1.5}$$

converges in law to the O-U process as $V \to \infty$. Indeed, this is formally proven in [3].

We note that $\{\, X_V(t) : t \geq 0\,\}$ has discrete support defined by

$$r(m) := \sqrt{\frac{2}{V}}m - \sqrt{2V}, \quad m = 0, 1, ..., 2V. \tag{1.6}$$

The correspondence between the states of $N_V(t)$ and those of $X_V(t)$ is summarized in Table 1.1, where

$$\eta_V(x) := \left\lceil \sqrt{\frac{V}{2}}x \right\rceil. \tag{1.7}$$

Table 1.1: State Conversions

Process	State Conversion		State Space
	$x \in \mathbb{R} \to m \in \mathcal{N}$	$m \in \mathcal{N} \to x \in \mathbb{R}$	
$N_V(t)$	$\eta_V(x) + V$	m	$\mathcal{N} = \{0, 1, ..., 2V\}$
$X_V(t)$	$\sqrt{\frac{2}{V}}\eta_V(x)$	$r(m)$	$\left\{-\sqrt{2V}, ..., \sqrt{2V}\right\}$

The following two theorems of [3] are relevant to this paper. For the O-U process $\{\, X_{\mathrm{OU}}(t) : t \geq 0\,\}$, its initial state is denoted by $X_{\mathrm{OU}}(0) = x_0$.

Theorem 1.1 ([3]) *For any $x_0, x \in \mathbb{R}$, let $m := V + \eta_V(x_0)$ and $n := V + \eta_V(x)$. Let $T_V(m,n) := \inf\{\, t : X_V(t) = r(n)\,|X_V(0) = r(m)\,\}$ and $T_{\mathrm{OU}}(x_0, x) := \inf\{\, t : X_{\mathrm{OU}}(t) = x|$ $X_{\mathrm{OU}}(0) = x_0\,\}$. Then, $T_V(m,n)$ converges in law to $T_{\mathrm{OU}}(x_0, x)$ as $V \to \infty$.*

Theorem 1.2 ([3]) *Let m be as in Theorem 1.1. Let $M_V(m,\tau) := \max\limits_{0\leq t\leq\tau}\{X_V(t)|X_V(0) = r(m)\}$ and $M_{\mathrm{OU}}(x_0,\tau) := \max\limits_{0\leq t\leq\tau}\{\, X_{\mathrm{OU}}(t)\,|\,X_{\mathrm{OU}}(0) = x_0\,\}$. Then, $M_V(m,\tau)$ converges in law to $M_{\mathrm{OU}}(x_0,\tau)$ as $V \to \infty$.*

2 Uniformization Procedure of Keilson and First Passage Times and Historical Maximum of Markov Chains

Let $N(t)$ be a temporally homogeneous Markov chain in continuous time defined on $\mathcal{N} := \{\, 0, 1, 2, ..., N \,\},\ N \leq \infty$. The process is governed by a set of hazard rates $\{\, \nu_{mn} \,\}$ where ν_{mn} is the transition rate from state $m \in \mathcal{N}$ to state $n \in \mathcal{N}$. Then, the infinitesimal generator $\underline{\underline{Q}}$ of $N(t)$ is given by

$$\underline{\underline{Q}} := -\underline{\underline{\nu}}_D + \underline{\underline{\nu}}\ , \tag{2.1}$$

where

$$\underline{\underline{\nu}} := [\, \nu_{mn} \,]\,; \quad \underline{\underline{\nu}}_D := \mathrm{diag}\,[\nu_1, ..., \nu_N]\,; \quad \nu_m := \sum_{n \in \mathcal{N}} \nu_{mn}\ . \tag{2.2}$$

The transition probability matrix $\underline{\underline{P}}(t) := [\, p_{mn}(t) \,]$, where $p_{mn}(t) := \mathrm{P}\{\, N(t) = n \,|\, N(0) = m \,\}$, satisfies the Kolmogorov's matrix differential equation given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{\underline{P}}(t) = \underline{\underline{Q}}\,\underline{\underline{P}}(t). \tag{2.3}$$

It then follows that

$$\underline{\underline{P}}(t) = \mathrm{e}^{t\,\underline{\underline{Q}}}\ . \tag{2.4}$$

The process is said to be *uniformizable* if its hazard rates $\{\, \nu_{mn} \,\}$ are bounded in the sense that $\nu_m \leq \nu$ for all $m \in \mathcal{N}$ for some $0 < \nu < \infty$, see Keilson [2]. For a uniformizable chain with a constant ν, let $\underline{\underline{a}}_\nu$ be a matrix defined by

$$\underline{\underline{a}}_\nu := \underline{\underline{I}} - \frac{1}{\nu}\,\underline{\underline{\nu}}_D + \frac{1}{\nu}\,\underline{\underline{\nu}}\ . \tag{2.5}$$

It is clear that the matrix $\underline{\underline{a}}_\nu$ is stochastic, i.e., $\underline{\underline{a}}_\nu \geq \underline{\underline{0}},\ \ \underline{\underline{a}}_\nu\,\underline{1} = \underline{1}$. From (2.1) and (2.5), one has $\underline{\underline{Q}} = -\nu\,\left(\underline{\underline{I}} - \underline{\underline{a}}_\nu\right)$. Substituting this into (2.4), it then follows that

$$\underline{\underline{P}}(t) = \exp\left\{\, -\nu\,t\,\left(\underline{\underline{I}} - \underline{\underline{a}}_\nu\right)\,\right\} = \sum_{k=0}^{\infty} \mathrm{e}^{-\nu\,t}\frac{(\nu\,t)^k}{k!}\underline{\underline{a}}_\nu^{\,k}. \tag{2.6}$$

It should be noted that $\underline{\underline{P}}(t)$ can be computed via (2.6) independently of ν satisfying $\nu \geq \sup_m \nu_m$. Furthermore, since the expression involves only nonnegative numbers, the computational procedure is very stable, enabling one to deal with a fairly large state space, say, in the order of 10,000. In what follows, we describe computational algorithms for evaluating distributions of first passage times and the historical maximum of the underlying Markov chain based on (2.6).

Let $G \subset \mathcal{N}$ be a set of "G"ood states and define a set of "B"ad states by $B := \mathcal{N} \setminus G$. Of interest is the first passage time from a good state $m \in G$ to the bad set B defined by

$$T_{m,B} := \inf\{\, t \mid N(t) \in B,\, N(0) = m \,\}\ . \tag{2.7}$$

For computing the distributions of such first passage times, we introduce the lossy process $N^*(t)$ obtained from the original process $N(t)$ by making all the states in B absorbing. More specifically, the transition probability matrix $\underline{\underline{P}}^*(t)$ of the lossy process is given by

$$\underline{\underline{P}}^*(t) := \begin{pmatrix} \underline{\underline{P}}_{GG}(t) & \underline{\underline{P}}_{GB}(t) \\ \underline{\underline{O}} & \underline{\underline{I}} \end{pmatrix}. \tag{2.8}$$

It is clear that the first passage time $T_{m,B}$ is greater than τ if and only if $N(t)$ does not reach B during the period $[0,\tau]$ starting with $N(0) = m \in G$. From the definition of the lossy process, the latter probability can be expressed as

$$\mathrm{P}\{\, N(t) \in G \text{ for all } t \in [0,\tau] \,|\, N(0) = m \in G \,\} = \mathrm{P}\{\, N^*(\tau) \in G \,|\, N^*(0) = m \in G \,\}. \tag{2.9}$$

Consequently, the survival function of the first passage time T_{mB} for $m \in G$ is given by

$$\overline{S}_{m,B}(\tau) := \mathrm{P}\{\, T_{m,B} > \tau \,\} = \mathrm{P}\{\, N^*(\tau) \in G \,|\, N^*(0) = m \in G \,\} = \underline{u}_m^\top \underline{\underline{P}}_{GG}(\tau)\underline{1}, \tag{2.10}$$

and the distribution function $S_{m,B}(t)$ by

$$S_{m,B}(t) := 1 - \overline{S}_{m,B}(t). \tag{2.11}$$

Applying (2.6) and (2.8), one can see that

$$\underline{\underline{P}}_{GG}(t) = \sum_{k=0}^{\infty} \mathrm{e}^{-\nu\, t} \frac{(\nu\, t)^k}{k!} \underline{\underline{a}}_{\nu:GG}^{\,k}\ . \tag{2.12}$$

From (2.10) and (2.12), it then follows that

$$\overline{S}_{m,B}(t) = \sum_{k=0}^{\infty} \mathrm{e}^{-\nu\, t} \frac{(\nu\, t)^k}{k!} \underline{u}_m^\top \underline{\underline{a}}_{\nu:GG}^{\,k} \underline{1}. \tag{2.13}$$

Hence, $\overline{S}_{m,B}(t)$ and $S_{m,B}(t)$ can be readily computed via (2.13) through repeated vector-matrix multiplications.

When the underlying Markov chain $N(t)$ is a birth-death process, all the states are readily ordered and the historical maximum process may be of interest. Let upward and downward transition rates be defined by

$$\nu_{mn} = \begin{cases} \lambda_m & \text{if } n = m+1,\ m \geq 0 \\ \mu_m & \text{if } n = m-1,\ m \geq 1 \\ 0 & \text{otherwise} \end{cases}\ . \tag{2.14}$$

Let $M(m,\tau)$ be the historical maximum of the birth-death process $N(t)$ in the time interval $[0,\tau]$ given that $N(0) = m$, i.e.,

$$M(m,\tau) := \max_{0 \leq t \leq \tau} \{\, N(t) \,|\, N(0) = m \,\}. \tag{2.15}$$

From the dual relationship between the first passage time and the historical maximum, one sees that

$$F_{m,\tau}(n) := \mathrm{P}\{M(m,\tau) \le n\} = \mathrm{P}\{T_{m,n+1} > \tau\} = \overline{S}_{m,n+1}(\tau). \tag{2.16}$$

Consequently, the distribution function of the historical maximum is given by

$$F_{m,\tau}(n) = \begin{cases} 0 & \text{if } n < m \\ \overline{S}_{m,n+1}(\tau) & \text{if } n \ge m \end{cases}, \tag{2.17}$$

where $\overline{S}_{m,n+1}(\tau)$ is the survival function of the first passage time from m to $n+1$, which is actually the first passage time from m to $B = \{n+1, n+2, ..., N\}$ in (2.10).

3 O-U Process with One Absorbing Boundary

In this section, by using the convergence results and the uniformization procedure reviewed in the preceding sections, a numerical algorithm is given for evaluating the survival (or equivalently, distribution) function of the first passage times of the modified O-U process with one absorbing boundary. While the uniformization procedure based on (2.13) involves repeated vector-matrix multiplications, the algorithm developed in this section requires only vector computations since the Ehrenfest process defined in (1.1) is a birth-death process.

Let $\{N_{2V}(t) : t \ge 0\}$ be the Ehrenfest process on $\mathcal{N}_V = \{0, ..., 2V\}$ governed by the upward and downward transition rates specified in (1.1). Since the Ehrenfest process is defined on a finite state space, it is automatically uniformizable. For $m < n$, let $G = \{0, ..., n-1\}$ and consider the lossy process $N^*_{2V}(t)$ obtained from $N_{2V}(t)$ by making all the states in $B = \{n, ...2V\}$ absorbing. Since $N_{2V}(t)$ is a birth-death process and hence is lattice continuous, it is sufficient to consider $N^*_{2V}(t)$ only on $\{0, ..., n\}$ by making state n absorbing, provided that the process starts with $N^*_{2V}(0) = m \in G$. Since the good set G is on a lower side, we denote the corresponding stochastic matrix on $\{0, ..., n\}$ by $\underline{\underline{a}}^*_{V(\mathrm{L})}$. This matrix can be obtained via the uniformization procedure as specified in (2.5) and is given by

$$\underline{\underline{a}}^*_{V(\mathrm{L})} := \begin{array}{c} \begin{array}{cccc} 0 & ... & n-1 & n \end{array} \\ \left(\begin{array}{ccc|c} & & & 0 \\ & \underline{\underline{a}}_{V(\mathrm{L}):GG} & & \vdots \\ & & & 0 \\ & & & \frac{\lambda_{n-1}}{V} \\ \hline & \underline{0}^\top & & 1 \end{array}\right) \end{array}, \tag{3.1}$$

where

$$
\underline{\underline{a}}_{V(\mathrm{L}):GG} = \frac{1}{V}
\begin{array}{c}
\begin{array}{cccccc} {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle \cdots} & {\scriptstyle n-2} & {\scriptstyle n-1} \end{array} \\
\left(\begin{array}{cccccc}
0 & \lambda_0 & 0 & \cdots & 0 & 0 \\
\mu_1 & 0 & \lambda_1 & \cdots & 0 & 0 \\
0 & \mu_2 & 0 & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & 0 & \lambda_{n-2} \\
0 & 0 & 0 & \cdots & \mu_{n-1} & 0
\end{array} \right)
\begin{array}{c} {\scriptstyle 0} \\ {\scriptstyle 1} \\ {\scriptstyle 2} \\ \vdots \\ {\scriptstyle n-2} \\ {\scriptstyle n-1} \end{array}
\end{array}
. \tag{3.2}
$$

When G is on an upper side, i.e., $G = \{n+1, ..., 2V\}$, the corresponding stochastic matrix denoted by $\underline{\underline{a}}^*_{V(\mathrm{U})}$ is obtained similarly as

$$
\underline{\underline{a}}^*_{V(\mathrm{U})} := \left(\begin{array}{c|c}
\begin{array}{c} {\scriptstyle n} \\ 1 \end{array} & \begin{array}{c} {\scriptstyle n+1 \quad \ldots \quad 2V} \\ \underline{0}^\top \end{array} \\
\hline
\begin{array}{c} \frac{\mu_{n+1}}{V} \\ 0 \\ \vdots \\ 0 \end{array} & \underline{\underline{a}}_{V(\mathrm{U}):GG}
\end{array} \right), \tag{3.3}
$$

where

$$
\underline{\underline{a}}_{V(\mathrm{U}):GG} = \frac{1}{V}
\begin{array}{c}
\begin{array}{cccccc} {\scriptstyle n+1} & {\scriptstyle n+2} & {\scriptstyle n+3} & {\scriptstyle \cdots} & {\scriptstyle 2V-1} & {\scriptstyle 2V} \end{array} \\
\left(\begin{array}{cccccc}
0 & \lambda_{n+1} & 0 & \cdots & 0 & 0 \\
\mu_{n+2} & 0 & \lambda_{n+2} & \cdots & 0 & 0 \\
0 & \mu_{n+3} & 0 & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & 0 & \lambda_{2V-1} \\
0 & 0 & 0 & \cdots & \mu_{2V} & 0
\end{array} \right)
\begin{array}{c} {\scriptstyle n+1} \\ {\scriptstyle n+2} \\ {\scriptstyle n+3} \\ \vdots \\ {\scriptstyle 2V-1} \\ {\scriptstyle 2V} \end{array}
\end{array}
. \tag{3.4}
$$

In either case, one sees from (2.12) that

$$
\begin{cases}
\underline{\underline{P}}^*_{V(\mathrm{L}):GG}(t) = \displaystyle\sum_{k=0}^{\infty} \mathrm{e}^{-Vt} \frac{(Vt)^k}{k!} \left(\underline{\underline{a}}_{V(\mathrm{L}):GG} \right)^k \in \mathbb{R}^{|G| \times |G|} \\
\underline{\underline{P}}^*_{V(\mathrm{U}):GG}(t) = \displaystyle\sum_{k=0}^{\infty} \mathrm{e}^{-Vt} \frac{(Vt)^k}{k!} \left(\underline{\underline{a}}_{V(\mathrm{U}):GG} \right)^k \in \mathbb{R}^{|G| \times |G|}
\end{cases}
. \tag{3.5}
$$

The survival function $\overline{S}_{x_0,x}(\tau)$ of the first passage time of $X_{\mathrm{OU}}(t)$ from x_0 to x is then approximated by the survival function $\overline{S}_{V:m,n}(t)$ of the first passage time of $N_V(t)$ from m

to n where $m = \eta_V(x_0) + V$ and $n = \eta_V(x) + V$, which is obtained from (3.5) as

$$\overline{S}_{V:m,n}(t) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \mathrm{e}^{-Vt} \frac{(Vt)^k}{k!} \underline{u}_m^{\top} \left(\underline{\underline{a}}_{V(\mathrm{L}):GG} \right)^k \underline{1}, & \text{for } m \in G = \{0, 1, ..., n-1\} \\ \displaystyle\sum_{k=0}^{\infty} \mathrm{e}^{-Vt} \frac{(Vt)^k}{k!} \underline{u}_m^{\top} \left(\underline{\underline{a}}_{V(\mathrm{U}):GG} \right)^k \underline{1}, & \text{for } m \in G = \{n+1, ..., 2V\}. \end{cases} \tag{3.6}$$

For the historical maximum $M_V(m, \tau) := \max_{0 \le t \le \tau} \{ X_V(t) \,|\, X_V(0) = r(m) \}$, the distribution function $F_{V:m,\tau}(n)$ satisfies the following dual relation as (2.16):

$$F_{V:m,\tau}(n) = \mathrm{P}\{M_V(m, \tau) \le r(n)\} = \mathrm{P}\{T_{V:m,n+1} > \tau\} = \overline{S}_{V:m,n+1}(\tau). \tag{3.7}$$

The distribution function $F_{V:m,\tau}(n)$ of the historical maximum of the O-U process can be computed from (2.17) and (3.6).

By exploiting the structure of any birth-death process, the computation for (3.6) can be simplified. Let $\underline{\underline{b}}$ be a matrix of the form

$$\underline{\underline{b}} = \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & n-2 & n-1 \end{array} \\ \left(\begin{array}{cccccc} 0 & \eta_0 & 0 & \cdots & 0 & 0 \\ \xi_1 & 0 & \eta_1 & \cdots & 0 & 0 \\ 0 & \xi_2 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & \eta_{n-2} \\ 0 & 0 & 0 & \cdots & \xi_{n-1} & 0 \end{array} \right) \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ n-2 \\ n-1 \end{array} \in \mathbb{R}^{n \times n}. \tag{3.8}$$

For any n-dimensional real vector $\underline{z} := (z_0, z_1, ..., z_{n-1})^{\top} \in \mathbb{R}^n$, let $\underline{z}^0$ and $\underline{z}^1 \in \mathbb{R}^{n-1}$ be defined by $\underline{z}^0 := (z_0, z_1, ..., z_{n-2})$ and $\underline{z}^1 := (z_1, z_2, ..., z_{n-1})$, respectively. We also define an operator $\otimes$ by $\underline{w} \otimes \underline{y} = (w_1 y_1, w_2 y_2, ..., w_n y_n)$. Then, for $\underline{\eta} := (\eta_0, \eta_1, ..., \eta_{n-2})$, and $\underline{\xi} := (\xi_1, \xi_2, ..., \xi_{n-1})$, one has

$$\underline{z}^{\top} \underline{\underline{b}} = (0, \underline{z}^0 \otimes \underline{\eta}) + (\underline{z}^1 \otimes \underline{\xi}, 0) \in \mathbb{R}^n. \tag{3.9}$$

We are now in a position to describe an algorithm for computing the survival function $\overline{S}_{V:m,n}(t)$ in (3.6), where a generic symbol $\underline{\underline{a}}_{V:GG}$ is employed for $\underline{\underline{a}}_{V(\mathrm{L}):GG}$ and $\underline{\underline{a}}_{V(\mathrm{U}):GG}$.

Algorithm 3.1 (Survival Function of the First Passage Time of the O-U Process from x_0 to x)

Input :

- $\triangleright$ V : parameter to describe the range $[x_{\mathrm{L}}, x_{\mathrm{U}}]$ of the O-U process by $2V + 1$ points
- $\triangleright$ $n \in \mathcal{N}_V$: the absorbing state with $B = \{n\}$ where $n = \eta_V(x) + V$

▷ G : the good set consisting of all the states on either the lower side or the upper side of n

▷ $m \in G$: the state from which $N(t)$ starts where $m = \eta_V(x_0) + V$

▷ τ : future time as the argument of the survival function

▷ $\varepsilon_{\max}, \varepsilon_{\min}$: parameters for stopping criteria for the series expansion of (3.6)

1) Set $s_{m,n} \leftarrow 0$, $k \leftarrow 0$ and $\underline{x} \leftarrow \underline{u}_m$.

2) Set $K = \max\left\{k : \mathrm{e}^{-V\tau}\frac{(V\tau)^k}{k!} < \varepsilon_{\max}\right\}$ and $k_0 = \min\left\{k : \mathrm{e}^{-V\tau}\frac{(V\tau)^k}{k!} > \varepsilon_{\min}\right\}$

3) LOOP1: $\underline{x}^\top \leftarrow \underline{x}^\top \underline{\underline{a}}_{V:GG}$.

4) If $k < k_0$, set $k \leftarrow k + 1$ and go to LOOP1.

5) LOOP2: $s_{m,n} \leftarrow s_{m,n} + \mathrm{e}^{-V\tau}\frac{(V\tau)^k}{k!}\underline{x}^\top \underline{1}$.

6) If $k < K$, set $\underline{x}^\top \leftarrow \underline{x}^\top \underline{\underline{a}}_{V:GG}$, $k \leftarrow k + 1$, and go to LOOP2.

7) Stop.

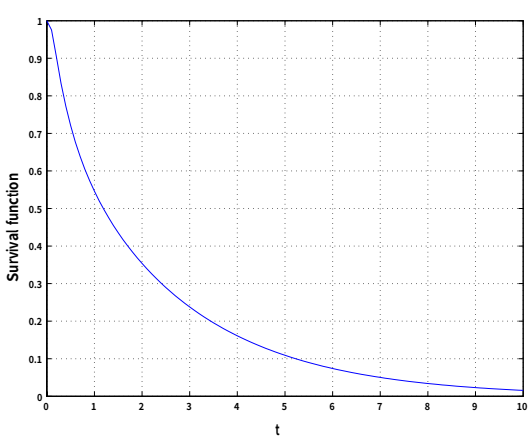
Remark : For computational stability in evaluating the sequence $\left\{\mathrm{e}^{-V\tau}\frac{(V\tau)^k}{k!}\right\}_{k=1,2,\ldots}$, we used the following recurrence formula of $b(V,k,\tau) := \ln \mathrm{e}^{-V\tau}\frac{(V\tau)^k}{k!}$:

$$b(V,k,\tau) = b(V,k-1,\tau) + \ln\frac{V\tau}{k}.$$

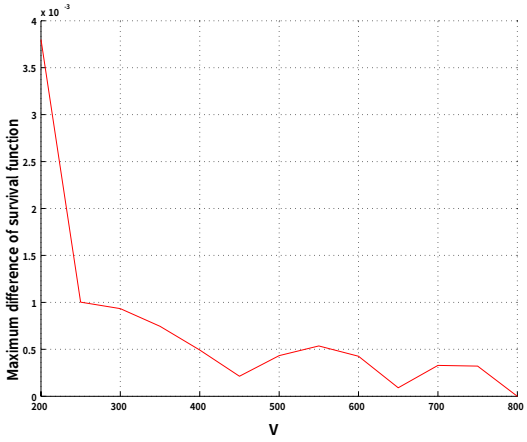
Figure 3.1(a) shows the survival function of the first passage time $T_V(m,n)$ of $X_V(t)$ from $m = \eta_V(0) + V = V$ to $n = \eta_V(1) + V$ for $V = 200$. A sequence of such survival functions converges in law to that of the first passage time $T_{\mathrm{OU}}(0,1)$ of $X_{\mathrm{OU}}(t)$ from 0 to 1 as $V \to \infty$. In Figure 3.1(b), this convergence is demonstrated by plotting $\left\|\overline{S}_{V:m,n} - \overline{S}_{800:m,n}\right\|_\infty = \sup\left\{\left|\overline{S}_{V:m,n}(\tau) - \overline{S}_{800:m,n}(\tau)\right| \,\middle|\, \tau \in [0,10]\right\}$ from $V = 200$ to $V = 800$ with step size of 50, and the supremum is taken with step size of $\Delta t = 0.1$. One observes that almost 4-digit accuracy is attained with speed at $V = 800$. The convergence is not monotone because the relative location of $x = 1$ within a discretized interval of the width $\Delta x = \sqrt{\frac{2}{V}}$ does not change monotonically as V increases.

To examine the convergence behavior from a different angle, the median value of the first passage time is computed as a function of V. Formally, this is defined as $\tau^*(x_0, x) := \overline{S}^{-1}_{V:m,n}(0.5)$, where $m = \eta_V(x_0) + V$ and $n = \eta_V(x) + V$. We call $\tau^*(x_0, x)$ the median time. Table 3.1 shows the computed median time $\tau^*(x_0, x)$ of the approximating process $X_V(t)$ from x_0 to a boundary point x for $x_0 = 0, 0.5$ and $x = 1, 2$. Only the results for V satisfying $x_0 = \sqrt{\frac{2}{V}}\eta_V(x_0)$ are shown. From this table, we see that the median time can be computed with 3-digit accuracy.

We next turn our attention to the historical maximum of $X_V(t)$, which approximates that of $X_{\mathrm{OU}}(t)$. Figure 3.2(a) displays the convergence of the distribution functions of the



(a) Survival Function $S_{V:m,n}(t)$ with $V = 200$



(b) Convergence of $S_{V:m,n}(t)$ to $S_{800}(t)$ where $V = 200, 250, ..., 800$

Figure 3.1: Survival Function of the First Passage Time of $X_V(t)$ ($m = \eta_V(0) + V = V$ and $n = \eta_V(1) + V$)

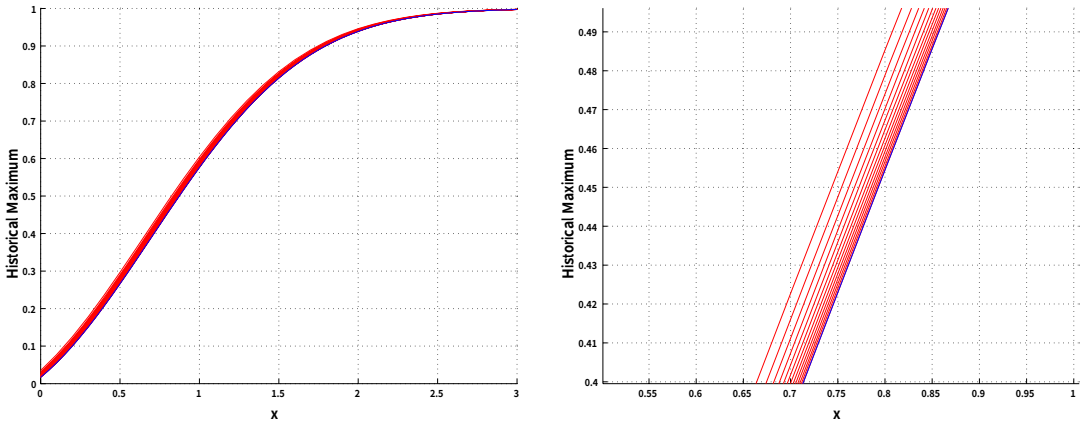
Table 3.1: Median Time of First Passage Time of X_{OU} ($\Delta t = 0.1$)

V	$\Delta x = \sqrt{\frac{2}{V}}$	$\tau^*(0,1)$	$\tau^*(0,2)$	$\tau^*(0.5,1)$	$\tau^*(0.5,2)$
200	0.1	1.18772	7.24733	0.38715	6.38354
800	0.05	1.18912	7.25101	0.38748	6.38650
3,200	0.025	1.18947	7.25192	0.38757	6.38723
5,000	0.02	1.18951	7.25203	0.37882	6.38732
20,000	0.01	1.18956	7.25218	0.37889	6.38737

historical maximum of the process with $x_0 = 0$ by varying V from 200 to 800. The enlarged view is provided in Figure 3.2(b). One sees that the speed of convergence is slower for the historical maximum than the first passage time. Table 3.2 shows the median point $x^*(x_0, \tau) := F^{-1}_{x_0,\tau}(0.5)$ of the historical maximum distribution until time τ when starting from a given point x_0 for $x_0 = 0$, 0.5 and $\tau = 1$, 10. From this table, we see that the median point can be computed with 4-digit accuracy.

4 O-U Process with Two Absorbing Boundaries

In this section, modified O-U processes with two absorbing boundaries are considered. Let x_1 and x_2 be the down and the upper boundaries respectively and define $\overline{S}_{x_0,(x_1,x_2)}(t) = \mathrm{P}\left\{T_{x_0,(x_1,x_2)} > t\right\}$ where $T_{x_0,(x_1,x_2)}$ is the first passage time of the modified O-U process from $x_0 \in (x_1, x_2)$ to either x_1 or x_2. The corresponding approximation $\overline{S}_{V:m,(n_1,n_2)}(t)$ with



(a) Overview (b) Enlarged View

Figure 3.2: Convergence of Distribution Function of Historical Maximum of $X_V(t)$ ($x_0 = 0$ and $V = 200, ..., 800$)

Table 3.2: Median Point of Historical Maximum $M(x_0, \tau)$ of $X_{\mathrm{OU}}(t)$

V	$\Delta x = \sqrt{\frac{2}{V}}$	$x^*(0,1)$	$x^*(0,10)$	$x^*(0.5,1)$	$x^*(0.5,10)$
200	0.1	0.92412	2.17776	1.24186	2.22067
800	0.05	0.92337	2.17765	1.24095	2.22057
3,200	0.025	0.92315	2.17759	1.24081	2.22052
5,000	0.02	0.92313	2.17758	1.24076	2.22052
20,000	0.01	0.92311	2.17758	1.24076	2.22052

$m = \eta_V(x_0)+V$, $n_1 = \eta_V(x_1)+V$ and $n_2 = \eta_V(x_2)+V$ can be evaluated via the uniformization procedure as for the case of one absorbing boundary. The stochastic matrix $\underline{\underline{a}}^*_V$ of interest becomes

(4.1)
$$\underline{\underline{a}}^*_V := \begin{array}{c} \begin{matrix} n_1 & n_1{+}1 & .... & n_2{-}1 & n_2 \end{matrix} \\ \left(\begin{array}{c|c|c} 1 & \underline{0}^\top & 0 \\ \hline \frac{\mu_{n_1+1}}{V} & & 0 \\ 0 & & 0 \\ \vdots & \underline{\underline{a}}_{V:GG} & \vdots \\ 0 & & 0 \\ 0 & & \frac{\lambda_{n_2-1}}{V} \\ \hline 0 & \underline{0}^\top & 1 \end{array} \right) \end{array},$$

where

$$
\underline{\underline{a}}_{V:GG} = \frac{1}{V}
\begin{array}{c}
\begin{array}{cccccc} n_1{+}1 & n_1{+}2 & n_1{+}3 & \cdots & n_2{-}2 & n_2{-}1 \end{array} \\
\left(\begin{array}{cccccc}
0 & \lambda_{n_1+1} & 0 & \cdots & 0 & 0 \\
\mu_{n_1+2} & 0 & \lambda_{n_1+2} & \cdots & 0 & 0 \\
0 & \mu_{n_1+3} & 0 & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & 0 & \lambda_{n_2-2} \\
0 & 0 & 0 & \cdots & \mu_{n_2-1} & 0
\end{array} \right)
\end{array}
\begin{array}{c} n_1{+}1 \\ n_1{+}2 \\ n_1{+}3 \\ \vdots \\ n_2{-}2 \\ n_2{-}1 \end{array}
. \tag{4.2}
$$

From (2.13), it then follows that

$$
\overline{S}_{V:m,(n_1,n_2)}(\tau) = \sum_{k=0}^{\infty} \mathrm{e}^{-V\tau} \frac{(V\tau)^k}{k!} \underline{u}_m^{\top} \left(\underline{\underline{a}}_{V:GG} \right)^k \underline{1}, \text{ for } m \in G = \{n_1+1, ..., n_2-1\}. \tag{4.3}
$$

For the historical maximum $M_V^+(m,\tau) := \max\limits_{0 \le t \le \tau} \{\, |X_V(t)| \,|\, X_V(0) = r(m) \,\}$, the distribution function $F_{V:m,\tau}^+(n_1,n_2)$ satisfies the following dual relation as before:

$$
\begin{aligned}
F_{V:m,\tau}^+(n_1,n_2) &= \mathrm{P}\,\{\, r(n_1) \le M_V(m,\tau) \le r(n_2) \,\} \\
&= \mathrm{P}\left\{ T_{m,(n_1-1,n_2+1)} > \tau \right\} \\
&= \overline{S}_{V:m,(n_1-1,n_2+1)}(\tau),
\end{aligned} \tag{4.4}
$$

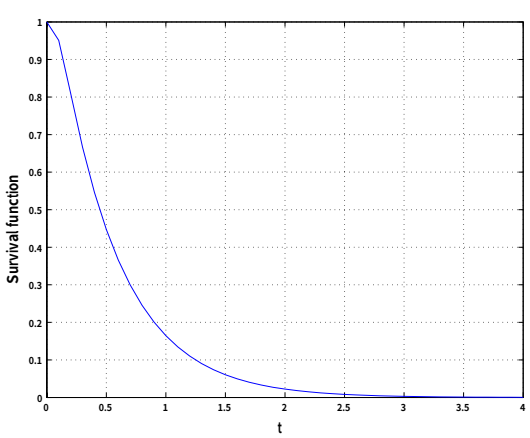
where $n_2 = 2V - n_1 \ge V$. Consequently, corresponding to (2.16), it follows that

$$
F_{V:m,\tau}^+(n_1,n_2) = \begin{cases}
\overline{S}_{V:m,(n_1,n_2)}(\tau) & \text{for } m \in \{n_1+1, ..., n_2-1\} \\
0 & \text{for } m \in \{0, ..., n_1-1\} \text{ or } m \in \{n_2+1, ..., 2V\} \\
1 & \text{for } m = n_1 \text{ or } n_2
\end{cases} . \tag{4.5}
$$

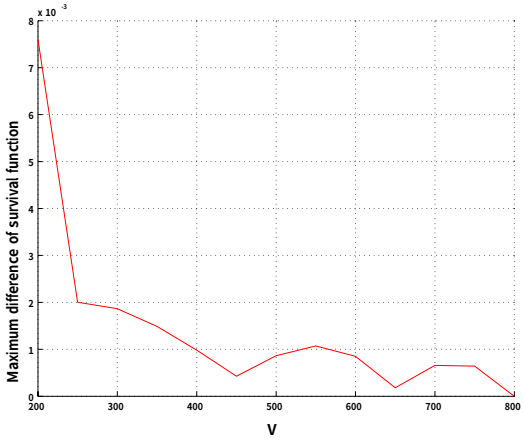
Both $\overline{S}_{V:m,(n_1,n_2)}(\tau)$ and $F_{V:m,\tau}^+(n_1,n_2)$ can be readily computed by an algorithm similar to Algorithm 3.1. Because of this similarity, the description of the algorithm is omitted here.

It should be noted that the first passage time of the absolute value process $|X_{\mathrm{OU}}(t)|$ is a special case with $x_1 = -x$ and $x_2 = x$ for $x > 0$. Let $T_{\mathrm{OU}}^+(x_0,x)$ be the first passage time of $|X_{\mathrm{OU}}(t)|$ defined by $T_{\mathrm{OU}}^+(x_0,x) := \inf\{\, t : |X_{\mathrm{OU}}(t)| = x \,|\, X_{\mathrm{OU}}(0) = x_0 \,\}$ for $x \ge 0$. The corresponding survival function is denoted by $\overline{S}_{\mathrm{OU}:x_0,x}^+(\tau) := \mathrm{P}\left\{ T_{\mathrm{OU}}^+(x_0,x) > \tau \right\}$. Figure 4.1(a) shows $\overline{S}_{\mathrm{OU}:x_0,x}^+(\tau)$ with $V = 200$ for $x_0 = 0$ and $x = 1$, and Figure 4.1(b) demonstrates the speed of convergence of such survival functions as V varies from 200 to 800 with step size of 50. Almost 4-digit accuracy is attained with speed at $V = 800$. As for Figure 3.1(b), the convergence is not monotone.

Corresponding to Table 3.1, the median times of $|X_V(t)|$ are exhibited in Table 4.1 for $x_0 = 0$, $x_1 = -1, -2$ and $x_2 = 1, 2$. One observes that the median time can be computed with 3-digit accuracy.



(a) Survival Function $S_{V:m,n}(t)$ with $V = 200$



(b) Convergence of $S_{V:m,n}(t)$ to $S_{800:m,n}(t)$ where $V = 200, 250, ..., 800$

Figure 4.1: Survival Function of the First Passage Time of $|X_V(t)|$
$(m = \eta_V(0) + V = V$ and $n = \eta_V(1) + V)$

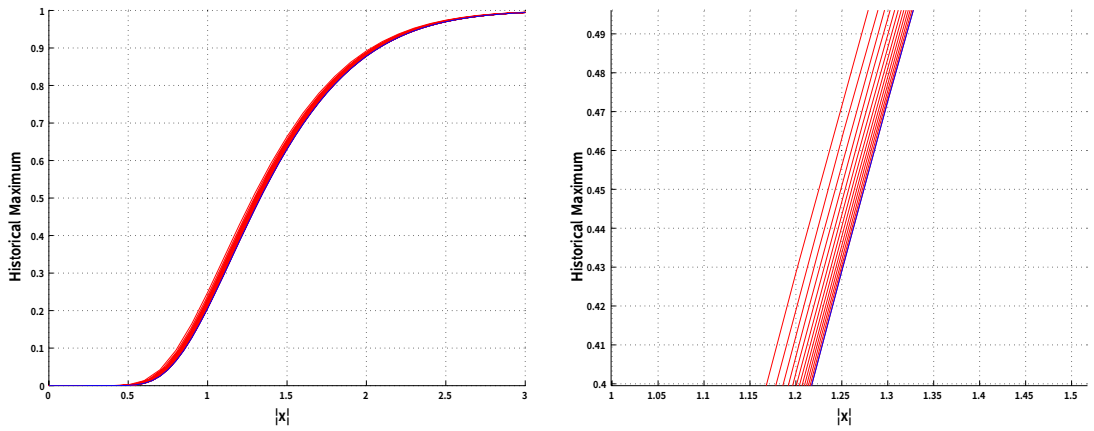
Table 4.1: Median Time of First Passage Time of $|X_V(t)|$

V	$\Delta x = \sqrt{\frac{2}{V}}$	$\tau^*_{0,(-1,1)}$	$\tau^*_{0,(-2,2)}$	$\tau^*_{0.5,(-1,1)}$	$\tau^*_{0.5,(-2,2)}$
200	0.1	0.44659	3.24198	0.30079	3.11018
800	0.05	0.44721	3.24366	0.30142	3.11172
3,200	0.025	0.44736	3.24408	0.30158	3.11211
5,000	0.02	0.44738	3.24413	0.30160	3.11215
20,000	0.01	0.44740	3.24419	0.30162	3.11218

Table 4.2: Median Point of Historical Maximum of $|X_V(t)|$

V	$\Delta x = \sqrt{\frac{2}{V}}$	$x^*_{0,1}$	$x^*_{0,10}$	$x^*_{0.5,1}$	$x^*_{0.5,10}$
200	0.1	1.38328	2.54814	1.44852	2.55399
800	0.05	1.38239	2.54823	1.44707	2.55409
3,200	0.025	1.38207	2.54837	1.44691	2.55422
5,000	0.02	1.38201	2.54840	1.44690	2.55424
20,000	0.01	1.38201	2.54840	1.44690	2.55424

Figure 4.2(a) shows the distribution functions of the historical maximum of the absolute value process for V from 200 to 800 with step size of 50. These graphs are enlarged in Figure 4.2(b) so as to see the convergence speed better. Table 4.2 shows the median value of the



(a) Overview (b) Enlarged View

Figure 4.2: Convergence of Distribution Function of Historical Maximum of $|X_V(t)|$ ($x_0 = 0$ and $V = 200, ..., 800$)

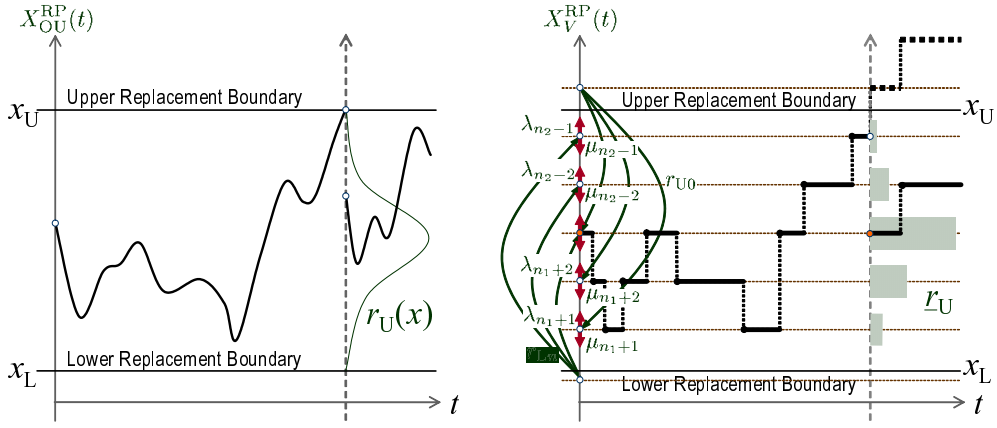
historical maximum of the absolute value process with 3-digit accuracy.

5 O-U Process with Two Replacement and Reflection Boundaries

In contrast with absorbing boundaries discussed in the previous two sections, a replacement boundary moves the process to a state in G according to a certain probability law as soon as the process reaches B. The purpose of this section is to establish a numerical algorithm to capture the dynamic behavior of modified O-U processes with such replacement boundaries. The relationship between a modified O-U process with one replacement boundary and that with two replacement boundaries is similar to the relationship for absorbing boundaries. Because of this, only the cases of two replacement boundaries are discussed here.

We say that $\{X_{\mathrm{OU}}^{\mathrm{RP}}(t) : t \geq 0\}$ has two replacement boundaries at x_{L} and x_{U} with replacement probability density functions $r_{\mathrm{L}}(x)$ and $r_{\mathrm{U}}(x)$ respectively if an instantaneous replacement to state $x \in (x_{\mathrm{L}}, x_{\mathrm{U}})$ occurs according to $r_{\mathrm{L}}(x)$ or $r_{\mathrm{U}}(x)$ as soon as the process reaches x_{L} or x_{U}, respectively. Figure 5.1(a) illustrates the movement of a modified O-U process with two replacement boundaries. The movement of the approximating process $\{X_V^{\mathrm{RP}}(t) : t \geq 0\}$ is depicted in Figure 5.1(b), where the replacement probability vectors $\underline{r}_{\mathrm{L}}$ and $\underline{r}_{\mathrm{U}}$ are employed instead of $r_{\mathrm{L}}(x)$ and $r_{\mathrm{U}}(x)$.

It should be noted that replacements for $X_V^{\mathrm{RP}}(t)$ occur as soon as the process reaches either $r(n_1)$ or $r(n_2)$ starting from $r(m)$ where $m = \eta_V(x_0) + V$, $n_1 := \eta_V(x_{\mathrm{L}}) + V$, and $n_2 := \eta_V(x_{\mathrm{U}}) + V$. As in (1.5), the relationship between $X_V^{\mathrm{RP}}(t)$ and the associated Ehrenfest



(a) $X_{\rm OU}^{\rm RP}(t)$ (b) $X_V^{\rm RP}(t)$

Figure 5.1: Two Processes with Double Replacement Boundaries

process $N_{2V}^{\rm RP}(t)$ is given by

$$X_V^{\rm RP}(t) = \sqrt{\frac{2}{V}}\, N_{2V}^{\rm RP}(t) - \sqrt{2V}. \tag{5.1}$$

It can be readily seen that $N_{2V}^{\rm RP}(t)$ has the transition probability matrix $\underline{\underline{P}}^{\rm RP}(t)$ given via the uniformization procedure as

$$\underline{\underline{P}}^{\rm RP}(t) = \sum_{k=0}^{\infty} {\rm e}^{-V\,t} \frac{(V\,t)^k}{k!} \underline{\underline{a}}_V^{{\rm RP}\,k}, \tag{5.2}$$

where

$$\underline{\underline{a}}_V^{\rm RP} = \left(\begin{array}{c|c|c} 0 & \underline{r}_{\rm L}^{\top} & 0 \\ \hline \frac{\mu_{n_1+1}}{V} & & 0 \\ \underline{0} & \underline{\underline{a}}_{V(n_1+1:n_2-1)} & \underline{0} \\ 0 & & \frac{\lambda_{n_2-1}}{V} \\ \hline 0 & \underline{r}_{\rm U}^{\top} & 0 \end{array}\right) \tag{5.3}$$

and

$$\underline{\underline{a}}_{V(n_1+1:n_2-1)} = \frac{1}{V} \begin{array}{c} \begin{array}{ccccccc} n_1{+}1 & n_1{+}2 & n_1{+}3 & \cdots & n_2{-}2 & n_2{-}1 \end{array} \\ \left(\begin{array}{cccccc} 0 & \lambda_{n_1+1} & 0 & \cdots & 0 & 0 \\ \mu_{n_1+2} & 0 & \lambda_{n_1+2} & \cdots & 0 & 0 \\ 0 & \mu_{n_1+3} & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & \lambda_{n_2-2} \\ 0 & 0 & 0 & \cdots & \mu_{n_2-1} & 0 \end{array}\right) \end{array} \begin{array}{c} n_1{+}1 \\ n_1{+}2 \\ n_1{+}3 \\ \vdots \\ n_2{-}2 \\ n_2{-}1 \end{array}. \tag{5.4}$$

Clearly, $\underline{\underline{a}}_V^{\mathrm{RP}}$ is ergodic and so is $N_{2V}^{\mathrm{RP}}(t)$ and $X_V^{\mathrm{RP}}(t)$. Hence, of interest is to compute the time dependent tail state probability defined as

$$\overline{G}_V^{\mathrm{RP}}(t,x) := \mathrm{P}\left\{X_V^{\mathrm{RP}}(t) > x = r(n)\right\} = \mathrm{P}\left\{N_{2V}^{\mathrm{RP}}(t) > n\right\} = \sum_{k=n+1}^{n_2} p_{mk}^{\mathrm{RP}}(t), \tag{5.5}$$

where $\underline{p}_m^{\mathrm{RP}}(t) := \left(p_{mn_1}^{\mathrm{RP}}(t), ..., p_{mn_2}^{\mathrm{RP}}(t)\right)^\top$ is computed from (5.2) by $\underline{p}_m^{\mathrm{RP}}(t)^\top = \underline{u}_m^\top \underline{\underline{P}}^{\mathrm{RP}}(t)$. As $V \to \infty$, $\overline{G}_V^{\mathrm{RP}}(t,x)$ converges to $\overline{G}_{\mathrm{OU}}^{\mathrm{RP}}(t,x) := \mathrm{P}\left\{X_{\mathrm{OU}}^{\mathrm{RP}}(t) > x = r(n)\right\}$.

Figure 5.2 shows $\overline{G}_V^{\mathrm{RP}}(t,x)$ with $V = 800$, $x_{\mathrm{L}} = -2$, $x_{\mathrm{U}} = 2$, and $t = 0.1, 0.2, ..., 0.6$, as well as the ergodic distribution

$$\overline{G}_V^{\mathrm{RP}}(\infty,x) = \sum_{k=n+1}^{n_2} e_k^{\mathrm{RP}}, \tag{5.6}$$

where the ergodic vector $\underline{e}^{\mathrm{RP}} := \left(e_{n_1}^{\mathrm{RP}}, ..., e_{n_2}^{\mathrm{RP}}\right)^\top$ is obtained from (5.3) by solving $\underline{e}^{\mathrm{RP}\top} = \underline{e}^{\mathrm{RP}\top} \underline{\underline{a}}_V^{\mathrm{RP}}$ with $\underline{e}^{\mathrm{RP}\top}\underline{1} = 1$. The two replacement probability vectors are taken to be a binomial distribution $r_{\mathrm{L}k} = r_{\mathrm{U}k} = \binom{n}{k} 0.5^k\, 0.5^{n-k}$, $k = 0, 1, ..., n = n_2 - n_1 - 1$. One can see that $\overline{G}_V^{\mathrm{RP}}(t,x)$ converges to the ergodic distribution $\overline{G}_V^{\mathrm{RP}}(\infty,x)$ as t increases.

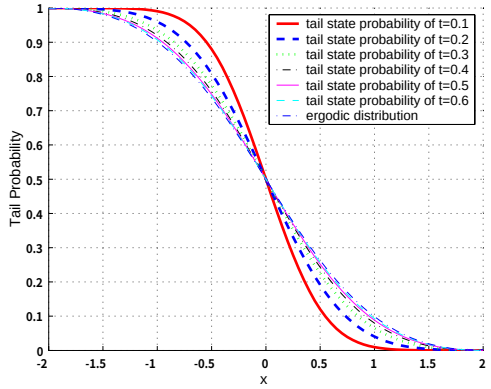


Figure 5.2: Convergence of Tail State Probability with Two Replacement Boundaries ($V = 800$, $x_{\mathrm{L}} = -2$, $x_{\mathrm{U}} = 2$)

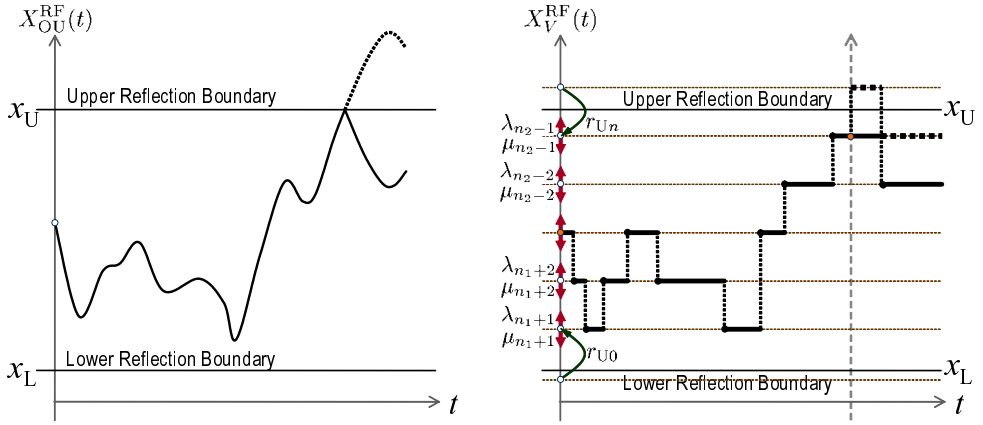
A modified O-U process with two reflection boundaries at x_{L} and x_{U}, denoted by $\{X_{\mathrm{OU}}^{\mathrm{RF}}(t) : t \geq 0\}$, is a special case of $\{X_{\mathrm{OU}}^{\mathrm{RP}}(t) : t \geq 0\}$ with two replacement boundaries at x_{L} and x_{U}. More specifically, the approximating process $\{N_V^{\mathrm{RF}}(t) : t \geq 0\}$ has the transition probability matrix $\underline{\underline{P}}^{\mathrm{RP}}(t)$, which is obtained as

$$\underline{\underline{P}}^{\mathrm{RF}}(t) = \sum_{k=0}^{\infty} \mathrm{e}^{-V\,t} \frac{(V\,t)^k}{k!} \underline{\underline{a}}_V^{\mathrm{RF}\,k}, \tag{5.7}$$

where

$$\underline{\underline{a}}_V^{\mathrm{RF}} = \left(\begin{array}{c|c|c} 0 & \underline{r}_{\mathrm{L}}^\top & 0 \\ \hline \frac{\mu_{n_1+1}}{V} & & 0 \\ \underline{0} & \underline{\underline{a}}_{V(n_1+1:n_2-1)} & \underline{0} \\ 0 & & \frac{\lambda_{n_2-1}}{V} \\ \hline 0 & \underline{r}_{\mathrm{U}}^\top & 0 \end{array}\right) \tag{5.8}$$

with $\underline{r}_{\mathrm{L}} = (1, 0, ..., 0)^\top$, $\underline{r}_{\mathrm{U}} = (0, ..., 0, 1)^\top$, and $\underline{\underline{a}}_{V(n_1+1:n_2-1)}$ given by (5.4). Figure 5.3(a) illustrates the movement of $X_{\mathrm{OU}}^{\mathrm{RF}}(t)$ and the movement of the approximating process $X_V^{\mathrm{RF}}(t)$ is depicted in Figure 5.3(b).



(a) $X_{\mathrm{OU}}^{\mathrm{RF}}(t)$ (b) $X_V^{\mathrm{RF}}(t)$

Figure 5.3: Two Processes with Double Reflection Boundaries

Similarly to (5.5) and (5.6), one has

$$\overline{G}_V^{\mathrm{RF}}(t,x) := \mathrm{P}\left\{X_V^{\mathrm{RF}}(t) > x = r(n)\right\} = \mathrm{P}\left\{N_{2V}^{\mathrm{RF}}(t) > n\right\} = \sum_{k=n+1}^{n_2} p_{mk}^{\mathrm{RF}}(t), \tag{5.9}$$

and

$$\overline{G}_V^{\mathrm{RF}}(\infty,x) = \sum_{k=n+1}^{n_2} e_k^{\mathrm{RF}}, \tag{5.10}$$

where $\underline{p}_m^{\mathrm{RF}}(t) := \left(p_{mn_1}^{\mathrm{RF}}(t), ..., p_{mn_2}^{\mathrm{RF}}(t)\right)^\top$ and $\underline{e}^{\mathrm{RF}} := \left(e_{n_1}^{\mathrm{RF}}, ..., e_{n_2}^{\mathrm{RF}}\right)^\top$ are computed from $\underline{p}_m^{\mathrm{RF}}(t)^\top = \underline{u}_m^\top \underline{\underline{P}}^{\mathrm{RF}}(t)$ and $\underline{e}^{\mathrm{RF}\top} = \underline{e}^{\mathrm{RF}\top} \underline{\underline{a}}_V^{\mathrm{RF}}$ with $\underline{e}^{\mathrm{RF}\top}\underline{1} = 1$, respectively. As before, $\overline{G}_V^{\mathrm{RF}}(t,x)$ converges to $\overline{G}_{\mathrm{OU}}^{\mathrm{RF}}(t,x) := \mathrm{P}\left\{X_{\mathrm{OU}}^{\mathrm{RF}}(t) > x = r(n)\right\}$ as $V \to \infty$.

Figure 5.4 shows $\overline{G}_V^{\mathrm{RF}}(t,x)$ and $\overline{G}_V^{\mathrm{RF}}(\infty,x)$ with $V = 800$, $x_{\mathrm{L}} = -2$, $x_{\mathrm{U}} = 2$, and $t = 0.1, 0.2, ..., 1.0$. One can see that $\overline{G}_V^{\mathrm{RF}}(t,x)$ converges to the ergodic distribution $\overline{G}_V^{\mathrm{RF}}(\infty,x)$ as t increases.

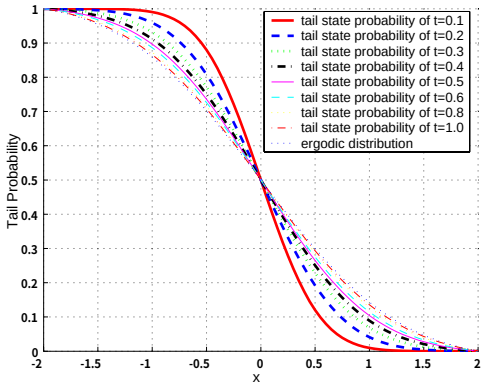


Figure 5.4: Convergence of Tail State Probability with Two Reflection Boundaries ($V = 800$, $x_{\mathrm{L}} = -2$, $x_{\mathrm{U}} = 2$)

6 Concluding Remarks

In this paper, computational algorithms are developed for capturing the dynamic behavior of modified O-U processes with one or two of absorbing, replacement or reflection boundaries. It is shown that, using the Ehrenfest approximation of Sumita, Gotoh and Jin [3] combined with the uniformization procedure of Keilson [2], such dynamic behaviors as the first passage time, the historical maximum and the time dependent tail state probabilities as well as the ergodic probabilities can be computed with speed and accuracy. The numerical procedures developed in this paper would provide useful tools for evaluating a class of derivatives involving modified O-U processes with various boundaries. This study is in progress and will be reported elsewhere.

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